Mathematical Logic
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Lecture - 21
Adequacy of Tableaux

Suppose you are given with a finitely generated tree and also you know that length of each path is finite there, then our question was does it mean that or does it imply that, the whole tree is finite. It is clear that if number of paths is also finite then the tree has to be finite, but we are not given that the number of paths is finite; we are only given that it is finitely generated, right? So we thought that it needs a proof, that is the reason.

Once we see that there is a proof, see sometimes infinite things are easy to handle than the finite things. So we will take the contrapositive of the statement; we will pose the problem as if it is a finitely generated infinite tree; does it imply that there exists one infinite path in the tree or not? This is the contrapositive statement. Let us try to prove that.

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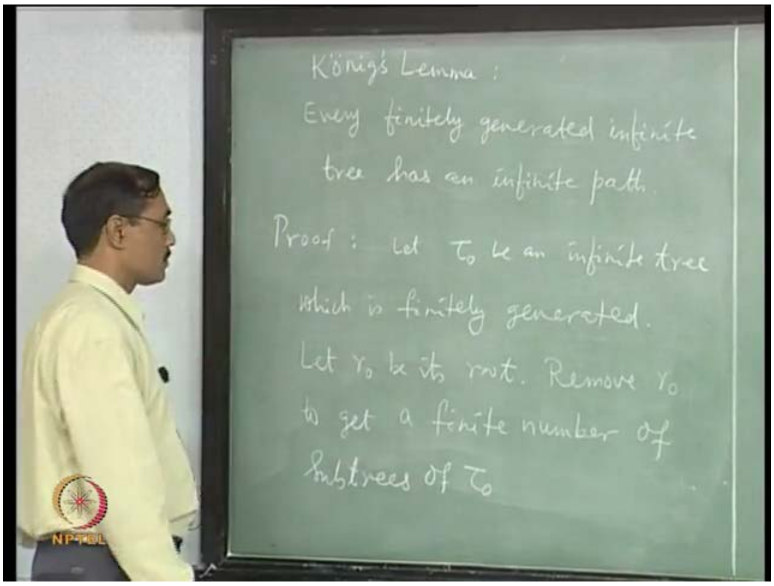

Every finitely generated infinite tree has an infinite path, this is what we want to prove. This is called Konig's Lemma. Of course, we have realized the truth of it, but now you want to a proof because, it is not, it does not follow obviously; so we need a proof. How do you go around a proof? You want to really find out a path which, whose length should be infinite. Now, think of the tree itself. Suppose the tree is given; it is finitely generated. We know, it is finite; now remove the root from it. Once you remove the root, you will get a finite number of subtrees because it is finitely generated; root has only finite number of children. Right? Now, the children of the root become the roots of the new subtrees and there are finite number of subtrees, fine.

The question is what can be about the finiteness, infiniteness of the subtrees? At least one of them has to be infinite, because if all of them are finite, then total number will become finite right. See, you are concerned about finiteness or infiniteness of trees that is why you are taking it. So, there is at least one infinite tree there; so take one such. There, remove the root of the tree. Again, you get the number of subtrees, some finite number of subtrees. There, one infinite tree, at least one infinite tree is there. So, take one such; continue the process. What happens is, at each n-th stage, you will get that there is another in the n plus 1 -th stage, a subtree which is infinite.

Now, take the sequence of the subtrees. It is an infinite sequence, because of that proposition “for each n, you get there is a stage n plus 1, which is having an infinite sub-tree. Therefore, this sequence of subtrees is infinite, right? Once you take this sequence of subtrees, take the roots of all those subtrees; you get one infinite path. That is the proof. Is it clear? We will not write it, or you want to write it? We will write it.

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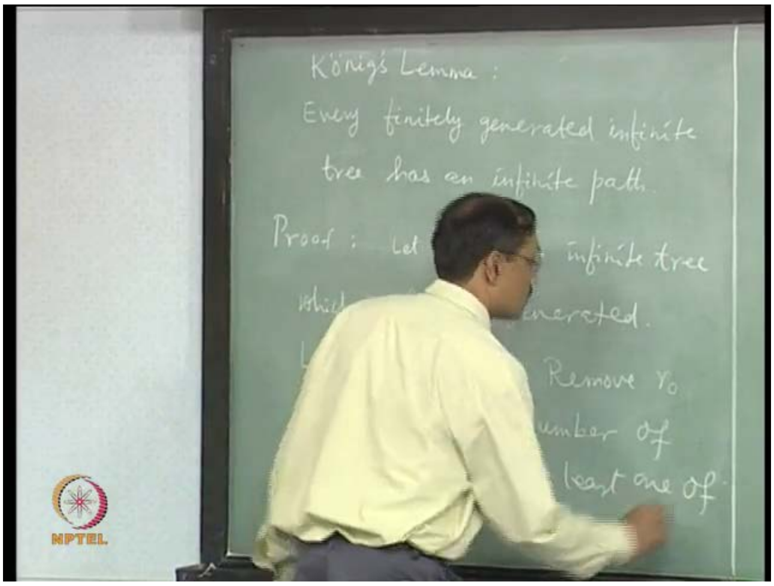

Let tau 0 be an infinite tree which is finitely generated. Then what we do, let r0 be its root. Now, remove r0 to get a finite number of subtrees of tau 0. At least one of them is an infinite tree since tau 0 is infinite. Then what we want is take that, take one such or one subtree which is infinite.

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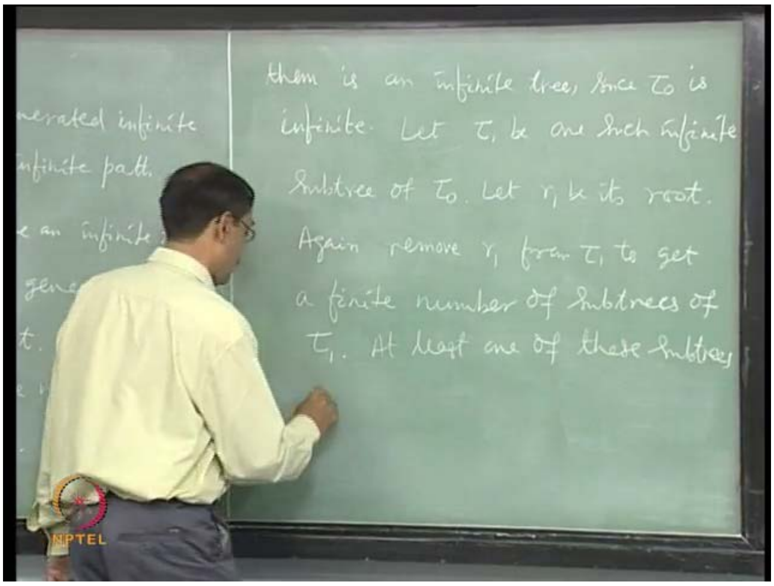

Let tau 1 be one such infinite subtree of tau 0, let r1 be its root. Again remove r1 from tau 1 to get a finite number of subtrees of tau 1. At least one of these subtrees is infinite since tau 1 is infinite.

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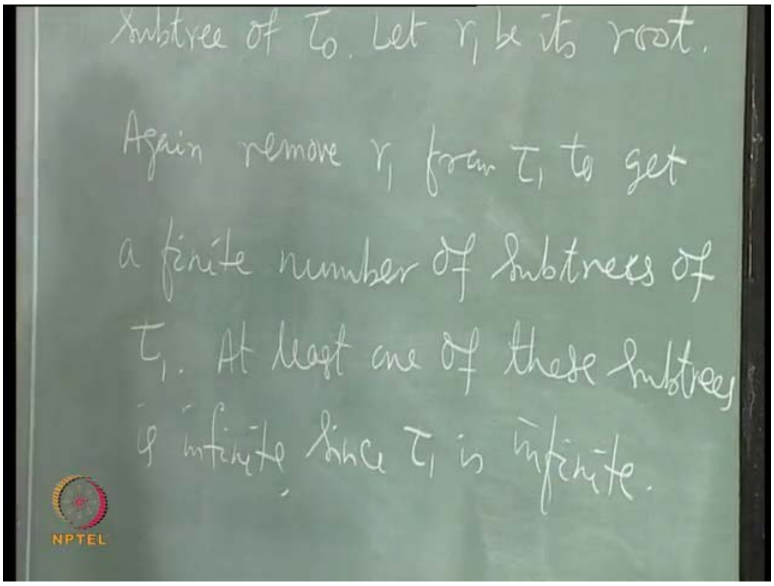

Then, let tau 2 be one such infinite subtree of tau 1 having root as r2. I will not write the index of stages here.

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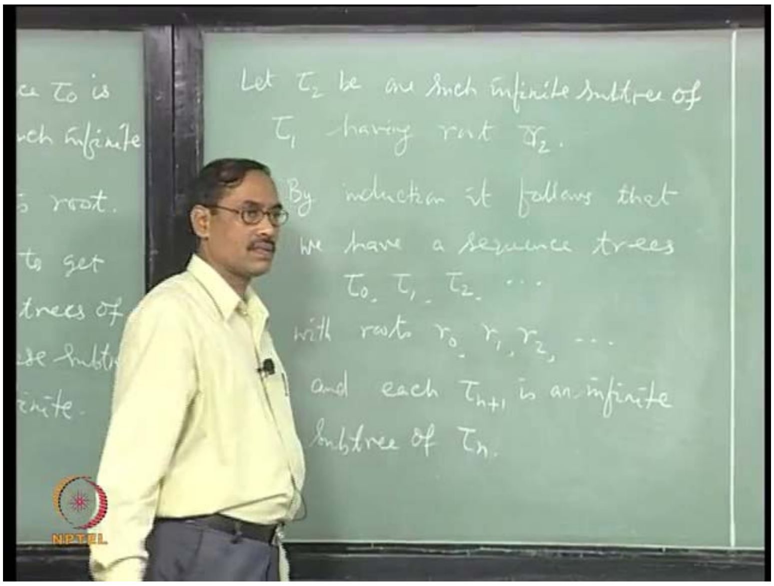

By induction it follows that we have a sequence of trees tau 0, tau 1, tau 2, and so on with roots r0, r1, r2, and so on; and each tau n plus 1 is a subtree of, is really an infinite subtree of tau n. This sequence is an infinite sequence. Once we do not write anything, it means infinite sequence, otherwise you will write finite sequence. Then the path r0, r1, r2, and so on, is an infinite path in tau 0. That is the end of the proof.

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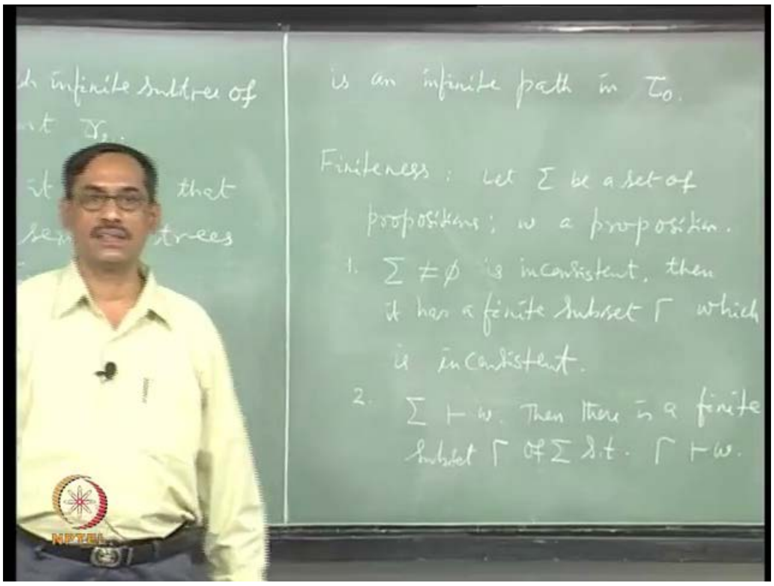

See, usually induction proofs hold only for finite things, finite stages; there, a property is satisfied by some n. Whatever be that n, that property will be true. Now, you are telling that there is an infinite thing; it is a construction. This means your inductive step is important. It says for every n there is another n which exists; therefore, the path is infinite; so it needs that argument. Once this is over, we can come back to our original question. You take its contraposition which says that if a tree is finitely generated and all its paths are finite then the tree has to be finite; that is what we wanted.

We can state our finiteness theorem for the tableaux. Suppose some set of propositions is inconsistent. Then you can find one finite subset of it which is also inconsistent, that is what finiteness theorem for PC was. Since a proof is of finite length, you take only all the propositions used in that proof in PC. So that itself entails the last one, right? That was the scenario in PC. But here, it is not a sequence of propositions; there, it was easy because it is a sequence.

Now, you have paths and the whole tree is there. You have to take care of the tree itself, right? And there, it is easier, because from the premises each one follows in a proof. Here, it is not done. It may follow, it may not follow, one proposition that is coming. So, you have to find some relation between them; a similar relation, to connect with the semantics also, first let us see the finiteness.

Finiteness says that: let sigma be a set of propositions, w a proposition; then what happens, we will have two formulations as earlier. We take sigma is inconsistent then it has a finite subset which is inconsistent. It should be nonempty, right? Let us start with a nonempty set. Suppose sigma is nonempty set, which is inconsistent. Then it has a finite subset, say, gamma which is inconsistent. This is for consequence or inconsistency. Similarly, you can have the formulation for entailment. If you think of reductio ad absurdum. But, here reductio ad absurdum is inside the definition itself. When you say sigma entails something it means you have to consider sigma union not w and tableau for it, fine? So it is there inside. Then we can say sigma entails w, then what happens, there is a finite subset gamma of sigma, such that gamma entails w. This entailment is now in PT, by analytic tableau, propositional analytic tableau not by PC, as earlier. We are not writing it explicitly.

How do you proceed, just see what is written, then you can find out it is not difficult. Suppose, you take this, sigma is inconsistent. Once it is inconsistent, you have to consider only the systematic tableau, say, it is ordered as written, some ordering is there. Now in the systematic tableau you see that it is closed because it is inconsistent, right?Each path, closed path, has to be finite; it is finitely generated, right? Now, Konig's Lemma says the whole tableau is finite. If the tableau is finite, then take all the premises occurring on the tableau so that the tableau itself is a tableau for that set of premises. You do not have to go to sigma, right. Whatever premises from sigma are occurring in the tableau, the tableau is for that, right; and that set is finite because it is a subset of the set of propositions in the tableau that is your gamma, is it clear?

Student: Gamma will be a finite proper subset, or not necessary, just a subset.

Sigma can be finite itself; so everything may be used, it is just a p comma not p.

Student: We are just saying finite.

Finite, even if sigma is infinite, you will always get a finite that is what we want. It is really crucial to think about infinite to finite, finite to finite does not give any meaning, it may be, there will be many more propositions in sigma which are not used in the tableau; that is fine. But we are not speaking about that; it is really infinite to finite reduction. So, how do you go about the proof now?

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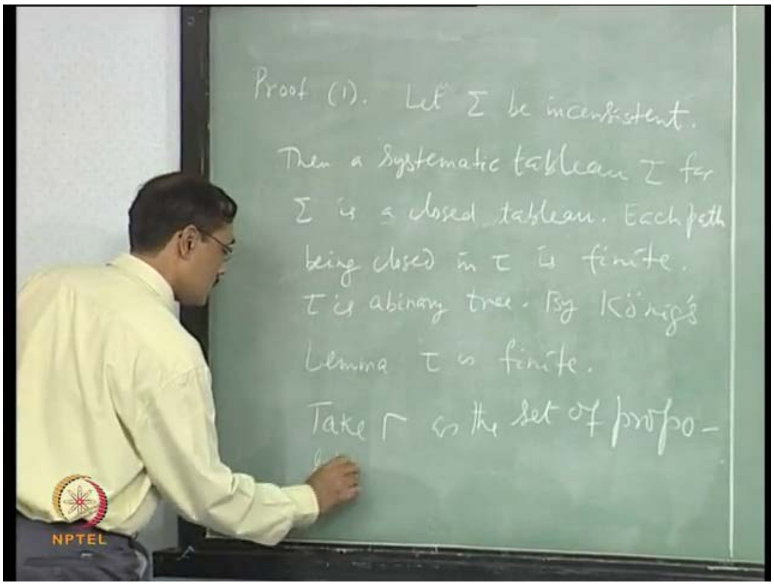

One, let sigma be inconsistent then a systematic tableau tau for sigma is a closed tableau. It is a closed tableau, means each path in tau is a closed path. That means, each path being closed in tau is finite, tau is also binary, right, it is finitely generated. So, tau is a binary tree. By Konig's Lemma, tau is finite. So, tau is finite means, number of nodes in tau is finite. Once this is done, what we do; you have to construct gamma, fine. Take gamma as the intersection of tau with sigma. Right, that is what we thought. We will not write intersection; it is a tree; that is a set; so, we will say take gamma as the set of propositions from sigma that occur in tau. Then we see that, first thing is, gamma is a finite subset of sigma.

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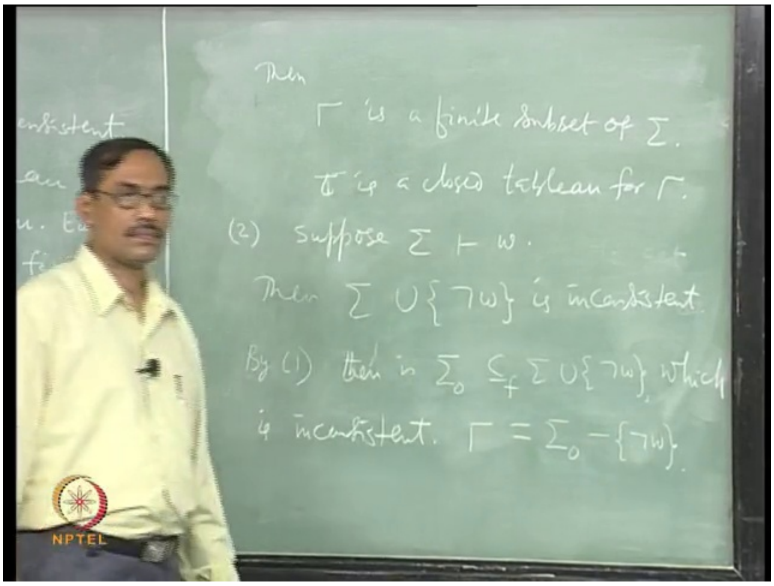

Second, tau is a closed tableau for gamma, that is all we have done. We want one finite subset gamma of sigma which is inconsistent; that is what we have got. Then second should be easy to prove. How to prove second? See, all the while we have not come to semantics yet; we are still in the tableau, in the proof system, consistency, entailment in the proof system only; we have to come to that slowly. Now, what about this? Suppose entails w in PT, that means, that is a closed tableau for sigma union not w, by definition. We want to produce a gamma of sigma such that gamma union not w has a closed tableau.

Student: Gamma union w sigma union w is only that.

Sigma union not w.

Student: Sigma union not w is only the previous.

Suppose sigma union not w we consider. It is inconsistent then?

Student: So, there exists at least one closed branch.

Ah there exists a, he wants to apply one. There exists a finite subset of sigma union not w which is inconsistent; then? Suppose that is sigma 0. Sigma 0 may have not w in it, may not have not w in it, entails it, because we want in that form.

Student: If sigma 0 is inconsistent, sigma zero union not w is also inconsistent.

Right.

Student: So, sigma zero entails.

That needs a proof; you have not proved one, but that is easy to prove; right. Just add not w at the root, with the same tableau; do not use it, how does it matter? At the root, what last proposition at root, but last. So systematic tableau has not used it. It will recover in next clause, but before that it is closed; so that itself is a tableau proof of that, then? Then that is sigma zero becomes your gamma. If it does not contain, right. Now, if it contains?

Student: So, if it contains then sigma union not w is inconsistent.

Inconsistent.

Student: He is saying sigma not w itself is.

It is there. Not w itself is there.

Student: Then you have sigma dash is just, just sigma not w is sigma dash union not w.

Take away that not w from it that is your gamma is that ok? Why do you have to take away? Because gamma should be a finite subset of sigma, not of sigma union not w, right? That is the problem.

Student: We also have to show that it also entails w.

It also entails w because that union not w is inconsistent; that is already there. All that you have to take care is that, to be a finite subset of sigma, not of sigma, union not w, there only some care is required. So, let us write the proof. Suppose sigma entails w then we get sigma union not w as inconsistent. By one there is, there is sigma 0, a finite subset of sigma union not w, which is inconsistent. By one we have a finite subset of sigma union not w which is inconsistent; right? Our plan is to start with gamma which is equal to sigma 0 minus not w.

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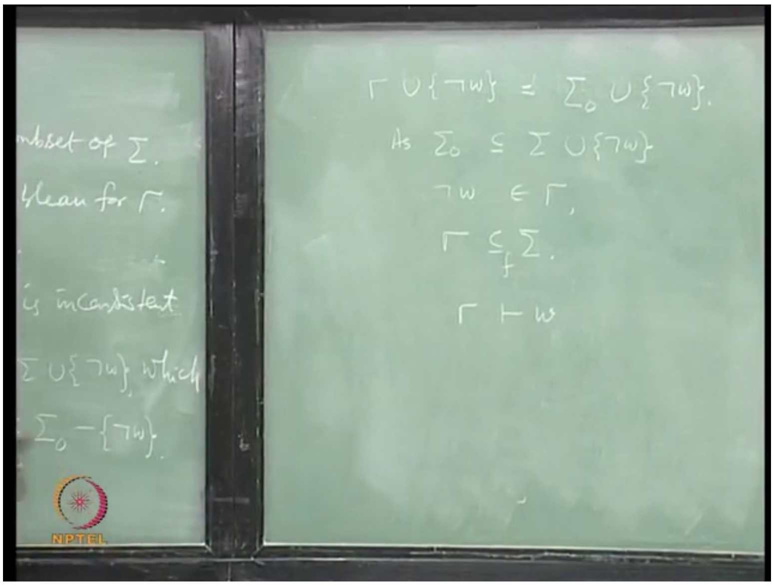

Then what we see is gamma union not w, is equal to sigma 0 union not w, as sigma 0 is a subset of sigma union not w and not w does not belong to gamma, gamma is a subset of sigma; is it clear? See, in sigma 0 there is a possibility that not w is there; right, but gamma is a subset of sigma 0, where not w is not there so all the propositions in gamma are from sigma 0, but that none of them is equal to not w.

And sigma 0 can have not w or propositions from sigma so not w is not there; so all that remains in sigma 0 are from sigma, gamma is a subset of that; so gamma has to be containing propositions from sigma only. Because only exception was not w and not w is not there. So, gamma is a subset of sigma. But gamma is also a subset of sigma 0; it is equal to sigma 0 minus w, which is finite. So, gamma is a finite subset of sigma. Gamma is a finite subset of sigma, gamma union not w equal to sigma 0 union not w, which is inconsistent; right. Therefore, gamma entails w. That is what we require.

Student: Not w does not belong to gamma.

Not w does not belong to gamma.

Student: You have written belongs to.

Does not belong to gamma. Here, we have to take care of that reasoning that gamma has to be a subset of sigma. When that happens, because not w does not belong to gamma; is it clear? So, second part is also proved. This is our finiteness of consequence or inconsistency. Now, if you see the contraposition, it says, if every finite subset of a nonempty set is consistent then the whole set is also consistent. It is something like your induction process because every finite things as they are consistent you see the tableau; it is, there exists an open path in every finite subtree. In every subtree of that tableau; in the tableau also has an open path, right. So, there really Konig's Lemma plays the role.

Our main concern was how to connect this proof theory notions to the semantics. Consistency and satisfiability should be connected somehow; that is what we want. Now, let us look at a rule, any tableau rule, we have motivated by the semantics itself, right, so it might look something like this.

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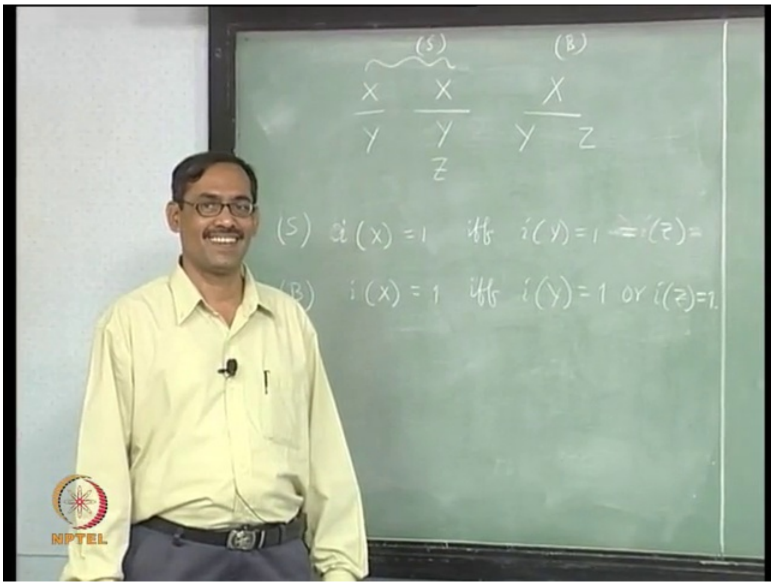

If you have a proposition x then it can give rise to some y for example, not not A gives A, right; it does not have more children than one. What happens with 'or'; it might be x to y, z; there are two children, which are stalked, right, or there may be two children who are branched out, right.

So, these first two types are called stalking rules; this is a branching rule. Suppose we have this kind of rule; all the three kinds of rules are there. Now, what was our semantic observation? x is true if both these are true; or same thing as, if x is true if and only if y is true, double negation for example, right, here x is true if and only if at least one of y or z is true, that is how we branched them, right. This is our first observation about the tableau rule. We have done it only for 'and', 'or', not for others, fine. So, you have to verify it for others also. It is true, that is easy because if you have implies for example, p implies q is equivalent to not p or q; then it is converted to or. So we can just use the equivalences to see it, right.

This is our observation: if it is a stalking rule like this, these two are the stalking rule; this is the branching rule; then what happens, if it is a stalking rule and i is any interpretation then you say i of x is 1 if and only if i of y is 1 and i of z is 1. Let us write equal to, right. And for the branching rules, you have i of x equal to 1 if and only if i of y equal to 1 or i of z equal to 1, or both may be true also. Please do not write either or, right. It is simpler; at least one of them is true, this is what happens. If you use induction and look at a tableau, what does it say? This, one of the rules and tableau is generated by using many such instances of the rules in the tableau. What does it say? There is no leaf, one of the. So, let us see an example, that will clarify.

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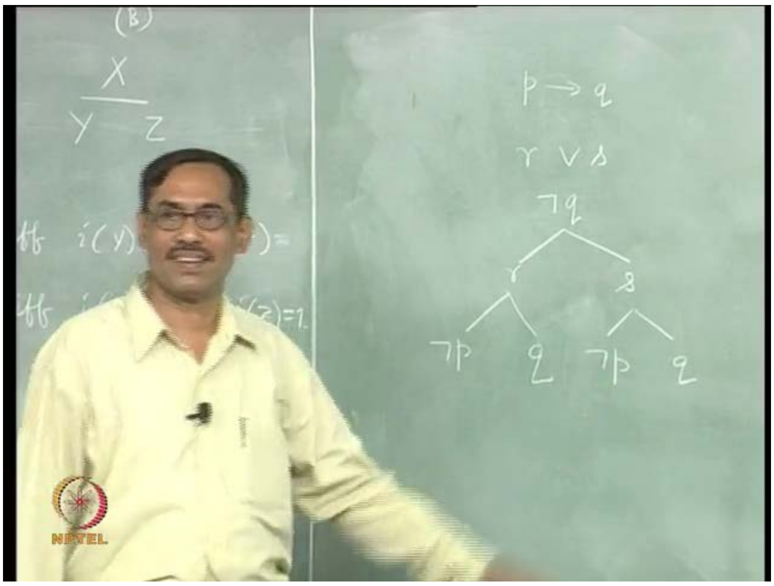

Let us see. Then we take say r or s, not q; let us try this. Now this, I can have r, s and with this, I have not p, q; I have not p, q. Now, let us look at this first. When I use the or tableau rule for the first time there, what happens? It says, branching rule; it says, r or s is true if and only if one of r or s is true, that is what it says; that is clear. Now, come to the next stage. I have used p implies q here. Now here, it says p implies q is true when at least one of not p, not q is true. When I take both of them, what does it say? It says both of them are true if and only if at least one of the paths is true, right, is it clear? It is not a leaf because those leaves earlier, they are in the path now; and the new leaves have come, right. So another leaf there will come in another path; this is what it should give. In the tableau, if you take all the propositions, all those propositions will be true if and only if at least one path of the tableau is true, is satisfiable.

Student: While building a systematic tableau then we stop and we get I mean inconsistency or to be continued in that path also.

We can close it.

Student: We close it.

Yes.

Student: Then any inconsistency if it exists, it will be a leaf right?

It will be closed.

Student: It will be closed.

The path will be closed. So, problem is, do not look at the leaf in tableau; you have to look at the path always.

Student: I am asking while looking at the leaf, can we say whether the path is closed or not.

No, the path will contain so many other things; now, we have to check whether the whole path contains a complimentary pair of literals or not; then only it will close. But one case, you can do that if bottom is coming in that right, if bottom is coming as a leaf, then yes, you have not stopped somewhere, so you get bottom as a stalking rule.

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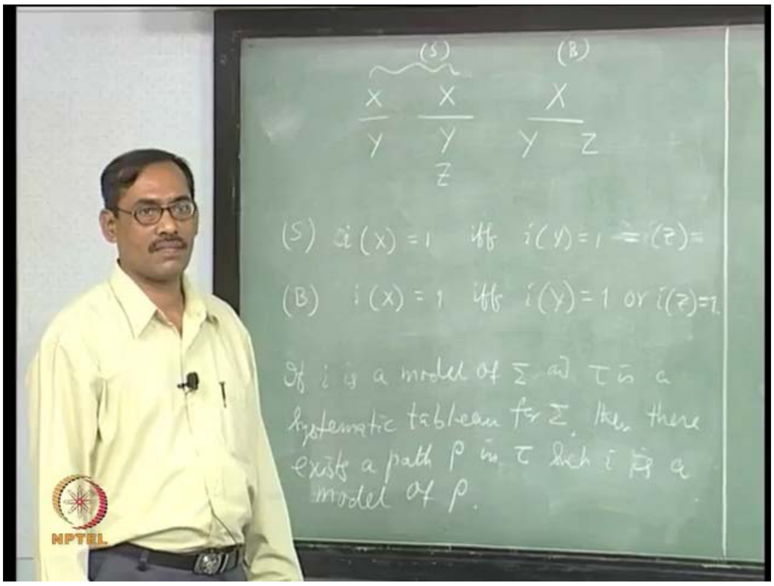

So, what we observe is, if i is a model of sigma and tau is a systematic tableau for sigma, then there exists a path, say, rho, in the tableau tau such that i is a model of rho. I will write the full sentence, this is what it says. And the proof will be by induction from s and v, essentially it is this. You would go for the proof take induction on the, on what? on the length of the longest path in the tableau; this we do not know, where it will occur. So, if it occurs, if it is closed, it is fine; longest path is closed, it is fine, does not matter.

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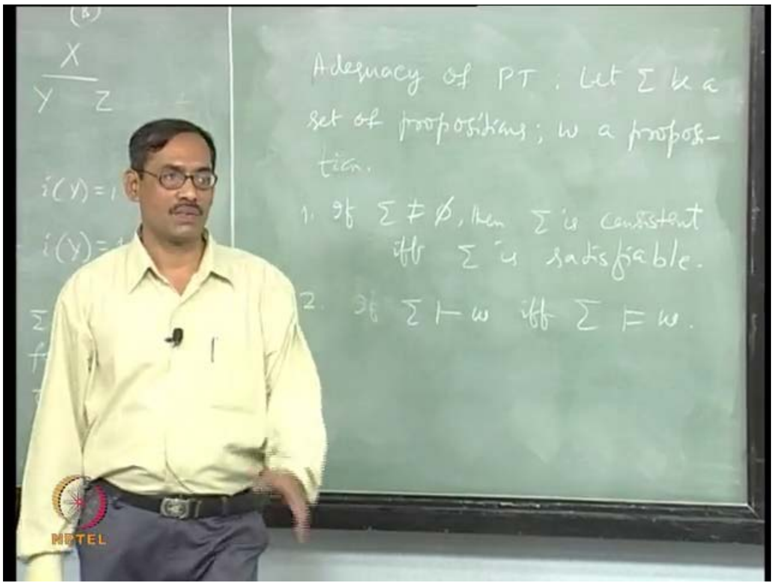

Because your, it is a conditional thing; if it is open then, form induction on that. We will not prove it now; I will leave it for you. Then we can really come to connect with the semantics; now you have started connecting with the semantics. This is adequacy of PT. We start with, sigma be a set of propositions, w a proposition. If sigma is nonempty then sigma is consistent if and only if sigma is satisfiable. You can see inconsistent if and only if it is unsatisfiable, they are the same thing.

And our second formulation is: with w, sigma entails w in PT if and only if sigma semantically entails w. Second one follows from the first easily because this if and only if sigma union not w is inconsistent by definition of tableau, and this if and only if sigma union not w is unsatisfiable by reductio ad absurdum; then it will follow from first. So, first one only we have to prove. Let us try.

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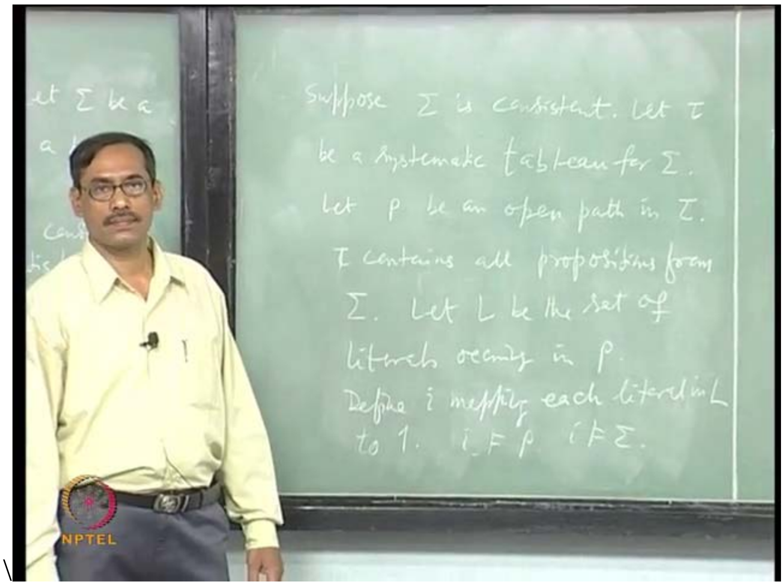

Suppose, sigma is consistent. We are concerned with tableaux, right? Once this is consistent there is one tableau which does not close, but we can consider the systematic tableau, right. So, let tau be a systematic tableau for sigma. There is an open path in tau because sigma is consistent, right. Every path does not close, there that is, why? It is consistent; otherwise it would have been inconsistent. So, let rho be an open path, there exists, so we are writing: let rho be open path in tau; then what happens?

This contains all the premises from sigma, because it is an open path, it is a systematic tableau, right; systematic tableau and open path; so it contains all the premises of sigma.

So, tau contains all propositions from sigma, but we want to show what? Sigma is satisfiable. We will show tau to be satisfiable, right; that is enough, monotonicity, it is a subset of tau. So, to show tau is satisfiable, first thing we see, it is an open path; so bottom does not belong to tau. It does not have a pair of, or a complimentary pair of literals.

Student: Rho.

Rho is the path; in rho we do not have bottom; we do not have a complimentary pair of literals, right? What you can do is, try to find out what are the literals used in that. You can define a function from there to anywhere, right, and it can be extended to a Boolean valuation, that is the procedure. Let L be the set of literals occurring in rho. Define one function from L, define i from L mapping each literal in L to 1.We need a partial function, right. If you need all others you say all other literals, you put 0, does not matter. Then what happens, this i is a Boolean valuation, because p, not p do not occur and we have the S-B rule, that is, under the Boolean valuations. So, once i is a Boolean valuation, as an i is also a model of rho again, because of that observation. So we see that i is a model of rho. Then i is a model of sigma.

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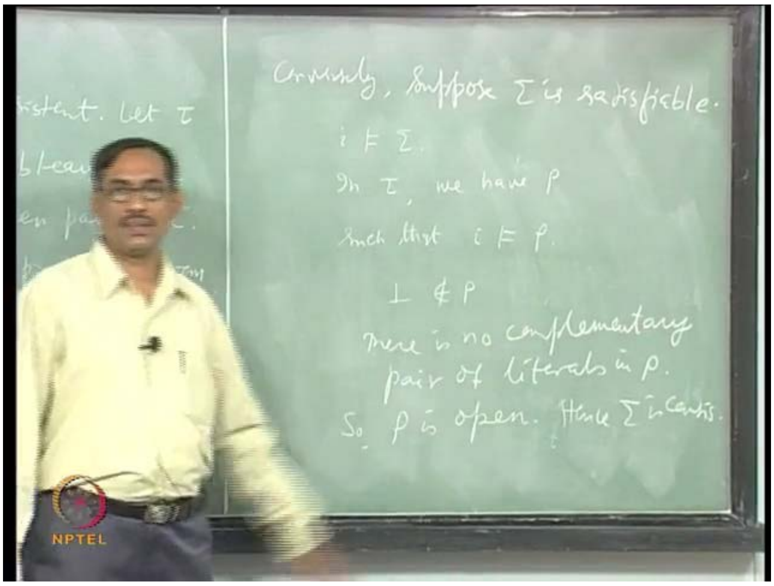

Conversely, suppose sigma is satisfiable. Our aim is to show that it is consistent; sigma should be consistent. Now, how do you proceed? We are stating the other way around; to show that it is consistent you have to say that. Take the systematic tableau for sigma in that ordering; then there is an open path, that is what we want to show. So, conversely suppose sigma is satisfiable. But if you take that finite, again you have to go via the finiteness, you have to again go via the finiteness. We have one i which is a model of sigma, fine. Then in tau, which is systematic tableau for sigma, we have rho such that i is a model of all the propositions of rho. Since i is a model of all the propositions in rho, rho cannot have bottom in it. If bottom occurs in rho, no interpretation can be a model of bottom. So bottom does not occur in rho, does not occur in rho and since it is a model of, there exists a model of rho. There cannot be complimentary pair of literals in it because both cannot be satisfied simultaneously. So, there is no complimentary pair of literals in rho. So, rho is open; it can be closed, so sigma is consistent.

This is the adequacy of tableau. It is easier than PC, right? In PC, you have to extend it to somewhere else; here you have the extension itself in the path. One such extension, it is not the deductive closure, but it is deduction of the literals, which literals can be there, can be true, so that this will happen, that is all.

Now, suppose you have proved only i is a model of sigma for finite sigma. Then how do you modify that rho, how do you come to this? You have only if i is a model of a finite sigma then you can have i is a model of that rho, in the systematic tableau. For finiteness, what you do? How do you take care, how do you modify the proof? Take it as an exercise. Think. It is not difficult, just a bit, you have to think it a bit, then you will get it. Again using finiteness theorem or do it through contradiction; say that it is satisfiable. But sigma is inconsistent then come down to finite set and apply that, right, that might be easier.

Let us summarize. Today what we have done is, starting from Konig's Lemma we had gone to finiteness theorem. Then we had seen how these stalking and branching rules allow us to connect semantics with the tableau. Specifically it is that theorem which helped us, the model of a proposition, if you take set of propositions and you take a tableau, systematic tableau, there is an open path there, right, of which there is a model. Then using that we come to adequacy of tableau. Now, compactness is clear. Again, compactness of propositional logic can be derived from finiteness theorem; is that ok? Because your entailment in tableau is same thing as semantic entailment; so finiteness theorem is simply translated to propositional logic as compactness theorem. Is that clear?