Mathematical Logic
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Lecture - 18
Adequacy of PC

Let us take one example on that and then we will raise an important issue about PC and you will try to answer that. This is the final example we take.

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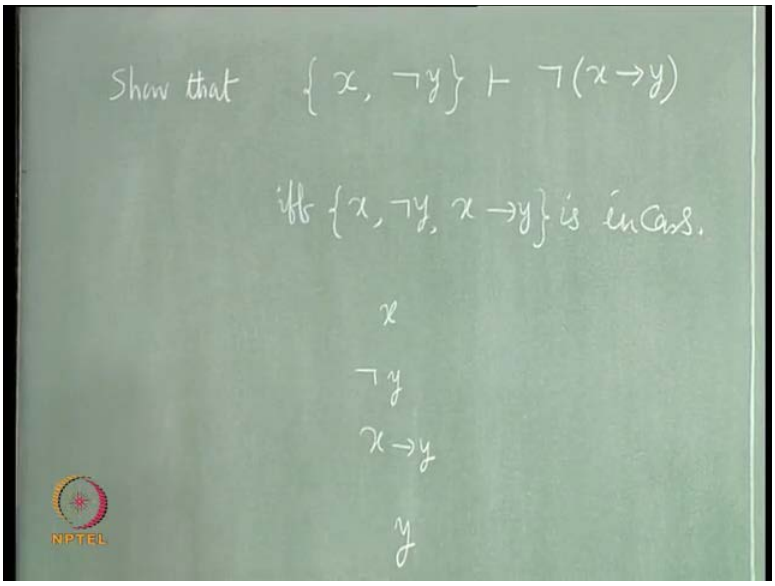

Show that it is not a difficult one, you can do it, I mean to tell you without any writing, yeah can you see? Just use reductio ad absurdum, you sort of set up, you say that this happens if and only if the set x, not y and x implies y is inconsistent; and this is inconsistent is obvious, fine? Because you can have a proof like x then not y, x implies y, all these are premises. Then use modus ponens, on x and x implies y to get y, right? So, y and not y are present; it is inconsistent, is that correct? This will be useful; we will see where.

First thing is, we will raise that important issue; the issue is this. We have started from defining the propositional logic itself, by first starting with the set of all propositions that could be general, generated syntactically, using the connectives and so on. Then we just carved out whatever are important or very interesting to have; they are the valid propositions or the valid consequences; and that came through defining the semantics by interpretation and models. Then we came to propositional calculus, where we just concentrated on two connectives instead of all the five, right? The other three could be introduced through deduct, definitions; or even the propositional constants, top and bottoms can be introduced through definitions, right? For example, top, we can introduce as p implies p, take any theorem, for that purpose, or even bottom, you can introduce by not of top, or not of p implies p, and so on. Then the question is, that is also another way of looking at a subset of the set of all propositions which use these two connectives, not and implies. The set of propositions is all those theorems of PC on one hand, in PL you have the valid propositions, and on the other hand you have PC, where you have the theorems, right? Do they match? That is the issue.

This matching can be brought in two parts, one is, whatever you did using PC, whether they are valid, and whatever you find to be valid in PL, whether they are theorems in PC, or not; is the issue clear? This is about validity and theoremhood. The same way you can ask the question about consequences. Suppose there is a PC consequences, that means a consequence in PC which, you have a proof for it. It is a provable consequence. Then you ask whether this provable consequence is a valid consequence in PL, and conversly, right?

Usually what happens is, if you have these later issue settled, then the first issue is also settled, is that clear? Suppose, you say that sigma entails w, entails in PC, right? Sigma entails w in PC; you also can prove that sigma semantically entails w. That consequence is also valid, then as a particular case you take sigma to be empty set, right? So, that settles the issue that if something is a PC theorem, it is also PL valid, is it clear? This is the reason we say the corresponding issues for consequences has stronger result. Let us formulate, give them some names.

So, one is, you say as soundness of PC. When you say soundness of PC, it is with respect to PL. if you take another logic, probably it is not sound. All that we have is PC and PL. We say with respect to PL, so this is if w is a theorem, once it is a theorem, is a theorem in PC. That is your standard notion we are following, right? Then w is also valid, this is called the soundness. When you say of strong soundness, soundness of PC with respect to PL. That will say, if sigma entails w in PC, then sigma entails w in PL. This is the way we will be using those words soundness and strong soundness.

Similarly, you can think of completeness, completeness of PC with respect to PL. That will look like if w is valid then it has a proof in PC and the same way you have strong completeness, which says if sigma entails w is a valid consequence in PL, then sigma entails w is provable in PC, right? Now, we think that it should hold otherwise you would not have done PC, because we have the valid ones, you want to see that they are still captured by some other formal game playing with the symbols, not exactly coming to the truth or falsity, that is what we wanted to see.

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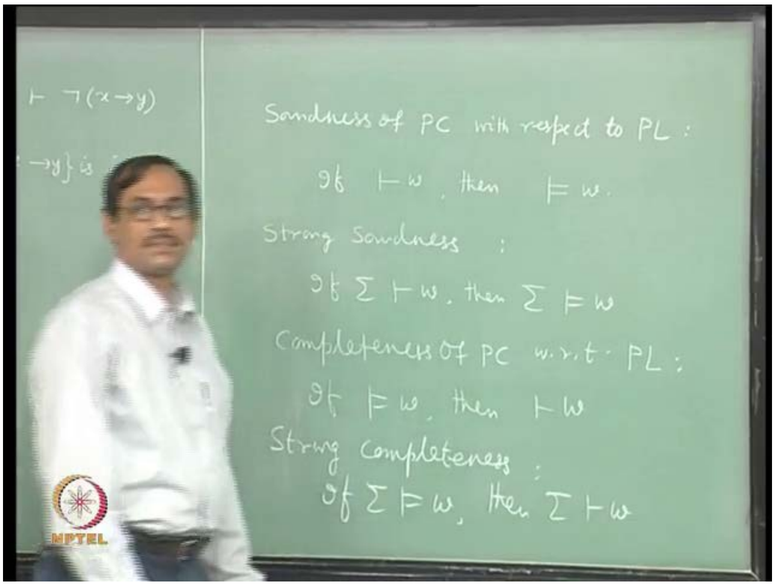

Now, to prove it what we do? See, here we are cheating a bit, we know that it is strongly sound, the system is strongly sound. You will not prove this, we will simply try this, prove it, then bring it as a corollary. The same way for this also, strong completeness may be a reformulation of it, but we will tackle strong completeness, strong soundness there, since we know that they are going to be true, that is why.

Now, for the strong soundness, let us see; sigma is a set of propositions, w is a proposition. Suppose that sigma entails w has a proof, then how are you going to prove that sigma entails w in PL?

Student: Axioms are valid.

And MP is valid, then?

Student: Then whatever we get from this is valid.

So, that gives you one step, by applying MP or taking the axioms, many steps? Induction. The proof is clear. It should hold. This is happening because each of the axioms in PC is a valid proposition in PL, that is to be checked, right? Then what you do, MP as a consequence is also valid consequence in PL. Then applying induction, right? Inductively you may not prove everything at a time, what you do is, you start with, suppose, sigma entails w, that means there is a proof of sigma entails w. Our proof of soundness is by induction on the number of proposition occurring in the proof. In fact, occurring any proof of any consequence not only of sigma entails w. That is what we are going to prove, sigma is a set there w is any proposition there, and you take any proof there.

Now, suppose it is a one line proof, that is your basis case of the induction. If it is one line proof, then w is either an axiom or it is a premise in sigma, right? You verify whether each axiom is valid or not, you verify by truth tables, let us say, crudely. Now, once that is over you say that sigma semantically entails w by what? If it is a premise, then sigma entails in PL, that is clear, and if it is an axiom then anything will entail it, monotonicity, right? That is basically monotonicity. So that step is done, that is step is done.

For the induction step, what you do? Suppose there is a proof with m propositions, it may not be that sigma, any gamma entails v, that is given and it is a proof of gamma entails v. For that you assume the induction hypothesis that gamma entails v in PL, right? You proceed to the induction step. Suppose, sigma entails w has a proof which has m plus 1 steps. So, we are using the strong induction, less than m plus 1, not only for exactly m, fine? Now, suppose it has m plus 1 number of propositions in a proof, fine. Then, w is the last line. How this w has been obtained? Again there will be cases, it can be a premise in sigma, it can be an axiom, it might have been obtained by an application of modus ponens, right? If it is axiom or it is a premise, it is just like the basis case, it is already done. Otherwise, it has been followed by MP. So, there is one proposition v such that v occurs in that proof earlier to this w, and there is also v implies w, which occurs earlier to it. Now, apply induction hypothesis on both of them. You get two different proofs possibly, not proofs, derivations.

So, you say that by induction hypothesis sigma entails v in PL, sigma entails v implies w in PL. Now, you apply modus ponens of PL, which says v and v implies w semantically entails w, that proves sigma entails w, is that clear? We will not write the proof, you do not get time, it is your liking, you can write at home. Then let us proceed to the completeness of PC with respect to PL. Again we will be tackling strong completeness.

So, that means if sigma entails w semantically in PL then sigma entails w in PC, that is what we are going to see, right? There are again two approaches here, one is, you start with sigma, find out all that can be deduced from sigma and proceed along with that, that is called the deductive closure of sigma. There is another approach which says, you just extends sigma to a consistent set, right? We will follow the second approach.

See, what happens. So for that purpose we will first reformulate this strong completeness, we want to bring consistency, unsatisfiability. Even, if you see strong soundness, if you reformulate it, how does it look? Well, first is sigma union not w is inconsistent, right? Sigma entails w means sigma union not w is inconsistent. So, you say if sigma union w is inconsistent, then sigma union not w is unsatisfiable, right? Which is equivalent to telling if sigma union not w is satisfiable, right, then sigma union not w is consistent, so you forget sigma union not w, take an arbitrary set even. We say, if sigma is any arbitrary set of propositions that is satisfiable, then it is consistent, you see, that is the reformulation of soundness.

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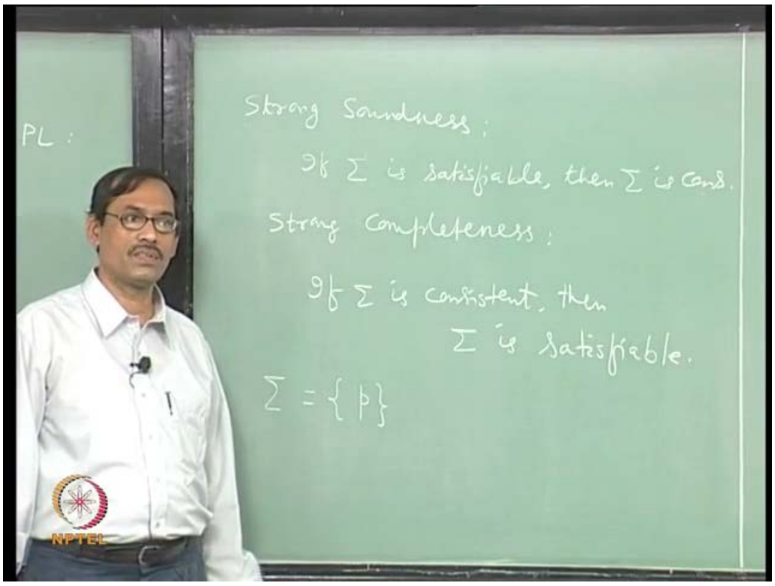

Let us write it, but we have proved already this. It says, if sigma is satisfiable, then sigma is consistent. Is it clear? Is the reformulation clear? If you look at this, it says, if it is satisfiable, then it is consistent. If you see by contraposition, it is, if sigma is inconsistent, then sigma is unsatisfiable. So, as a particular case sigma union not w is inconsistent implies sigma union not w is unsatisfiable, which is equivalent to reductio ad absudum in both PC and PL we are using. Equivalent to, if sigma entails w, then sigma entails w in PL, that is strong soundness.

Now, similar formulation is, strong completeness will be just converse of this, it will be if sigma is consistent then sigma is satisfiable. That also, you can see easily, it is not difficult; this thing is same thing as telling sigma is unsatisfiable implies sigma is inconsistent. So, take a particular sigma, sigma union not w, sigma union not w is unsatisfiable, then sigma union not w is inconsistent. We are just telling, sigma entails w semantically implies sigma entails w in PC syntactically, in PC, is it clear?

So, this is the formulation we are going to tackle it. Now, let us see what does this formulation say. Suppose, sigma is a single term set having a propositional variable, how does it look? Suppose, this is my sigma, so when I say this is consistent, it is consistent, right? Now, how do you say it is satisfiable? Well you can just assign one interpretation i, i of p equal to 1, it is satisfiable, right? But that is not what is inside this statement? Once you say this is consistent by deductions, you will see for example axiom one, which says p implies q implies p. So, we deduce q implies p from this also, so once p is consistent p along with its consequence, which is q implies p, should also be consistent, do you see the problem? Suppose, p is there, only p is there, now use axiom one, deduce q implies p from this, that is okey? It is deducible. Right? Look at the other side, you have p, and p also entails q implies p in PL, forget this proof, p implies q implies p as in PL, right? That is also satisfiable, if p is satisfiable, p along with q implies p is also satisfiable. The some model i, because by definition p entails q implies p, then i of p. If that is 1 then i of q implies p is also 1, so that set is also satisfiable, with the same model. Here in PC what happens, you have p along with that q implies p, that also should be consistent, if at all they are matching, right?

So, once you assume only p, it is not only p you are concerned with, you are concerned with all those things which can be deduced from it. Along with sigma, all of them should be consistent; because in consistency what happens, your assumption is you cannot deduce q and not q, both, from the same. That is assumed. Soundness says, if this is consistent, whatever you deduce from this also will be giving rise to the other side, soundnesses. If you deduce something from this that also along with this should be consistent. Because, on the other side it says only entailment, semantic entailment. All those will be entailed by this soundness.

So, when you look at it, look at the converse, if this theorem is proved, look at its converse. That is your completeness. It says along with p all the other things should also be consistent, is it clear? Because of the inter-play of soundness and completeness, this is what we are after. If p is consistent as a singleton, then along with p all its conclusions, that follow from p, should also be consistent, and there are infinitely many you have, because from this itself q implies p, q can be replaced by any proposition, right? If you use axiom 2 or axiom 3 something more; so there is potential an infinite set here, which should become consistent, fine? What else we need to do here?

If some set is consistent like p, we actually should produce one model of that set. We see that if it is a model it will be model of the whole set, along with that, all its conclusions. So, it is a big problem. First you realize what is the problem, then we will be tackling it. If it is very big, then you have to start in a bigger way. So, this is the bigger way we will be starting with, since this is coming up and we do not know what are the proposition which will be consistent with this set.

We will start with an enumeration of all propositions and find out if it is really consistent with the set. That is the big approach we are taking. Can you have an enumeration of all the propositions? Yes, can you have. Yes? The set of all propositions, can you have an enumeration? The enumeration will start, this is my enumeration, this is my second proposition and so on. If you can do that, it is an enumeration, yes. Can you do that? Yes, why yes? It is an infinite set, so the algorithm will not fill.

Does that give you the hints? The set is countable or not? It is so, it can be enumerated. That is it, is it? The set of all propositions is countable, set of symbols and then formulating it, allowing with the grammar, right? First, take the set of all symbols, then combine, take the set of all expressions possible from it, that is again countable, express some, each expression as a finite length. That setting, what you told each expression is of finite length, it is a finite sequence of symbols. The set of all finite sequence of symbols from a countable set will be countable, right? Therefore, the set of propositions is countable and then the set of PC propositions. We are concerned with only two connectives here, not all the other connectives, that is also countable. If it is countable, there is an enumeration, whichever way it does not matter, but we are going to fix the enumeration, whichever we have done, keep it to yourself. Give me an enumeration, accordingly I will proceed, that is the approach, fine?

Student: Countable, countable or finite? What it is? countable.

As you have defined, it is countable: p0, p1, p2, p3, and so on, it is countable. Now, let us take an enumeration of all the propositions, we start from there.

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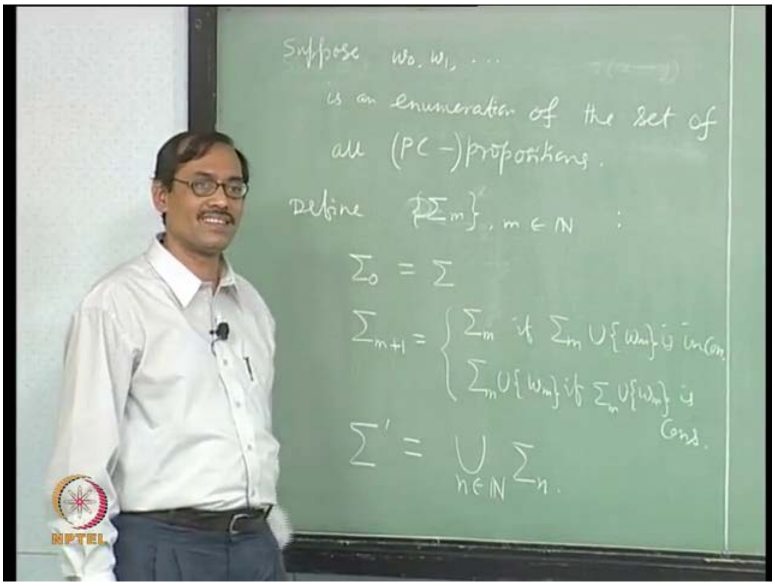

Suppose w0, w1 and so on is an enumeration of the set of all propositions, all in fact will not write after this PC, always it will be PC-propositions, that we are concerned with. From now onwards, up to some point of time, till we finish the completeness proof. So, it is the set of all PC propositions. Now, what we do, we start with sigma 0 and slowly define and extend, right? Define a sequence of sets of propositions, so define sigma. Let us write, say, sigma m, it is sequence propositions, we are defining for m in natural numbers as follows.

First take sigma 0 equal to sigma, to start with, next once it is defined, take sigma m plus 1 equal to sigma m, if sigma m union wm is inconsistent; and it is equal to sigma m union wm if otherwise, right? if sigma m union wm is consistent. That is the definition of the sequence of sets of proposition sigma m. Then finally, you take sigma prime which is our main object here, equal to union of all these things; is the construction clear?

We are constructing inductively; with sigma 0 as sigma, next sigma 1. How do you get sigma 1? Well, you go to w0, whatever enumeration we have taken, in that, take the first one, w0. Now, verify whether sigma union w0 is consistent or inconsistent. It is not really algorithmic. How to verify it is inconsistent? But it is a theoretical construction. If it is inconsistent, then do not include that w0, you will be happy with sigma 0, call it sigma 1, right? So, sigma 0 and sigma 1 are same, now w0 we are just omitting. If it is consistent, then include this w1, and formally, we say, call it sigma 1 and proceed, that is how the definition goes. Then finally, we write sigma, which is the union of all those sigma i. This is what it says; here is that.

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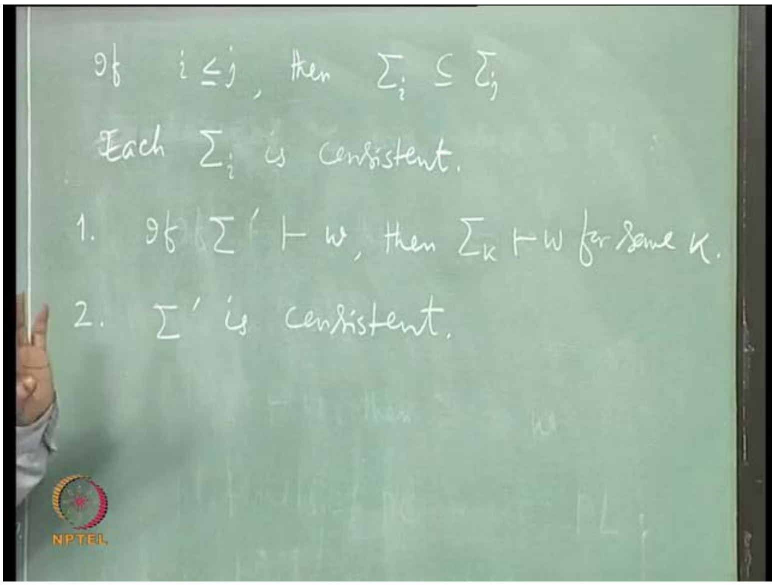

If i is less than j or equal to j, then sigma i is a sub-set of sigma j, is it clear? Yes, you are not deleting anything, you are adding possibly, possibly. We do not know whether it will be a proper subset. It will be a subset, that is what we can say. Next, what we observe is, each sigma i is consistent. That observation is clear, each sigma is consistent. We need an inductive proof for that again, but it is clear from the construction, fine? That is how it is proceeding; whichever is i, take this or this, if it is consistent only we are keeping it, otherwise do not add. So, earlier was consistent, now also it is consistent. Our assumption is that sigma i is consistent.

What happens, but, we have really extended it too much, which is very large. It is something like limit of sigma n, n goes to infinity; you do not have the limit concept, you write as a union; that is the idea. Now, we will find out some nice properties of this sigma prime, see what happens. First property, well let us have a part, we should start from sigma is consistent; everywhere we are using it and because of this assumption this is consistent, fine?

So, first property is finiteness, you will take sigma prime. From sigma prime, you show w in some derivation, then you say that there is a k such that sigma k also entails w, can you see that? If sigma prime entails w whatever proposition w may be be, then sigma k entails w for some k; it comes from the finiteness of the proofs. See, once you say sigma prime entails w there is a proof of it, okay? In that proof, you have only premises from sigma prime are used, w may not be there in sigma prime, somewhere else possibly, we do not know, right? Well, there are premises from sigma prime used, now these premises are in the enumeration because this is a set of all propositions, there in the enumeration. Then take the highest index, some k not any, there is one k, there exists one k such that sigma k entails w. Take that highest index, take that as your k, that includes all these premises earlier. So, sigma k must entail w because there is a proof for it, the same proof holds for sigma k entails w, is it clear? Idea is clear.

I will repeat. suppose sigma prime entails w as a proof p, then take all the premises occurring in p, is that clear? Now, all those premises are from the sets sigma 0, sigma 1, sigma 2, sigma 3 and so on, from somewhere they belong, because sigma prime is a union of sigma k,s, right? Now, take the maximum index of that in our enumeration, define that as your k. Then sigma k contains all this premises, now sigma k entails w because the same proof P is a proof of sigma k entails w, because all the premises are from sigma k.

Student: We can take?

Sigma k may not be equal to sigma prime.

Student: For any k for it is equal will definitely.

Student: You do not know sigma's convergence.

I am not able follow your question, can you repeat?

Student: I am asking, if there exists k, is sigma k equal to sigma prime?

We do not know, right? That may not happen because suppose sigma is a finite set, then sigma k is always a finite set, but sigma prime is infinite, right? That may not happen. But what it says is that if sigma prime entails w, one proposition, then for that w you will have one k such that sigma k will entail w; let us see.

This property is clear, so we go to next property. Sigma prime is consistent, is it so? Can you see why is sigma prime consistent? No? Our construction says, each sigma i consistent. Why the union is consistent?

Student: Because they are not disjoining.

Student: There will be a.

See this concept is, this consistency can be taken to the limit that does not always happen, right? Whatever idea you say that may not be carried away to the limit.

Student: Actually sigma n.

Right.

Student: It is the last one which we.

This is only last one.

Student: So essentially the limit the, the n in the set means sigma prime.

Right, that is, that is what we have to show, is it there?

Student: That is not there.

That is what this sentence says.

Student: Yeah.

That is what exactly the sentence says, sigma prime is consistent, that consistency is carried over to the limit.

Student: Suppose if it is inconsistent.

Student: There will be some disjoint.

You have to give a proof, we should have another induction, now it is very crisp. Now, if sigma prime is inconsistent then what happens? I have sigma prime entails some u, sigma prime entails some not u, for some proposition u, right?

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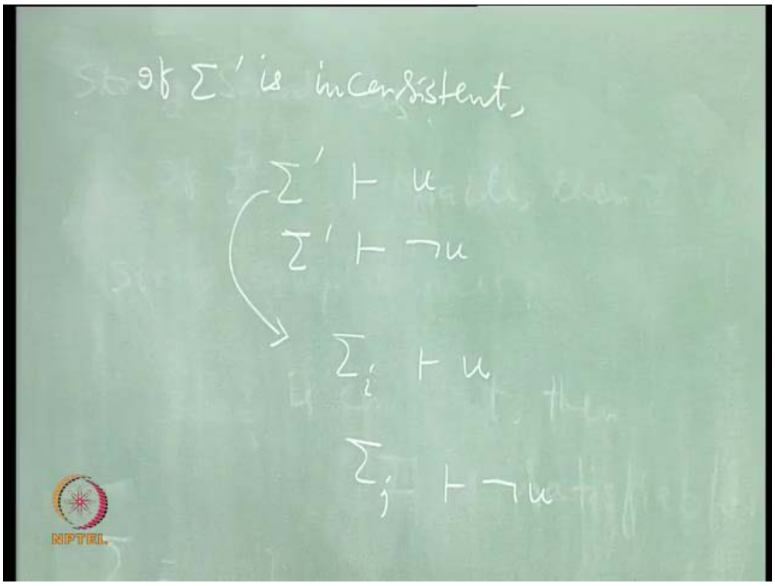

Now, by the property 1, I get sigma i entails u for some i, this i is fixed by that u and also sigma j entails not u. This j is fixed by not u, it need not be that i. Take the larger of i and j, call that k. Now, sigma k by monotonicity entails u, sigma k also entails not u, because sigma k is sigma j, both are subsets of that, right? Now, that sigma k entails both u and not u, sigma k is inconsistent, that contradicts our observation that each sigma is consistent, is it clear?

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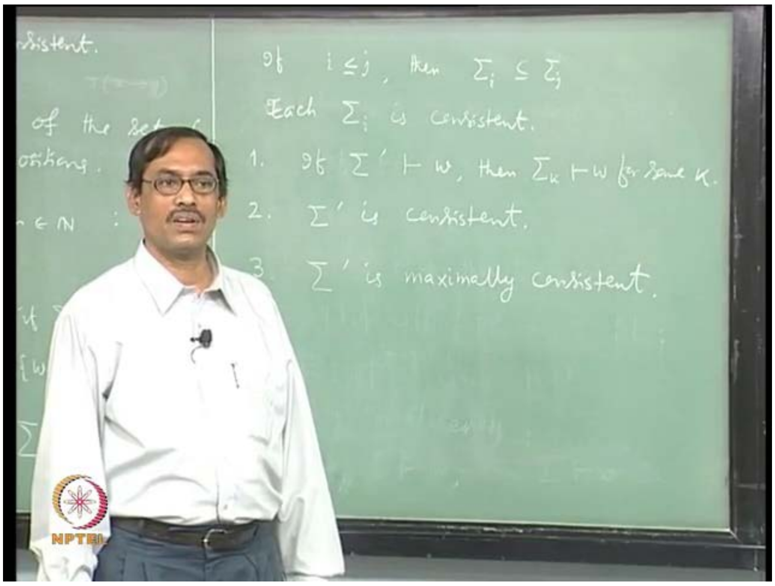

The construction is nice, it is not only consistent, it is maximally consistent. Maximally consistent means it is consistent, if we add another proposition to it, it becomes inconsistent.

That why, maximal is it. We have already proved that it is consistent; we have to only see that if you add something else which is not in sigma prime, then the new set becomes inconsistent, right? Let us write it first what do you want to show? Suppose, v does not belong to sigma prime, then we must show that sigma prime union v is inconsistent, this is what we are supposed to show.

Student: Something, so suppose sigma dash union that w j, if it is consistent then.

Student: Sigma dash sigma k will contain w j by the.

Student: Sigma j itself, I mean.

Student: Then sigma j contains w j.

Student: It should contain so that it means it should have already contained, but we are assuming it does not belong to the same. Let v equal to some w j in the enumeration of all the w's.

Ok, what you say is this v.

Student: Is some w j.

Is equal to w j like.

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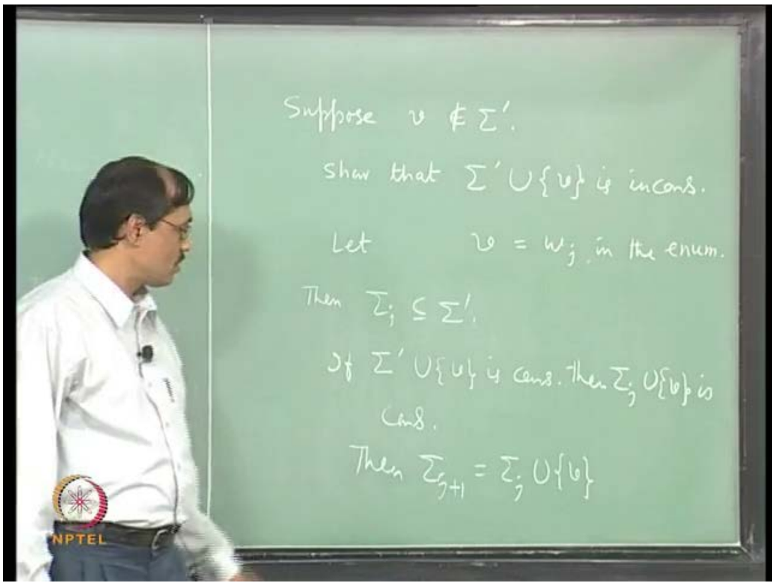

Student: Then sigma j.

Suppose v equal to w j.

Student: Then sigma.

Student: Sigma j is contained in sigma prime.

Then?

Student: Sigma j is contained in sigma prime; so if sigma prime.

Sigma j is a subset of sigma prime.

Student: Yeah.

Ok?

Student: So, sigma prime union v is consistent, then sigma j union we would have also been consistent.

Why?

Student: That is.

Monotonicity.

Student: Monotonicity.

Right, if a set is consistent, then its subset is also consistent, that is monotonicity. So sigma j union wj is consistent, then?

Student: Which means sigma j plus 1 contains wj.

Sigma j plus 1 will have a member wj, so?

Student: Sigma prime contains wj, w j.

Which is in contradiction to: v does not belong to sigma prime, is it clear? Let us write it. So, let v, v equal to wj in the enumeration. Then what we say, sigma j is a subset of sigma prime.

In fact, they need sigma j union v, but let us try with that; if sigma j is a subset of sigma prime and sigma prime union v is consistent then sigma j union v is also consistent, because sigma j union v is a subset of sigma prime union v. There is monotonicity. Then what happens, sigma j plus 1 should be equal to sigma j union v because v is equal to wj, right. Next, we argue: since sigma j plus 1 is a subset of sigma prime, v should have been in sigma prime, but v is not.

Student: Sigma entails w and we say sigma union w is also consistent, yeah.

Decide.

Student: Entails right?

Decide for yourself what will happen.

Student: It is not intersecting.

Student: Yes sir.

Why is it consistent?

Student: Because you cannot …

If it is inconsistent, then what happens, you have a proof of sigma union w entails u, sigma union w entails not u, right? In this proof, w has been possibly used, now you can eliminate that, because it follows from sigma. So, you can construct a proof of sigma entails u sigma entails not u, that is it.

Student: So, from that, we can say that if sigma dash, prime, entails w, then w does belong to sigma prime, right? That was the first thing I told there.

Ok, now prove this. Inside that I have given that proof, it will be quicker.

Suppose w belongs to sigma prime. Then sigma prime entails w, obvious? One line proof. Suppose, sigma prime entails w. Now, we have to say w belongs to sigma prime, right? Again, if w does not belong to sigma prime, then sigma prime union w is inconsistent, maximal consistency, right? If w does not belong to sigma prime, by maximal consistency, sigma prime union w is inconsistent, right? So, sigma prime entails not w, but sigma prime also entails w; sigma prime is inconsistent, which is wrong. It is quicker, right is it clear?

It says that sigma prime is its own deductive closure, whatever that can be deduced, it is already there, right? Whatever proposition w you take, for every proposition w, that is what it is written here, not written here, read it that way.

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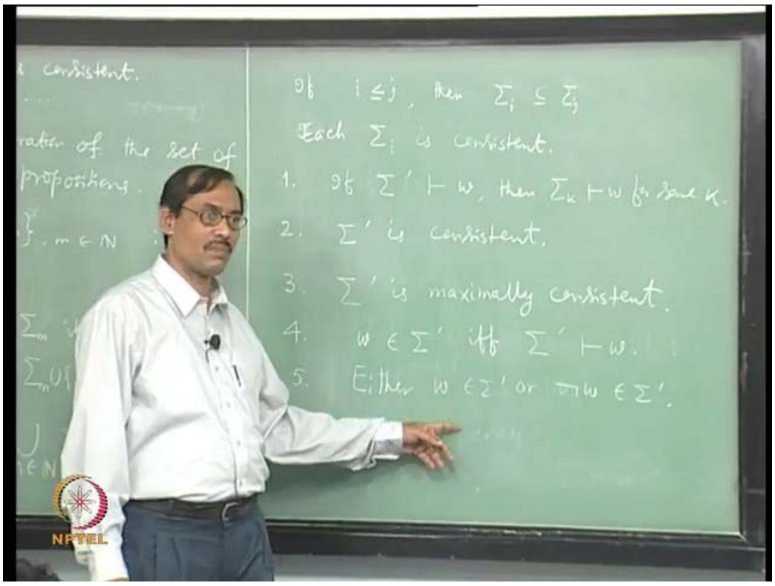

For any proposition w, for each proposition w, either w belongs to sigma prime or not, if w belongs to sigma prime, one of them will be there. It is too large in that sense, whatever proposition you say, either it is there or its negation is there; why is it so? Both cannot be there, right? One of them should be there, that is what we want, one of them should be there. How do you show?

Student: Union.

That’s what we want to show, right. First thing is, if w belongs to sigma prime, then not w is not belonging to sigma prime, why is it? So, 3 to 4, because once you say not w also belongs to sigma prime.

Sigma prime entails also not w, sigma prime entails w; sigma prime becomes inconsistent. So, this part is clear.

Now, if w does not belong to sigma prime then, maximal consistency says that sigma prime union w is inconsistent. Say, sigma prime entails not w, then not w belongs to sigma prime by 4, okay?

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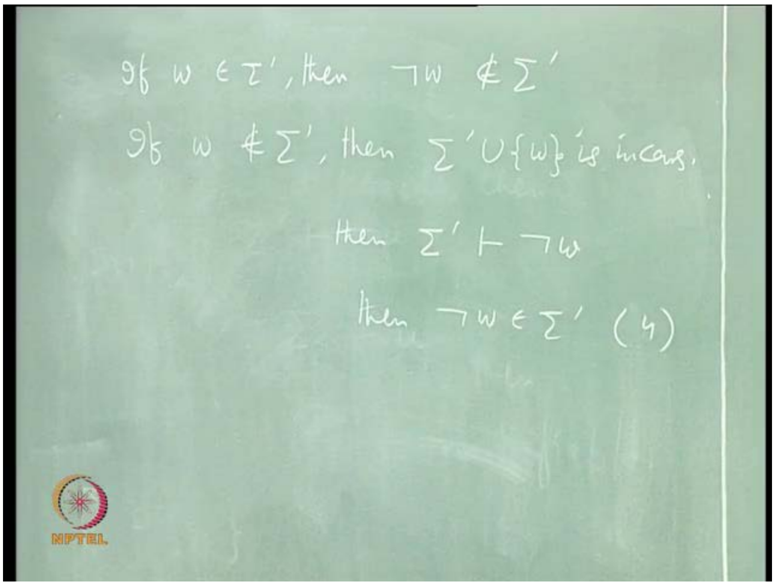

Once you prove maximal consistency, thing should faster. One more property is easy. See, you just use 4. Look at property 4. Belongs to or entailed by are same now. In sigma prime, once, you have to show p implies q belongs to sigma prime. You simply show: sigma prime entails p implies, and that follows because of Axiom 1 : q implies p implies q, we have already q in sigma prime. So, p implies q follows, so it belongs to can you see this, if does not belong to sigma prime then p implies q belongs to sigma prime.

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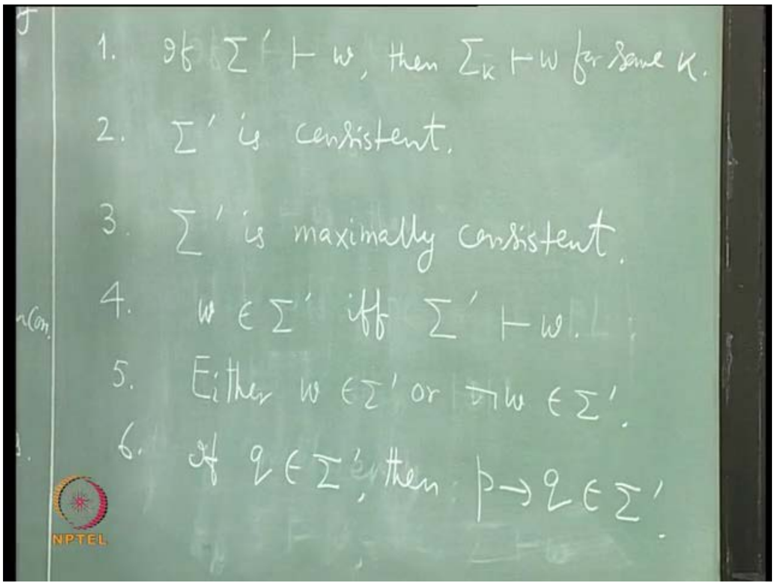

Similarly, once sigma does not belong to sigma prime, not p belongs to sigma prime. Now, not p implies p implies q implies q implies q is a theorem, you have already proved it. I get an axiom there, q implies p implies q we have a theorem, not p implies p implies q that proves this, something.

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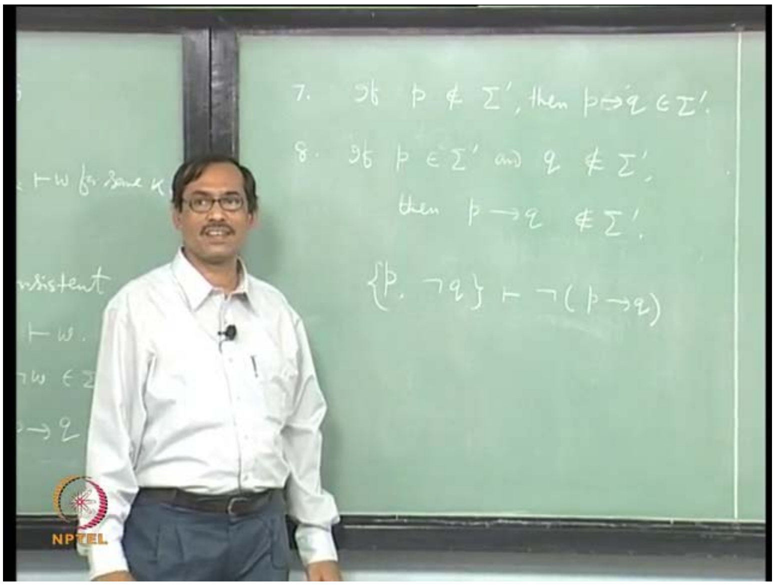

Well, today we have proved something. What was it? Look at your notes. That is it. Today we have proved this p not q entails not of p implies q. If p is there in sigma prime, q does not belong to sigma prime, means not p belongs to sigma prime. Then we have not of p implies q follows from sigma prime, then not of p implies q belongs to sigma prime. Then p implies q does not belong to sigma prime.

We have there all the properties of sigma prime. Then we will see how to use this in proving our main theorem, that should be two lines now, after this. Because if you see, if you see here, fifth one captures the negation symbol, and these: 3, 6, 7 and 8 capture the implies symbol. This is the semantics of implies, that is the semantics of negation. Now, we should be able to do it quickly.