Mathematical Logic
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Lecture - 17
Arguing with Proofs

We start with a set of propositions and also in it another proposition, then something happens about the consequence relation. It says you that sigma entails w if and only sigma union not w is inconsistent. Then similarly, without this negation sign, you see sigma entails not w if and only if sigma union w is inconsistent. That is how we had formulated it, we do not use exact words, but that is how it was.

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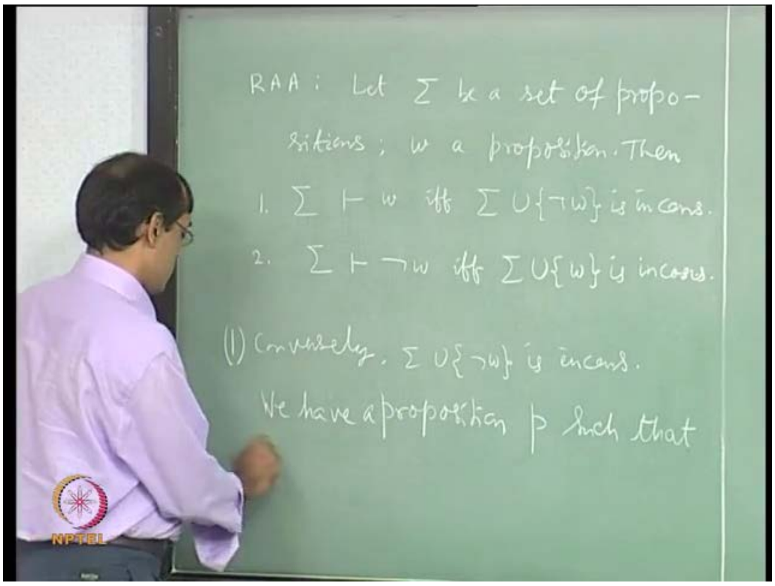

We wanted to show the first one, let us say, in this one part is bigger. The part is, if you assume sigma entails w, means? That means here we have a proof, where w is the last line and premises might have used from sigma alone. Then you add to that not w, because that is a new premise. Now, sigma union not w, we have already got a w. Now, you have not w again, but you have used premises from sigma union not w, since we have got both w and not w, sigma union not w is inconsistent, right? That is easy; that follows from line of our definition of proofs.

Now, you have to see the converse. Conversely, what happens is, we start with the converse part only, say sigma union not w is inconsistent.

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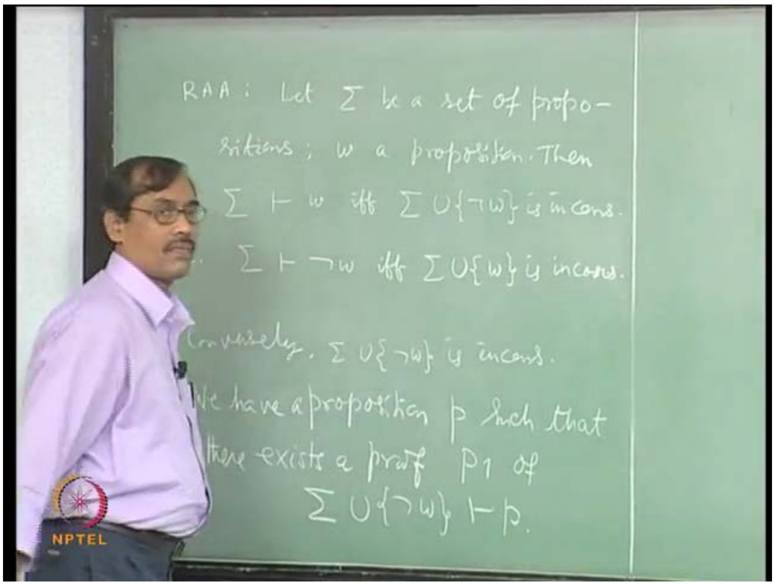

Suppose this happens. Our aim is to prove that sigma entails w, that should be the one to prove. And it does not look how to eliminate the not w from here because in your proof, not w possibly has been used. If it has not been used, then sigma itself is inconsistent that can be simpler, fine?

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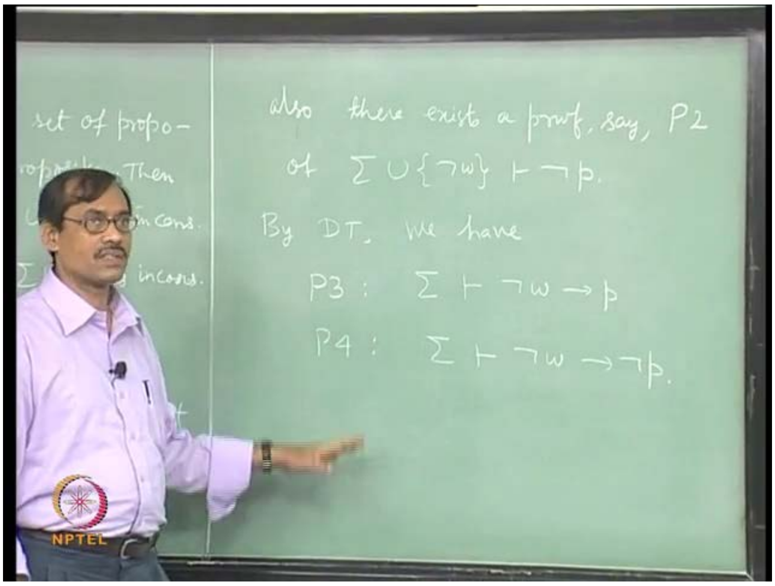

But let us take the realistic case. Once you say sigma union not w is inconsistent, it says what? Go back to the definition. There exists a proposition p such that, p follows, then not p also follows from the same premises, right? That means we have propositions, we have a proposition p, such that sigma union not w entails p. That means there is a proof of sigma union not w entails p, right?

That there exists a proof, call it P1 of sigma union not w entails p. Also there exists a proof say, P2 of sigma union not w entails not p. Somehow you have to eliminate this not w and get a proof of sigma entails w. We want, that should be obvious, how do we go about it?

Well apply deduction theorem and both of them that we have already proved. By deduction theorem, we have two proofs. Now, we have P3, let us say corresponding to this P1, which says that sigma entails not w implies p, right, deduction theorem, not w is a premise which can go to the other side with implication sign. This P3 proves sigma entails not w implies p. Similarly, P4, corresponding to P2, which says sigma entails not w implies not p. Now, I should, see, should have, the plan is not w implies not p implies not w implies p implies w, that is your A3, use that. That is the end of the proof, can you see that?

Your A3 says not w implies not p implies not w implies p implies w, right? We have already got it, so by MP, you get not w implies p implies w, again you have got it. So, you have got w, easy? Right, but you have to write it exactly, that is the only problem, okay? What we do, we construct a proof or we take P3, we take P4 and add some more lines, right?

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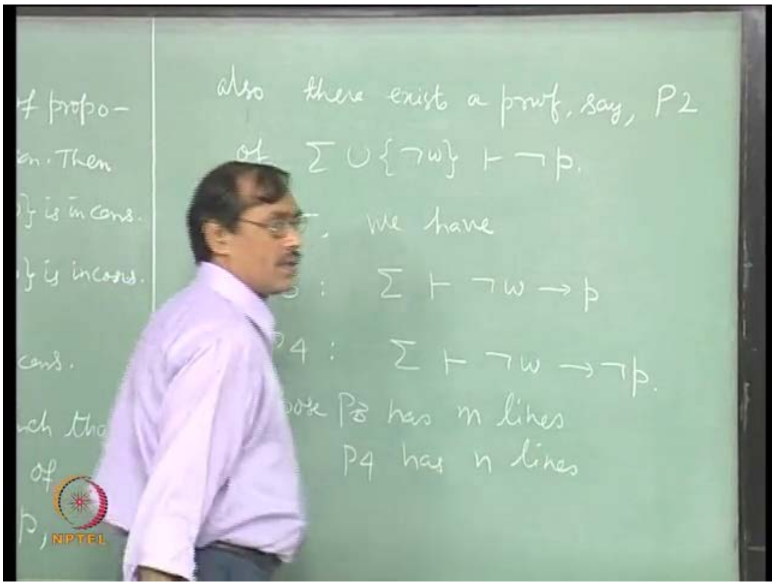

So, suppose P 3 has m lines and P 4 has n lines. Now, construct a proof, construct P as follows. So, 1 to m, let us write as P3. P 3 begins and P3 ends. It ends with what? not w implies p, next add p for it, that goes up to m plus n. Here we write say, P4 begins and P4 ends, and it ends with what? w implies not w implies not p, you just add the proofs. Next line what we do, m plus n plus 1, we write Axiom 3. Next m plus n plus 2, that gives not w implies p implies w. This is from line m plus n plus 1; and then you have m plus n, we write the other way around, right? You are following? m plus n, for MP?

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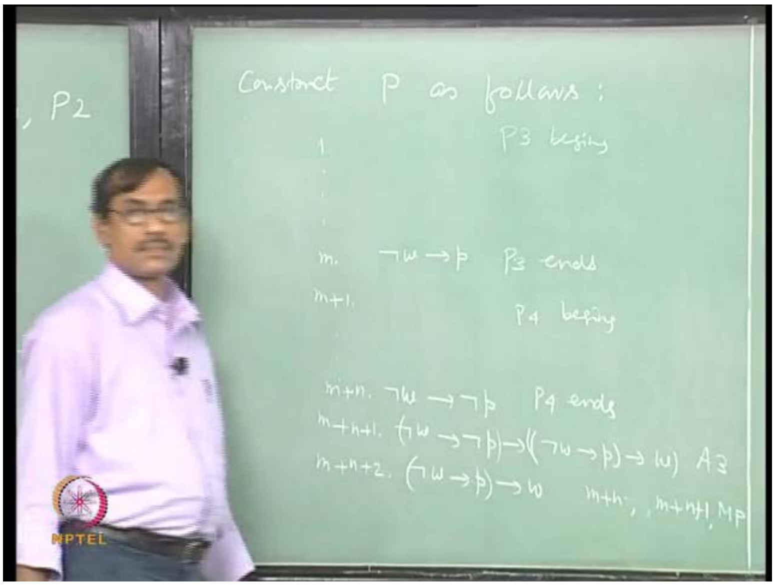

Next m plus n plus 3, so you want to use not w implies p. So, we write, our comment will be m plus n plus 2 and MP, which gives w, right? Now, what happens, we have to check whether sigma entails w or not, we have got w. Now, P3, it is sigma entails something. All the premises used are from sigma, in P4, all the premises are used from sigma. Here it is an axiom nothing else is used. It is sigma entails, right? Now, you see that P proves sigma entails w, it is only the details, idea is the same, clear? What about the second one? What about second one? Second part of reductio ad absurdum. Again one part will be easy there, yeah? Which part is easy? If sigma entails not w then?

Student: Then sigma union w also entails not w.

By monotonicity, right? And you have w now, w is a premise. It is a one line proof. Sigma union w entails w, and sigma union w entails not w. Therefore, sigma union w is inconsistent; that part is clear. It remains, only the converse part. So, conversely suppose sigma union w is inconsistent. There is a problem, it does not go as it is, because in A3, you have to start with not symbol. Now, if you follow the same procedure, it will say w implies not v, w implies p, not w, that can be brought here, but that is not an axiom.

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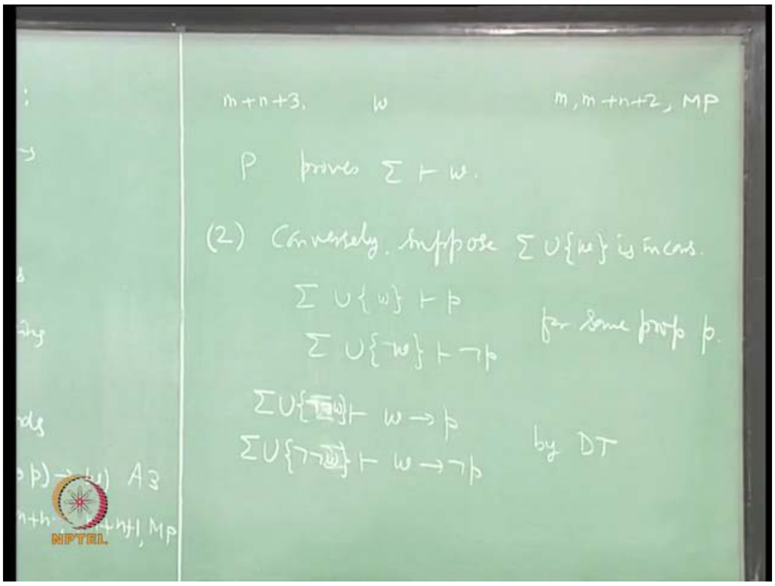

Suppose, you uniformly replace w by not w, you have to get not not w, right? Implies not p, then not not w implies p. Therefore, not w. It will go for not not w, right? What you need is, from w you can get not not w, if you can do that, that is enough, that is the special idea, somehow you have to do it, right?

Now, how you can say that w entails not not w? We will apply RAA, w entails not not w, if and only if w comma not w, right? is that ok? No, it does not. See, first part says sigma entails w. So, you are replacing with not w here, so sigma entails not w if and only sigma union. There is problem again, anyway. What we want is, you have to show somehow sigma union not not w is inconsistent, that is what this amounts to, right?

From these, assuming that sigma union w is inconsistent, you go for proving sigma union not not w is inconsistent, fine? If you can do that then probably it will be all right. Let us try to do that. How to show sigma union not not w is inconsistent? It should entail some p, it should entail some not p, fine? That is what we want. Now, from this what happens, sigma union w is inconsistent, we have some p such that sigma union w entails p, sigma union not w, sorry w, entails not p for some p, some proposition p; this is what we know, clear? Tthen we can use deduction theorem, can you? Will that help or not?

See, our aim is to show that sigma union not not w is inconsistent, right? This is what we want; and suppose if you want deduction theorem, then we should have sigma entails w implies p, sigma entails w implies not p, by deduction theorem. Then, by monotonocity we can say that sigma union not w, you want not not w, right, not not w entails w implies p. Similarly, you may say sigma union not not w entails w implies not p, that part is clear. But how to show that it is inconsistent? From w implies p, w implies not p, can you show it is inconsistent?

Yes. By deduction theorem, you do not have not w, here use monotonocity, you can add a premise, does not matter. You write and monotonocity, so what you get is w implies p, you get also w implies not p, right? Then how do you say that sigma union w is inconsistent? This is the problem. Suppose, you go to sigma union not not w and w. See, all that you wanted is, w implies not not w, that you wanted. So, how do you show w implies not not w?

Student: Is by union not not w union w is inconsistent.

That is ok.

Student: Then use that.

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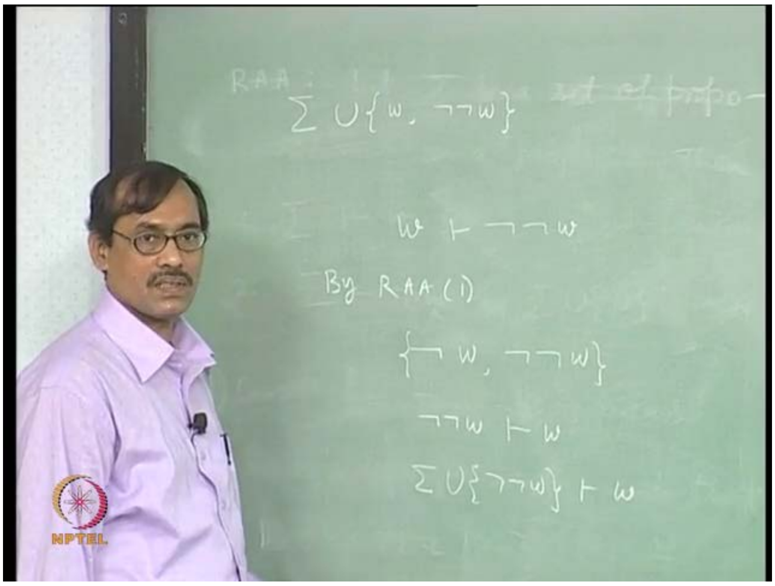

Yes, you are telling sigma union, sigma union w is inconsistent that we know, right? Then?

Student: w union not w is inconsistent.

That is also inconsistent, yes. Well, let us see how does it proceed. You wanted to show sigma union not not w is inconsistent, right? But you have sigma union w. You wanted to prove first w implies not not w, right? Is that right? So, w entails not not w. How do you prove this? No, use RAA, you have only to prove one. By these, what you want is w, there is the point.

Student: It is different not.

We will not take that way. What should we do? Can you take this not w? If this is inconsistent then what do you say, it will say not not w entails w. That is enough for us, you do not need this, we need this really. Why? Because, suppose you take sigma union not not w, that gives you w, right? By monotonocity and sigma union w is inconsistent, therefore sigma union not not w should be inconsistent. Is it clear? Line of proof is clear or not clear?

Well, you start with not w comma not not w. This is inconsistent, how is that? Because this entails not w. This also entails not not w, one line proofs. Now, my set is this, so from this set what can I conclude? not w.

Student: That alone, the set is inconsistent.

Yes, that set is inconsistent. Now, once this set is inconsistent, you use RAA one. This gives not not w entails w, clear? Then sigma union not not w entails w. So, you have w along with sigma also to be used. Our proposition is sigma, they are still there, right? But sigma union w is inconsistent, it gives p, it gives not p, by assumption. Therefore, sigma union not not w is inconsistent, it is clear.

Student: The last part is not clear.

Last part is not clear. See, from this, you get sigma union not not w entails w, this is clear. Now, what happens? Does it prove? Now, we will construct the proof that sigma union not not w should be inconsistent, right? So, what we will do, first take proof of this, get w, after getting w you adjoin proof of sigma union w entails p. Then add the proof of sigma union w entails not p. That gives a proof of sigma union not not w entails p as well as not p. Therefore, this is inconsistent, is it clear? Now, can you proceed to prove? Where should we start, this is all right. This is all right, see our main thing is to start with this set not w comma not not w, fine? You have to write it according to that. We say, sigma union not w, not not w is inconsistent. This is clear, or it is inconsistent, because from this set you get not w, from this set you also get not not w, right?

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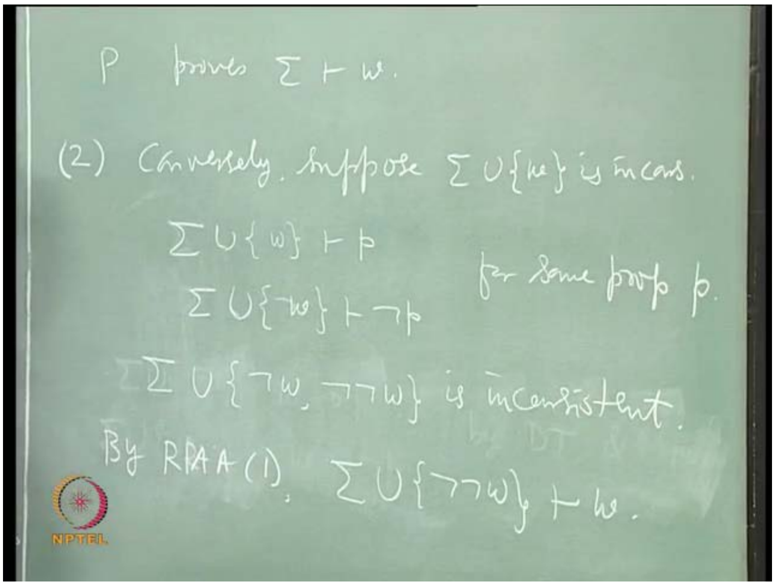

So, this is inconsistent. Once this is inconsistent by RAA1, we get sigma union not not w entails w, right? RAA1 says you add one not to the conclusion, going it to the premise, that will be inconsistent, right? You cannot delete, you can add; that is what it says. The second condition says, then also you can delete, fine, is that clear? This and this are equivalent, if and only if statement. You get sigma union not not w entails w, right? Now, look at this sigma union w entails p, sigma union w entails not p. What do you say? If you want deduction theorem, you can still use it, if you want to use deduction theorem, you can still use it.

Student: You will get omega if you like.

Or as it is also, you can proceed. Let us see deduction theorem, if you are taking that proof. Let us number this first. For this, I say star. Now, from star by deduction theorem we get sigma entails w implies p, and sigma entails not w, w implies not p, okay? Now, call these the proof, which proof? sigma entails w implies p. Let P2 be the proof which shows sigma entails w implies not p. Similarly, you take a proof of this: sigma entails not not w entails w. Suppose P3 is our proof, which shows sigma union not not w entails w. Now, you construct the proof P, how does it look?

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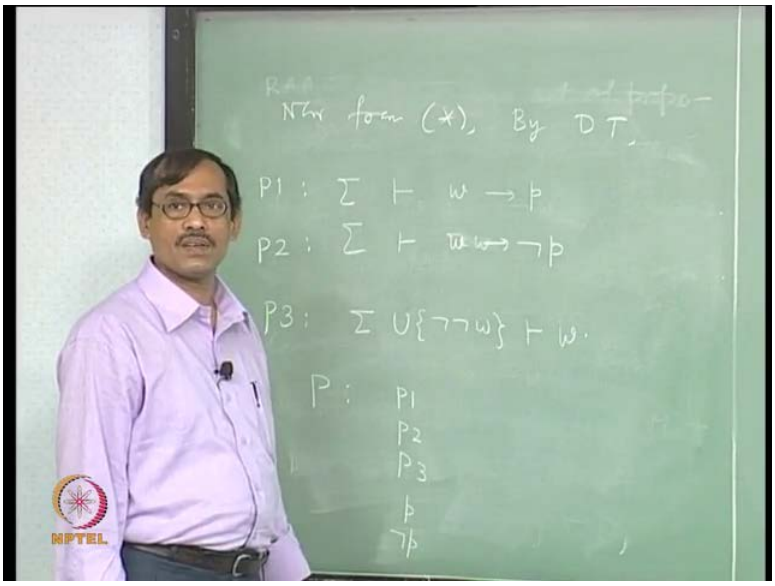

You take P1, next P2, next P3, next what you do? No, just write p, then write not p, that is the end of the proof, right? This p follows from last line of P1 and last line of P3 by MP, last line of P1, last line of P3 there w implies p and w from which by m p you get this p, right? And this not p follows from last line of P2 and last line of P3, is that clear? So, that is also by MP, and this is our proof which shows sigma union not not w is inconsistent, is it clear?

Yes, so all that we do is, you have to write it exactly. Suppose, you have m lines, you write 1 to m. Suppose, it has n lines done, you start with m, 1 plus, m plus 1 to m plus n, next P3, suppose it has k lines, so m plus n plus 1 to m plus n plus k. Now, you give in P justification as m line, line m and line m plus n plus k, and P is that. Then, next one is m plus n, m plus n plus k and P, that is all, is it okay? So, what we have shown?

So, sigma union not not w is inconsistent, now by first part of R A sigma entails not w, clear? So, once you prove this meta theorems, you get some help in not proving many other theorems. You can just show that there are provable by using these meta theorems, right? So, you do not have to construct the proof. Now, we say that it is a proof, sometimes there exist becomes easier than proving it, then constructing the proof and you remember that example?

No? We discussed in the first class. The proof that, there exist irrational numbers x, y such that x to the power y is irrational.

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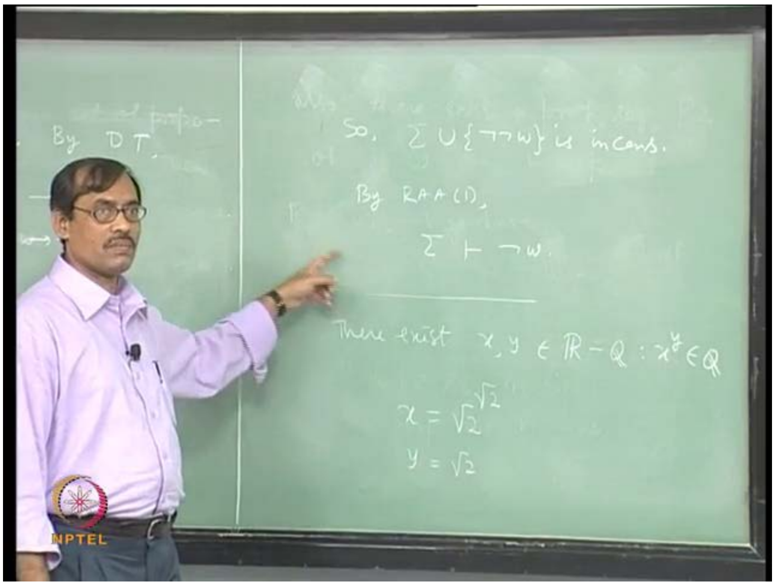

Let us see that. It is nice. We want to show that there exists x and y which are irrational such that x to the power y is irrational. You have to really produce something like e to the power log 2, right? So, e is irrational, log 2 is irrational, e to the power log 2, and but you have to prove e is irrational and log 2 is irrational. I do not want to do that, I know something 2 to the power, irrational, say root 2 and here root 2 is irrational. Let us try that, right?

Suppose you consider this number x equal to root 2 to the power root 2, right? Now, it is irrational or rational? It can be shown to be rational, right? But I do not know that proof also. Suppose, well I do not know whether it is rational or irrational, if it is rational I have got my x and y, x is root 2, y is root 2. If it is irrational then I will take my y as root 2, so that x to the power y will be equal to the root to the power root 2 into root 2, which is 2, which is rational, right? So, it bases on two cases, either root 2 to the power root 2 is rational or it is irrational, if it is rational it is done, if it is irrational then take to the whole to the power root 2, right?

So, you could prove this easily when this is something, existence, of the proof of existence of numbers, there exist, yes, there exists. You do not know what it is, when sometimes existence can be easier that is what we are going to do. Once you use the meta theorems it will show that there exists a proof, its provable, but you do not know what the proof is.

Probably you will be able to construct, but here it is constructible because all our proofs here are in the meta theorems. They are basing on construction, it is recursive, but its construction is there and so they are constructible still, but we have not constructed. When we prove that there exists a proof for it, that is it, so it can be easier. Let us see how it becomes easier, at least one of them, you know you have already proved during this proof of RAA, so it is this.

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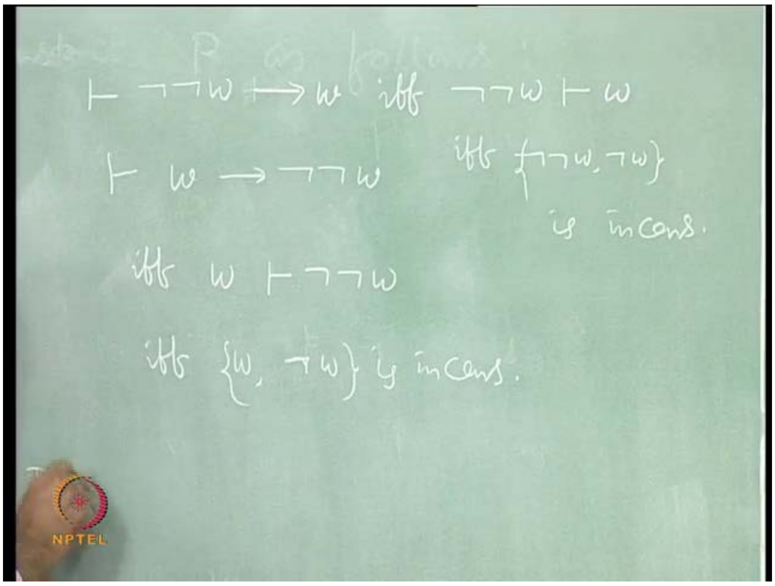

This already you have proofed, yes, by RAA1, it is, this also you have proved or not, that easier still, clear? All that you have to do is here, you show by deduction theorem, this one holds if and only not not w entails w.

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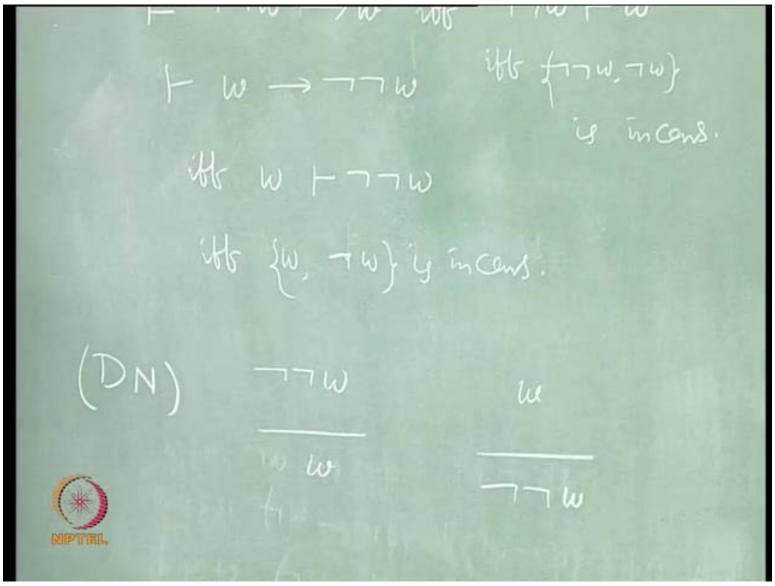

Next one if you go if and only if not not w, not w is inconsistent which is true, okay? For this one, you say these words: if and only if w entails not not w if and only if, you eliminate one, w, not w is inconsistent, things become easier, right? So, you can have a derived rule, now of double negation. These derived rules tell that if you have not not w, you can have w. If you have w you can form not not w. Also while proving theorems, you can go for derived rules, right? If it is an implication, if not an implication, you said it is a derived theorem. We do not have modus tolens in PC, can you form that modus tolens? What does modus tolens say?

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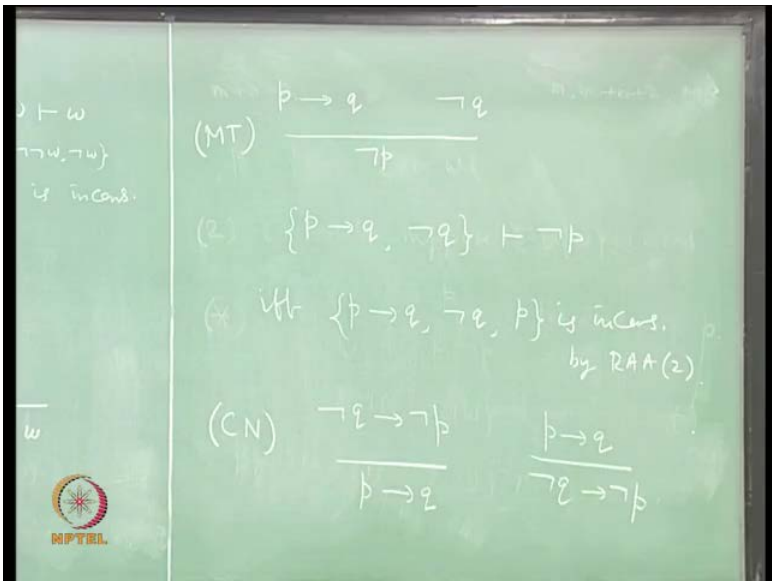

You have p implies q and not q, then derive not p, this is what modus tolens is. Once you write it as a P C consequence, you will be proving the theorem p implies q, not q entails not p, this should be provable. Now, how do you say that it is provable?

Well, apply deduction, eliminate that negation sign, instead of adding another, so that you have to choose which one; you have to do so that it will be eliminated. Now, say if and only if p implies q, not q, p is inconsistent by RAA. So, this is RAA2 second part of. Now, this should be easy, how to show it is inconsistent, right? Start with p, next take p implies q, so you get q, by modus ponens, which is a rule in the axiomatic system. You get not q as another line, because it is a premise. You get both q and not q, therefore it is inconsistent, okay?

Let us say another, say contraposition, how does it look? It has again two parts. Let us say, not q implies not p, that implies q, another is p implies q, q entails not implies p. Now, formulate it as a consequence, PC-consequence.

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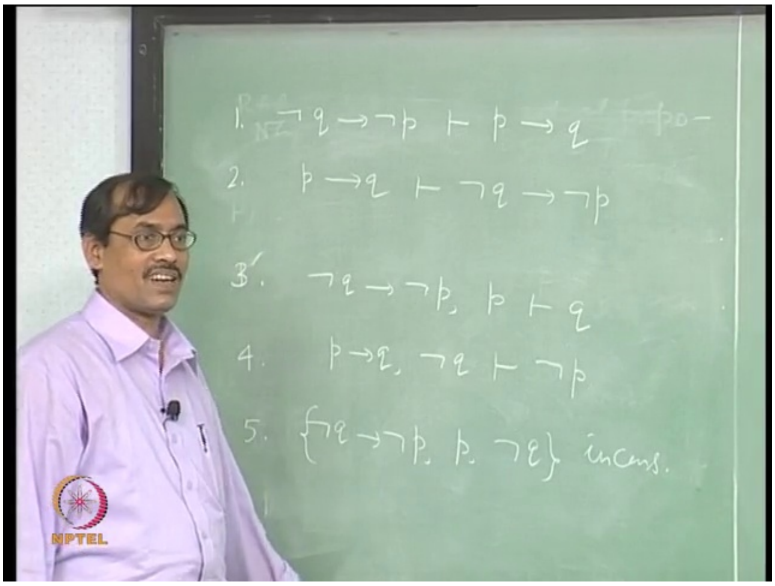

It would say not q implies not p entails p implies q, that is one part. Second part says, p implies q entails not q implies not p. Then, how to proceed? You have implies on the right side. First, it says apply deduction theorem, so one will be equivalent to proving by deduction theorem, will be what? Let us write there, all odds will go then this will say not q implies not p, p entails q and this one p implies q, not q entails not p.

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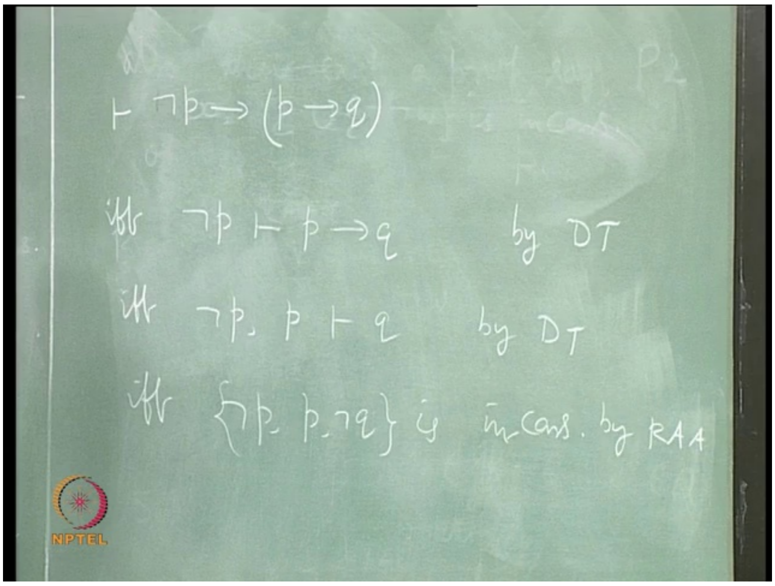

Now, how do you prove this? Direct. Is it coming by modus tolens? No, you will use negation, again let us do the other way. We will put again not q implies not p, p, not q inconsistent, by reductio ad absurdum theorem, and here, it is direct, directly modus tolens, right? Anything to do? And this is inconsistent because, modus ponens, not q, not q implies not p, therefore, not p; and you get p, so it is inconsistent. Things are quicker, happening easily now.

What about this law? not p implies, when p implies q; well, heuristic is, to the right of it there are implication signs. So, use deduction theorem, right? By deduction theorem, this is not p entails p implies q, there is axiom A1, that is axiom A1; then apply hypothesis, because you will be using contraposition. Yeah, there is no truth we are just proving, we are only proving. If we start with Axiom 1, you will be starting with not p implies not q implies not p. It is an axion, which axiom? Every axiom is by definition, a theorem, there is no truth.

See you want to construct the proof, isn't it? How do you construct the proof? You start with the axiom: not p implies not q implies not p, right? Next what you do? not q implies not p implies p implies q, contraposition, right? Next you use hypothesis not p implies q, right? So, you have to use all the laws.

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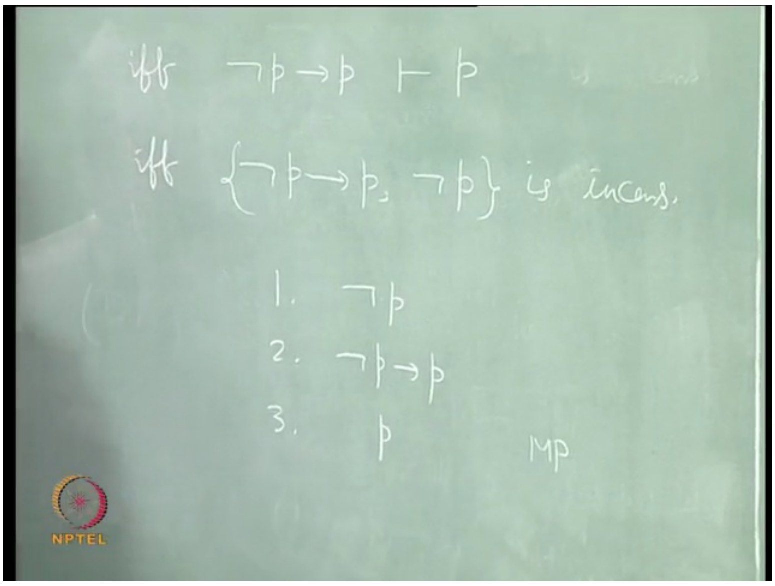

Now, what happens without using the laws, just use the meta theorems? This, we can say if not p, p entails q, by deduction theorem, look at this. It says something; it is easy. But it says something. This stage itself, it says from one inconsistent set everything follows, right? That is also called the paradox of material implication. The implication we have defined, that itself says this, so from an inconsistent thing everything follows. This happens here, because q with not sign is inconsistent by reductio ad absurdum theorem and this is true because you have a proof, not p, along with p, one line proof, fine? So, it is inconsistent.

So, about this law; this is, this was one of the laws, right? In semantics, Clavius' law, or something. How do you go about? Well, deduction theorem. This happens if not p implies p entails p.

Student: RAA.

Then RAA, not p implies p, not p is inconsistent. Is it inconsistent? Yes, you can give a proof, modus ponens gives p. You have already not p. The proof would look something like this, not p, 2, not p implies p, third will be p, by modus ponens. Now, look at first line and third line, this proves p and this prove not p. Therefore, the set of premises is inconsistent.

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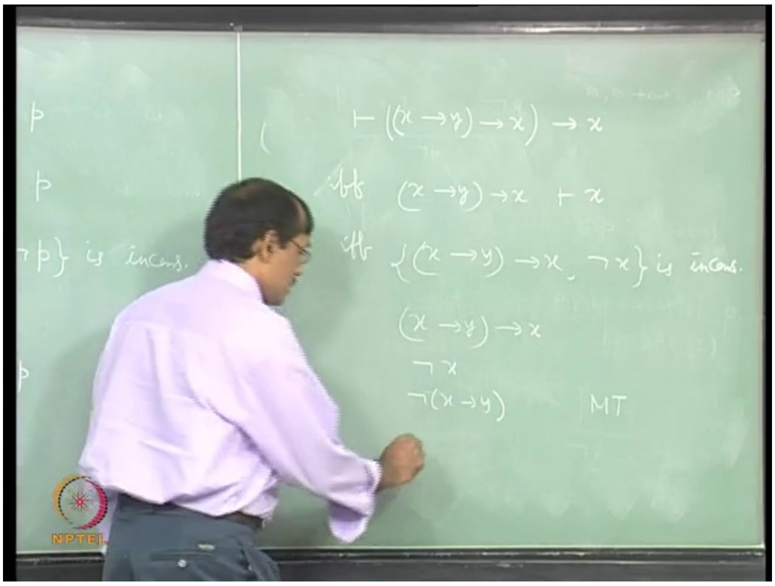

This also looks as absurd as paradox of material implication, this was also a law, Pierce' law. I think so, from x implies y, you get x, then you can really get x. So, how do you prove this? Again you want deduction theorem, this happens, if, then?

Student: Take modus tolens.

You have to apply modus tolens, so you get not of x implies y. Then? Not of x implies y.

Student: The negation not of x it is y, so or not will get sometime there is no r?

You do not have r or r? You have x implies y implies, x is a premises, right? You have, not x is a premises, then you say, you get not of x implies y, by modus tolens.

Student: They have two contradictions.

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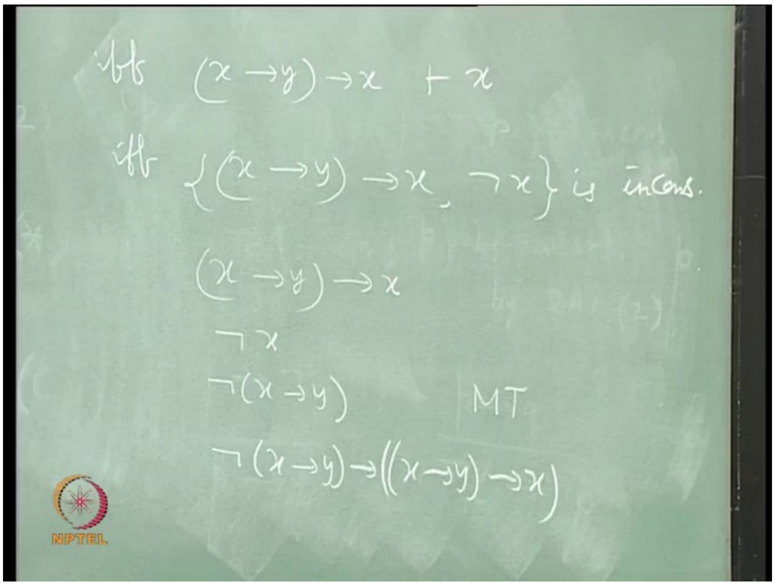

All right, what about this? It is not actual p. You have already proved it, okey? You have already proved it, so you can use it. Now that is a theorem, any p, any q, right? So, not of x implies y, that is my p here, x implies y, q is x.

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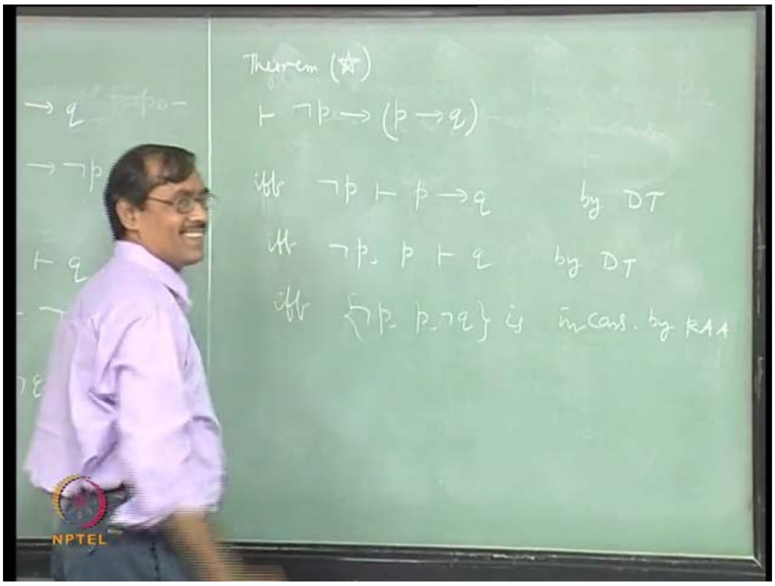

Okay? Now, from these two, I get x implies, implies x that is already there, x implies y implies x is already there, yes?

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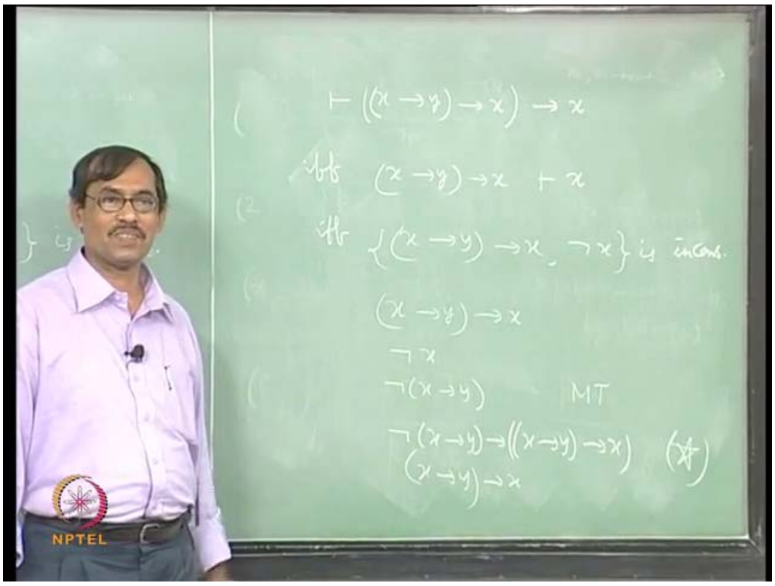

Student: Yeah, but normally, not x implies y.

So?

Student: not of x implies y.

So? These are not matching, we are just trying. How to proceed? These three, you have, you are trying only. But this gives x implies y implies x, which is already there, right? That does not do anything, no help, but you have not x instead of these. Suppose, I have taken not x, yeah? not x implies x implies something, x implies y, you can take that. Then I can get x implies y.

Student: The right side.

Then what will it give? x? inconsistent is it? Not because not of x implies is already there, is it clear? What should I do? Here, not x implies x implies y, you take it in that form, fine. Now, you have not x, we get x implies y by modus ponens and that is the end of the proof, which says I have here not of x implies, here x implies y. So, this will be inconsistent, it is clear.

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