Linear Algebra
By Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 8
Vector Spaces

So in today's lecture we look at the notion of a vector space. Almost all of us have done vector calculus calculus involving two variables three variables, for example look at vectors in the plane we have done vector addition and then multiplying a vector by a scalar we have for example this is called a linear combination of vectors we have also encountered linear combination of functions especially when we have discussed the notion of solutions of differential equations if y1 and y2 are two solutions of a linear differential equation then a linear combination alpha times y1 plus beta times y2 is also a solution of the differential equation.

The structure that formalizes this notion of linear combination etcetera is called a vector space this is a basic object of linear algebra. So what I will do is today give the definition of a vector space and then give a variety of examples just to tell you that vector spaces abound in the whole of mathematics. So these examples for instance will include the differential calculus example that I mentioned and also other examples, okay.

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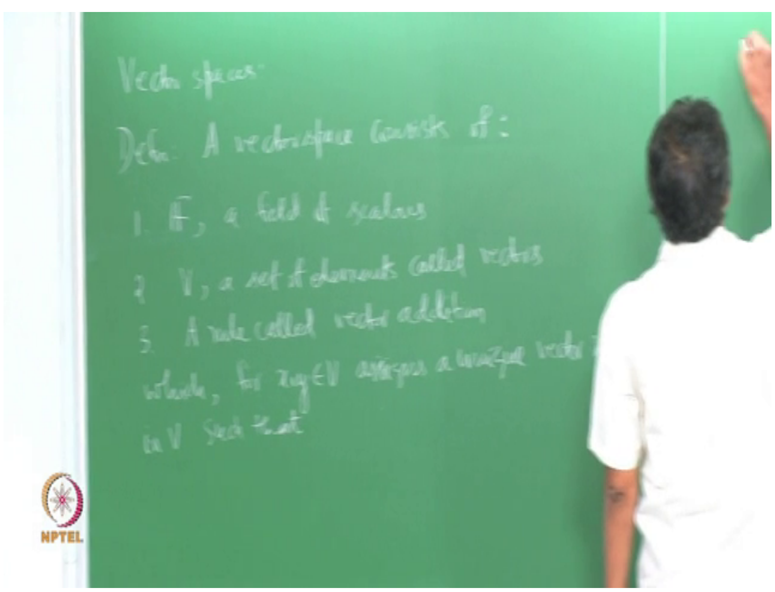

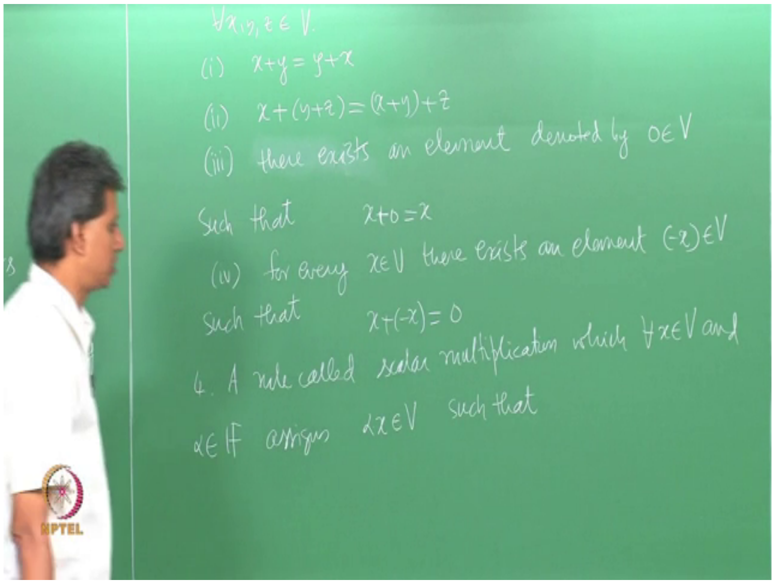

So let us look at first the notion of a vector space, so here is the definition a vector space consist of the following consist of the first object is field so we say vector space over a field the first object is if a field we say that it is a field of scalars in this entire course of linear algebra we will restrict our attention mostly R and in some examples we complex a field also, then we have V a set of elements V of course is non-empty set of elements these elements are called vectors then third aspect is a rule called vector addition a rule called vector addition this is a binary operation which for x, y in V assigns a unique vector which we will denote as x plus y this x plus y belongs to V, okay so there is a binary operation on V it is called the vector addition this binary operation satisfies the following conditions, okay for every x, y in V there is a unique x plus y which also belongs in V so it is closed to respect to this operation such that the following conditions are satisfied such that lets write these conditions here, first it is commutative x plus y equals y plus x second condition its associative x plus y plus z equals x plus y plus z so that is associativity.

Condition 3, for every, I am sorry it is a condition 3, condition 3 there exist an element denoted by 0 that is in V such that so this 0 acts like additive identity so we have this condition to be satisfied such that x plus 0 equals x, okay. See all these are for all x, y, z in V so such that this holds for all x in V and the last condition the fourth condition is that for every x in V there exist an element which we denote as minus x this minus x belongs to V such that this is like the negative additive inverse negative element such that x plus minus x equals the additive identity, alright? So these conditions hold for all x, y, z in V so this is with respect to the vector addition operation this four conditions must be satisfied.

So we have so a vector space has a field so there is field over which the vector space is defined there is V a set of vector satisfying these four conditions, now what is the interaction between the field elements and the vector elements that is given by scalar multiplication, so I will say four a rule called scalar multiplication which for every x in V and a scalar alpha in F which for every x in V alpha assigns alpha in F assigns a number alpha times x in V. So for every x in V alpha in F this product this scalar multiplication alpha times x must be in V and this scalar multiplication must satisfy the following four conditions so what are those conditions, remember the underlying remember that F is a field so F for instance has 0 as well as 1, 1 is the multiplicative identity.

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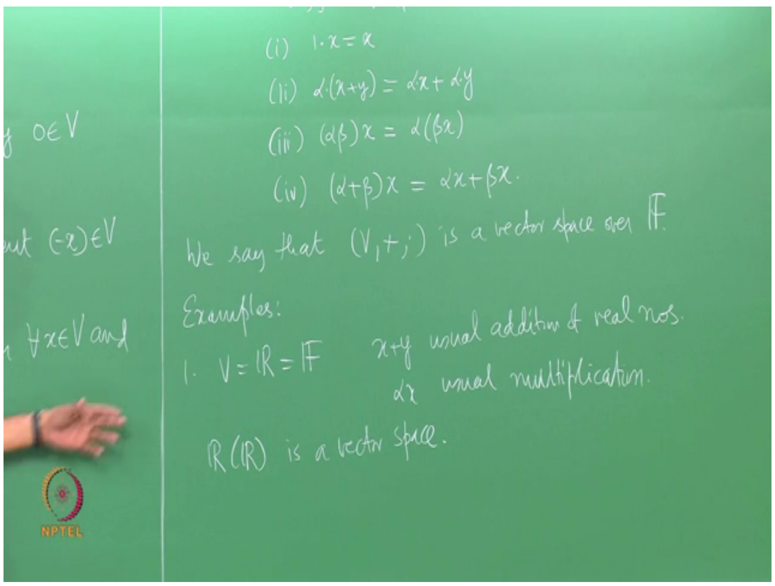

So the first condition relates uses the multiplicative identity so I will again write four conditions for the scalar multiplications 1 in 2 x equals x so for all x, y in V and for all alpha beta in F so I have this condition 1 into x equal to x, condition 2 alpha into x plus y equals alpha x plus alpha y I should actually write a dot here we will follow the convention of not using this dot alpha into x plus y is alpha x plus alpha y.

Condition 3, is if you look at alpha b alpha beta of x it says you do it repeatedly this is alpha into beta x and finally condition 4 is you take scalars alpha and beta look at alpha plus beta of x this must be alpha x plus beta x. So the scalar multiplication must satisfy these 4 conditions the vector addition must satisfy these 4 conditions the underlying set F must be a filed then V is called a vector space over F, so in this case we say that we say that V plus dot is a vector space over F so this is a formal definition of a vector space, alright?

Let us look at some examples, now before we proceed with the examples let us understand that the two examples that I told you in some sense as motivating examples satisfy these conditions example of two solutions coming from differential equation and the behavior of vectors in the plane for instance, so let us look at some examples I will give an abundance of examples just to illustrate that this is an absolutely fundamental object in mathematics not just in linear algebra.

So let us look at examples let us start with the simplest example you take V as R also the underlying field as R then we know that then addition is what x plus y is the usual addition usual addition of real numbers scalar multiplication is also the usual multiplication then this is a vector space V over F is a vector space that is we say that R is a vector space we will follow this notation only for this today's lecture we will simply say R is a real vector space, C is a complex vector space etcetera.

So R over R is a vector space as I said we can also say that R is a real vector space. So this is the basic example similarly one could define C as a vector space over itself. In general any field over itself is a vector space over that field.

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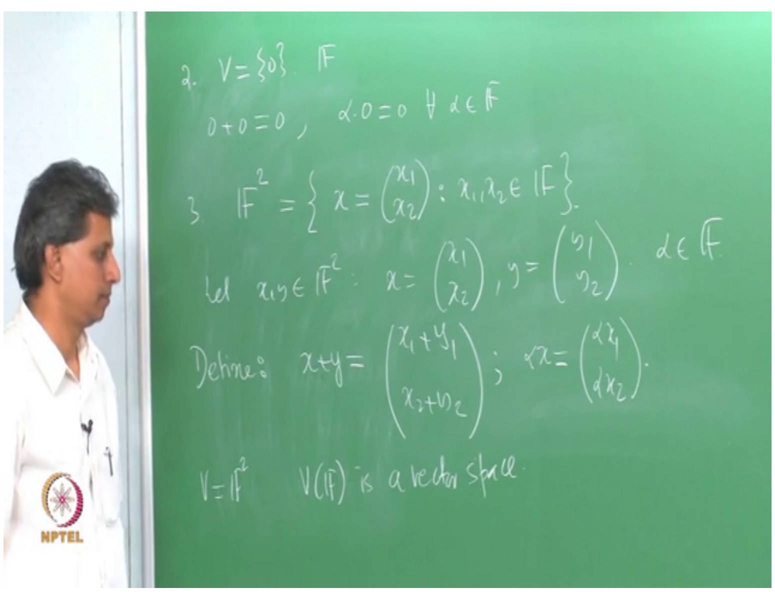

Second example is also trivial take V to be single term 0 and F is any field you have to define see you must define vector addition scalar multiplication vector addition is defined like this scalar multiplication alpha times 0 equal to 0 for all alpha in F then this is a vector space V over F is a vector space over the field F, okay these are trivial examples let us look at the first non-trivial example look at F2, F2 is what? F2 is the set of all vectors x written in this manner x equal to x1, x2 so I will write it as a column vector set of all x1, x2 such that x1 and x2 both belong to the underlying field so this is F2, okay set of all columns which (())(12:00) which have just two coordinates which have just two components, what is vector addition scalar multiplication once I specify that we can verify whether it is a vector space.

So let us take two elements x, y in F2 then x can be written as x1, x2 y can be similarly written y1, y2 let us also take alpha in the underlying field then x plus y we will define x plus y to be the column vector the natural way to define addition is x1 plus y1 that is the first coordinate x2 plus y2 is the second coordinate this is obviously a binary relation scalar multiplication alpha x you do it for each coordinate multiply the scalar alpha to each coordinate alpha x1 alpha x2 then it is an easy exercise to verify that F2 over F that is V equals F2 so V over F is a vector space with respect to this addition operation and this scalar multiplication, okay.

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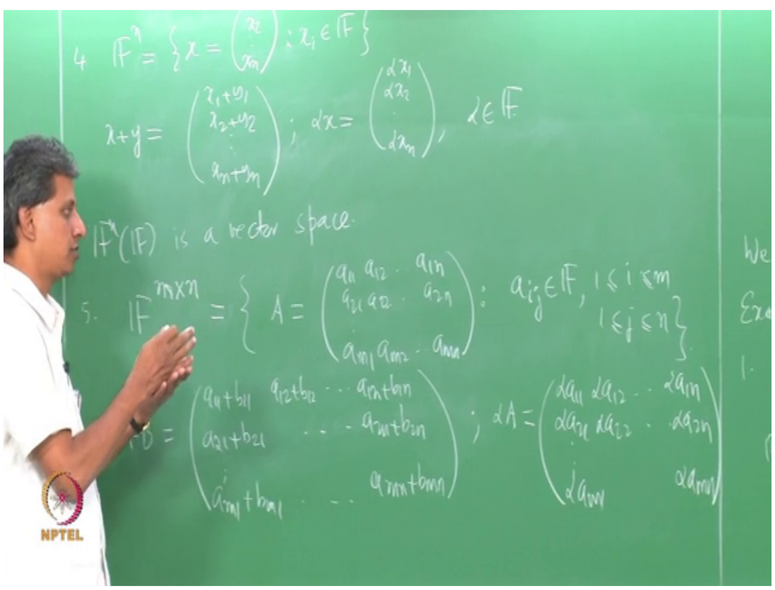

Now we can do a little more general so let us look at Fn, Fn this times is the set of all x that can be written as x1, x2 etcetera xn so there are n coordinates this times such that each xi comes from the underlying field this simply extends F2 that was given in the previous example how do you define addition as before coordinate wise x1 plus y1, x2 plus y2 etcetera xn plus yn.

So x is this y is y1, y2 etcetera yn addition is defined in this manner scalar multiplication is similarly for alpha in the underlying field then again we can show that Fn over F is a vector space with respect to these operations, okay so this is a little extension of the third example we can do a little more general let us look at F m cross n, now this m cross n should suggest the objects that we are considering here so what is the definition here this is the set of all x such that x has m into n coordinates that is one way of looking at it which will give rise to the same vector space as above a slightly different way of looking at F m cross n which is the set of all A which can be written as A11, A12, etcetera A1n so this is set of all m cross n matrices, matrices with entries coming from the underlying field which matrices which have m rows and n columns a1 etcetera a11 etcetera a1n a21, a22 etcetera a2n let me write the last row am1, am2 etcetera amn.

So these numbers aij they all come from F 1 less than or equal to i less than or equal to m 1 less than or equal to j less than or equal to n such an object is called a matrix so you collect the set of all matrices with the property that the matrices have m rows and n columns the entries of the matrix come from the underlying field F then again the operations of vector addition and scalar multiplication is natural, by the way here it is addition of matrices but still we refer to it as vector addition that is each element in F m cross n will be referred to as a vector, okay now this is the terminology that we will adopt. So a vector will not denote any more objects on the plane or objects on the three dimensional space they can now denote matrices we will do a little more general we will use vectors to denote polynomials, we will use vectors to denote solutions of differential equations etcetera, okay.

So let us look at vector addition scalar multiplication so take two matrices A and B A written in this manner B written in a similar manner then A plus B is coordinate wise addition, so A plus B is a11 plus b11, a12 plus b12 etcetera a1n plus b1n, a21 plus b21 etcetera a2n plus b2n similarly the last column last row am1 plus bm1 etcetera amn plus bmn. So this is how vector addition is defined scalar multiplication as before alpha times A will be you multiply each entry by the scalar alpha alpha a11, alpha a12 etcetera a1n, alpha a21 alpha a22 etcetera alpha a2n etcetera am1 amn, so this is how scalar multiplication is defined again one can verify that F m cross n over F is a vector space.

See for instance the 0 element will be the 0 matrix the matrix all of whose entries are 0 the negative element of an element A will be the matrix whose entries are the negatives of the corresponding entries of the matrix A etcetera.

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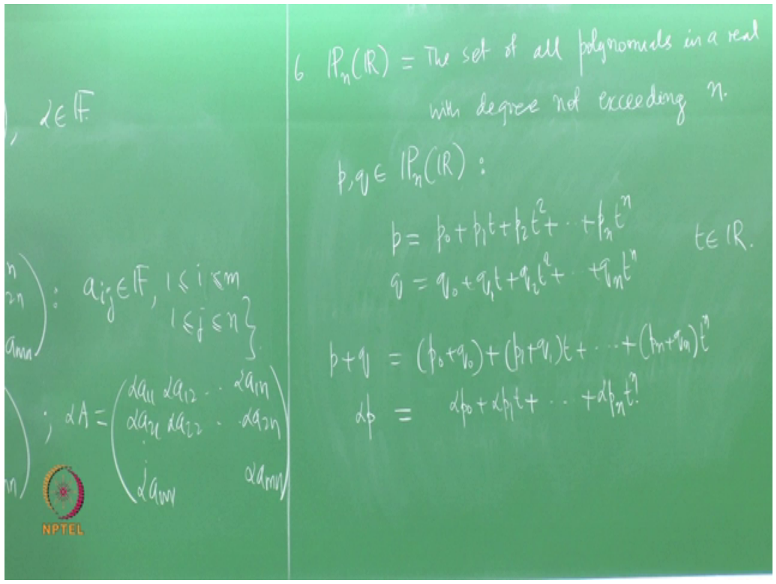

So this is another vector space example of vector space let us next look at Pnr for instance, what is Pnr? Pnr equals the set of all polynomials in a real variable in a real variable t with degree of the polynomial not exceeding the degree not exceeding the number n that is used as a subscript here.

So it is set of all polynomials of degree less than or equal to n polynomial in what polynomial in one single real variable t that is what is denoted by this R, okay. Now for example if Pq belong to Pn of R then these are polynomials I can write polynomials in the following manner P can be written as p not plus p1 p plus p2 t square etcetera plus pn p2n remember some of these constants p not, p1, p2 etcetera pn some of these constants could be 0 because it is a set of all polynomials of degree not exceeding n that degree could be less than n in which case pn is 0 for instance similarly q q not plus q1p plus q2 t square etcetera t is real variable, so I have two elements two vectors now so vectors we are using the name vector for a polynomial here.

So what is vector addition p plus q that is again the natural addition p plus q is defined as so a definition is p not plus q not plus p1 plus q1p plus etcetera plus pn plus qn t to the n, so this is the definition of addition vector addition scalar multiplication also you would simply bring it to the coefficients, so alpha times p is alpha p not plus alpha p1 t etcetera plus alpha pnp t power n then you can again verify that v is this pnr is a real vector space, okay so pnr is a real vector space vector space over the field of real numbers.

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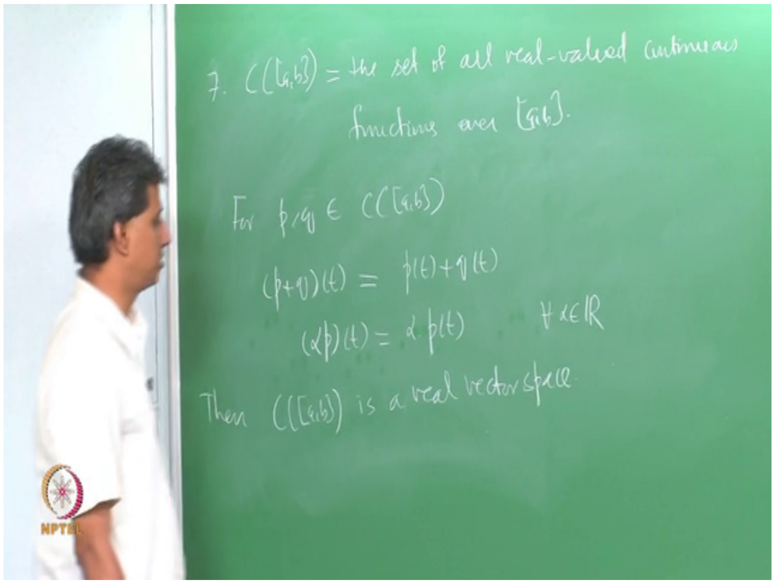

Let us do a little a more general let us now look at the space of continuous functions the space of real valued continuous functions on a closed interval AB the space of real valued continuous functions on the closed interval AB we should again so I am saying the set of all real valued continuous functions so the underlying field will be R we should again define vector addition scalar multiplication you know that is similar to what we have done earlier for the case of polynomials this is so called point wise addition so we will do a similar thing here for pq in C of AB define p plus question of t so p plus q is a new function whose definition is p plus q at the point t is p of t plus q of t and scalar multiplication is defined similarly alpha p is a new function it is just alpha times p of t this is the most natural way of defining it for all alpha in R for pq in C of AB.

So it can again be shown that C of AB is a real vector space if we replace underlying field by C and then consider complex values continuous functions then C of AB will be a complex vector space.

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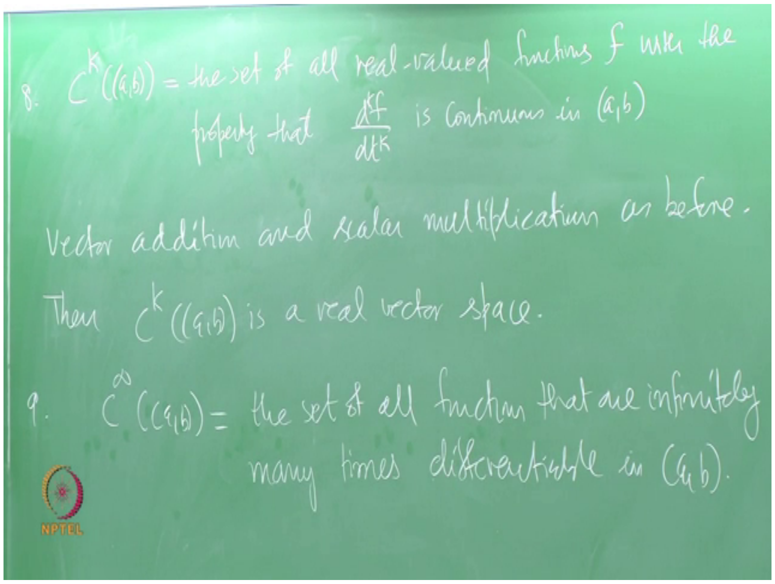

We can do a little more general let us denote Ck this time open interval to be the set of all functions real valued functions let me say F with the property that let us say that I will use the variable t for any point on the interval AB so it is F of t for instance with the property that dkf by dtk the kth derivative of f is continuous the property that dkf by dtk the kth derivative of f with respect to of course there is only one variable here the independent variable is t this is continuous in AB collect all such functions real valued functions having this property again vector additions scalar multiplication as before vector addition and scalar multiplication as before then this is a vector space is a real vector space we can do a little more general one looks at C infinity of A this is now the set of all functions real valued functions that are infinitely many times differentiable in the interval AB open interval AB same operations of vector additions scalar multiplication will tell you that this is a real vector space that is infinity AB, okay that is infinity AB let us consider a more let us consider another example I will give it as example 10.

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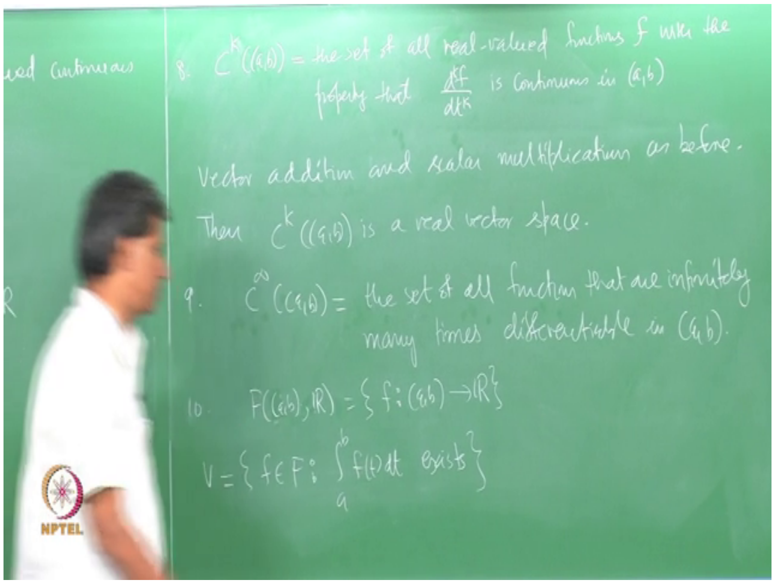

Let us consider let, okay let me use this notation, okay let me use the notation F of let me use the notation F of AB, AB comma R to be the set of all functions F from AB to R set of all real value functions depend on the open interval AB collect all those functions that is my capital F AB comma R.

Now in this look at V as the set of all F and F such that integral A to B F of t dt exists that is f is Riemann integrally that is set of all functions that are Riemann integrable then with respect to again usual addition and scalar multiplication we can show that V is a vector space when V is a real vector space. Now what is the meaning that this is a vector space if F Riemann integrable G is Riemann integrable then F plus G is Riemann integrable if F is Riemann integrable alpha is a real number then alpha times F is Riemann integrable, okay.

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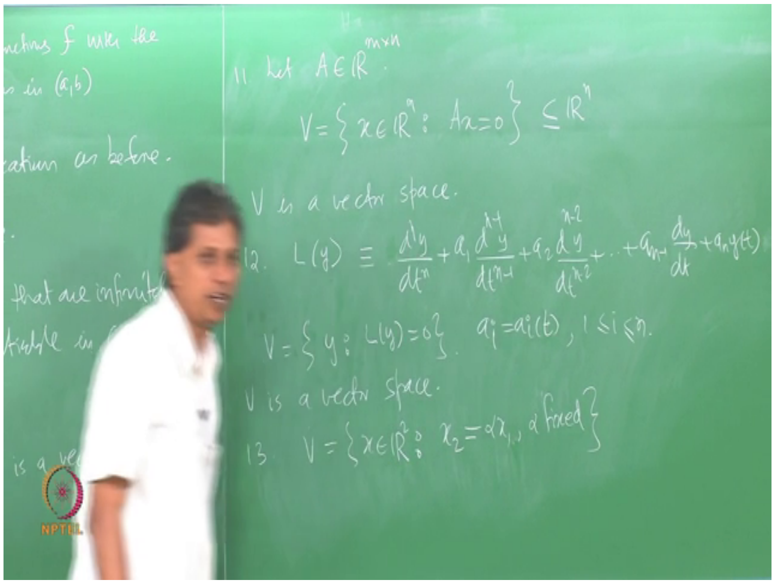

Example 11, suppose that we have a matrix A of order m by n so there are m rows n columns entries are real numbers defined V as the set of all vectors x in Rn such that Ax is 0 Ax equal to the 0 vector. Now remember A is an m cross n matrix x is n cross 1 any vector standing alone is a column vector so when I write a vector as it is it is a column vector, so the product here is m by n into n by 1 so it is a usual product of matrices so the resultant is m cross 1 so this 0 vector on the right hand side is a 0 vector which has m coordinates.

Now this is a vector space now this is a subset of Rn, Rn already has vector addition scalar multiplication defined it can be shown with respect to those operations that V is a vector space, again close to with respect to addition means what if x and y satisfy Ax equal to 0 and Ay equal to 0 then x plus y satisfies A of x plus y equal to 0 because A of x plus y is Ax plus Ay. Similarly, if x belongs to V then alpha times x also belongs to V because the alpha comes out here of this equation alpha comes out of this equation.

So this is a vector space we also have another example this the example that I gave as a motivating example let us look at the operator L defined on a function y which is at least n times differentiable y is a function of the independent variable let us say t define L of y so I am looking at this equation L of y equals L of y is defined by d to the n y by d t to the n plus a1 d to the n minus 1 y by dt n minus 1 plus a2d n minus 2y dt n minus 2 plus etcetera plus a n minus 1 dy by dt plus an y of t, let us so let L be defined in this manner L is a differential operator, consider V as the set of all y such that set of all y such that L of y equals 0, okay see I must mention here that a1 etcetera each of these is a function of the independent variable t each of these is a you can take them as constants in particular but in general they are functions of t.

Then look at V as a set of all y so this you need y to be at least n minus you need y to be n times differentiable in order for this to be defined, so collect all those functions then that satisfies then collect all those function then such a way such a set is a vector space if you consider L valued function then it is a real vector space. Let me give you a couple of more examples some intuition coming from geometry, so example 3 V is the set of all x in R2 so this is a subset of R2 in fact it is a vector space in its right set of all x in R2 such that x2 equals alpha times x1 alpha is a fixed number x2 is alpha times x1 alpha is a fixed number, okay.

So which means if you collect the set of all vector which have the property that the second coordinate is alpha times the first coordinate, okay. Now geometrically what does this set represent? This set represents the set of all points passing through the origin this set represents the set of all points lying on a straight line passing through the origin, okay it can be shown again that this is a vector space the set of all points lying on a particular straight line passing through the origin the slope is given by alpha.

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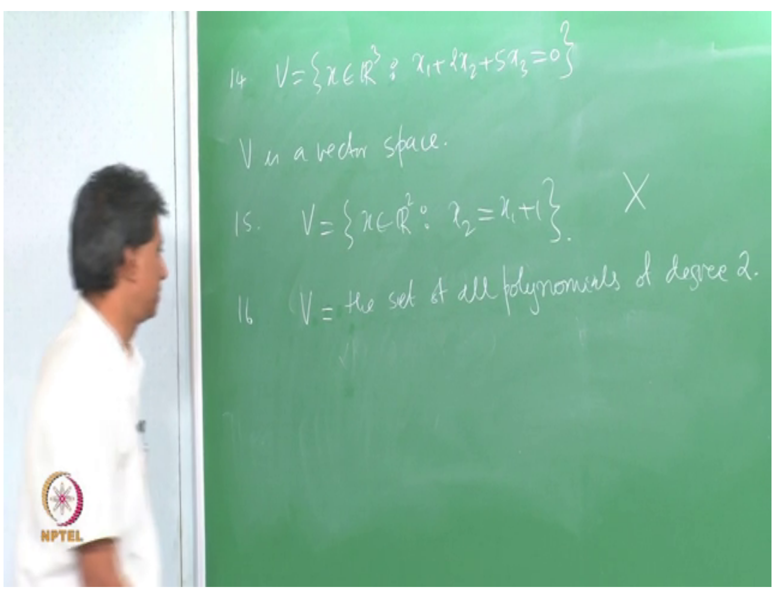

A little more general I will call that as example 14 is the set of all vectors in the three space that satisfy something like x1 plus 2a x2 plus 5 x3 equals 0 geometric interpretation this is the set of all points lying on a certain plain passing through the origin, okay this can again be shown to be a vector space, okay. So I hope the abundance of this examples illustrate the importance of the notion of a vector space, let me conclude the lecture by giving two by giving at least one example of a subset of R2 which is not a vector space.

Example 15, let us look at V as the set of all x in R2 such that x2 is x1 plus 1 the second coordinate is you add 1 to the first coordinate collect all such vectors this is again a straight line the only thing is that this straight line does not pass through the origin you can show that this is not a vector space, okay. Now you must observe that vector space must have 0, 0 must belong to the vector space now this one does not have 0, okay so this is not a vector space.

One final example, non-example so this is not a vector space one final example let us say V equal to the set of all polynomials of degree precisely 2 I do not say degree less than or equal to 2 degree precisely 2, I leave it for you to show that this is not a vector space it is not even close with respect to addition V is not a vector space, okay. Let me stop here in the next lecture I will discuss the notion of subspaces, examples of subspaces etcetera.