Linear Algebra
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Lecture 7
Invertible matrices, Homogeneous Equations Non-homogeneous Equations

I had written down this theorem yesterday, let us look at a proof of that, okay and also discuss other consequences how one could use element matrices and then as I mentioned yesterday a summary of how elementary matrices are related to invertible matrices then homogeneous equations non-homogeneous equation, okay that is what we will discuss today.

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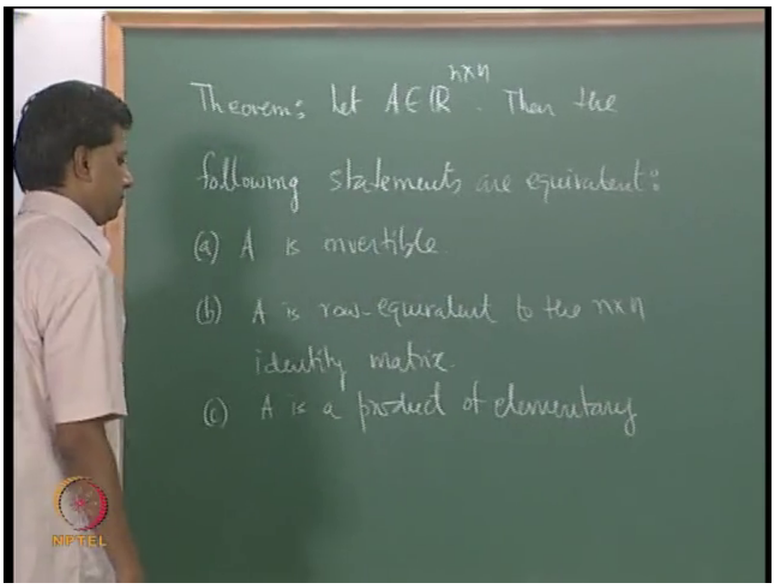

So let me write down the theorem once again I have a square matrix with real entries then the following statements are equivalent for a square matrix the following statements are equivalent.

First statement is A is invertible, second statement A is row equivalent to the n by n identity matrix that is a second statement, third statement is A is a product of elementary matrices, okay so we want to see how these statements are equivalent, okay let me proof so the usual schema will be to prove A implies B, B implies C and then C implies A, okay I will follow that idea then it would mean that these three statements are equivalent.

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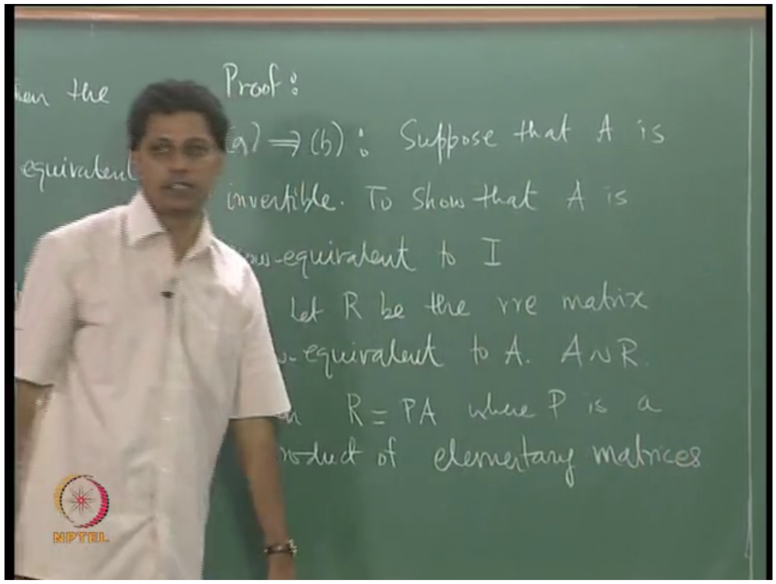

Proof first a implies b I want to show that if a is invertible then a is row equivalent to the n by n identity matrix remember the problem that I gave you yesterday if r is a row reduced echelon matrix which I know is invertible then r is equal to identity, okay suppose a is invertible let me just write down the statement A suppose that A is invertible we are required to proof that A is row equivalent to identity and write that statement also to show that A is row equivalent to the n by n identity matrix let us take R as the row reduced echelon form of A so let R be the row reduced echelon matrix row equivalent to A, see we know that every matrix can be reduced to a row reduced echelon form that form I am calling as R as before.

Then we had seen yesterday that, okay if R is a row reduced echelon matrix row equivalent to A it means A is equivalent to R, okay this is the notation for that we had seen yesterday that this means I can write R as P times A where P is a product of elementary matrices, if P is row equivalent to A then B is P times A that P is a product of elementary matrices that is what we proved yesterday, okay so R is equal to P times A where P is a finite product of elementary matrices this was demonstrated yesterday so I am making use of that here.

Now what I know is that a product of each elementary matrices is invertible product of invertible matrices is invertible that was proved yesterday, so P is invertible and given that A is invertible I know then that product P is invertible so R is invertible so R is identity matrix.

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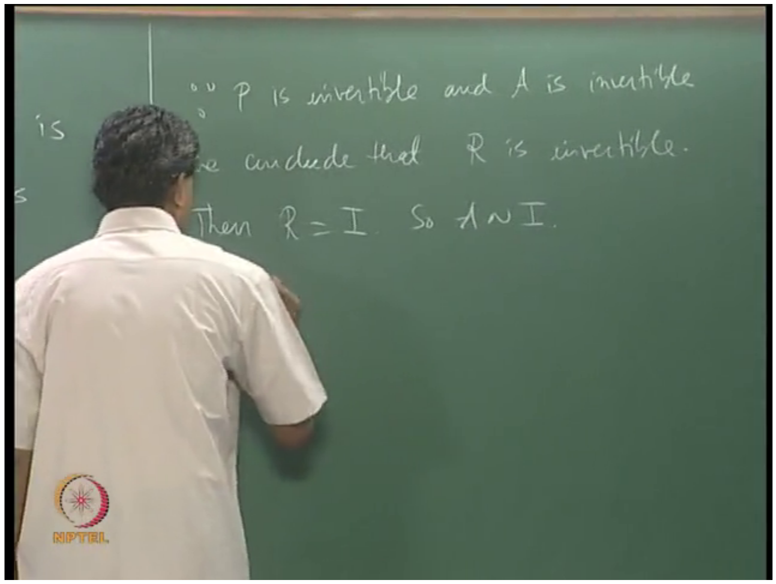

Since P is invertible and A is invertible we conclude that R is invertible but we know that if this happens R is a row reduced echelon matrix it is invertible then it must be equal to identity, okay. And so R is equal to I so I go back and substitute in this equation, okay from this can we directly say A is row equivalent to the identity matrix A is row equivalent to R that is what I have written down here R is identity so A is row equivalent to I that is statement B A is row equivalent to the identity matrix, okay so that proves A implies B, okay.

B implies C so you will see that the results today for instance will be kind of a summary of all that we have discussed till now, okay.

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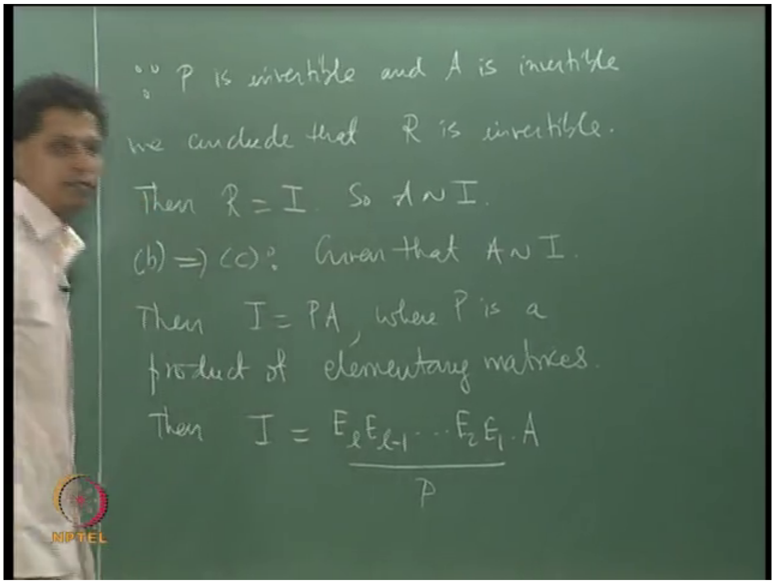

B implies C let us prove this next A is row equivalent to given that A is row equivalent to the n by n identity matrix I must show that A is a product of elementary matrices I will appeal to the same result that we proved yesterday if B is row equivalent to A then B is equal to P times A where P is a product of elementary matrices, then I equals P times A where P is a product of elementary matrices so remember I want to show that A is a product of elementary matrices if A is, is that okay? We are proving B implies C I want to show that A is a product of elementary matrices, okay.

So what I will now do is list identity is El El minus 1 etcetera E2 E1 times A that is the I know that P is a product of elementary matrices I have written down this product El El minus and etcetera E1 times A, okay please remember that this is P I know that I can write like this. Now what I know is that each elementary matrix is invertible so I will pre-multiply by first El inverse then El minus 1 inverse etcetera.

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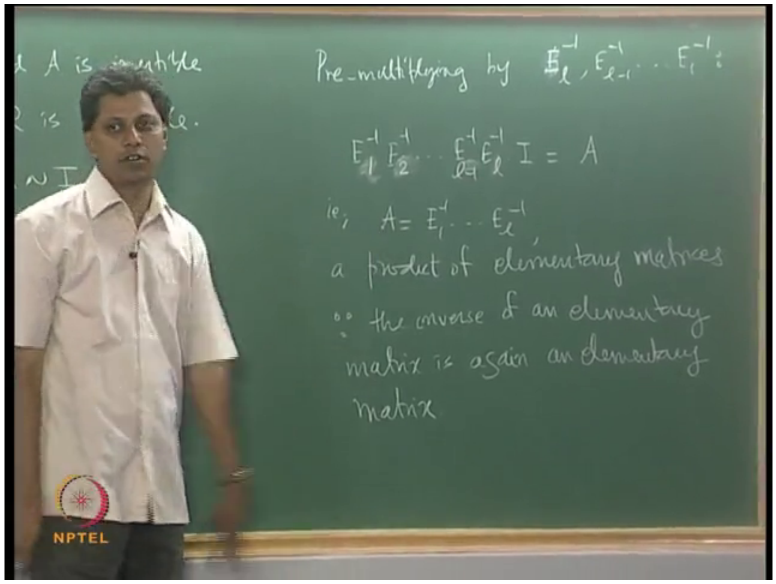

So let me just do the first step also write what I am doing pre-multiplying by Al inverse, sorry El inverse each elementary matrix is invertible I know each elementary matrix is invertible so I take this equation pre-multiply by El inverse for instance I get, okay I will do fully El inverse El minus 1 inverse etcetera E1 inverse pre-multiplying by this I get on the left El inverse El minus 1 inverse etcetera E2 inverse E1 inverse times identity is equal to A, you are right E1, E2, El minus 1 yes that is perfect.

So this what I have this means what A is equal to E1 inverse etcetera El inverse which is a product of elementary matrices because the inverse of each elementary matrix is another elementary matrix it is an elementary matrix of the same type does not matter it is an elementary matrix, okay. So I have written A as product of elementary matrices what is a reason since the inverse of an elementary matrix is again an elementary matrix, okay this last step because the inverse of any elementary matrix is again elementary matrix so I have written E as a product of A, okay so that is a that statement B implies C, okay.

Finally C implies A, C is A is the statement C is the matrix A is a product of elementary matrices statement A is capital A is invertible but this is something we have seen yesterday, okay C implies A we have already seen C implies A.

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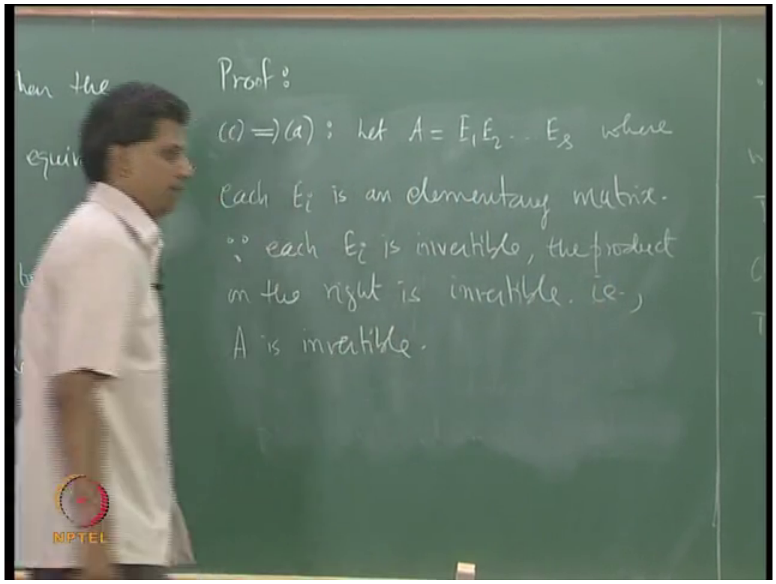

Let A be a product E1, E2 etcetera Es where each Ei is an elementary matrix I must conclude that A is invertible, okay but since each Ei is invertible the product on the right is invertible hence that is A is invertible, right away there is nothing C implies A follows from the fact that each elementary matrix is invertible and the fact that the product of invertible matrix (())(13:03), okay. So this is one summary connecting row equivalence elementary matrices invertible matrices.

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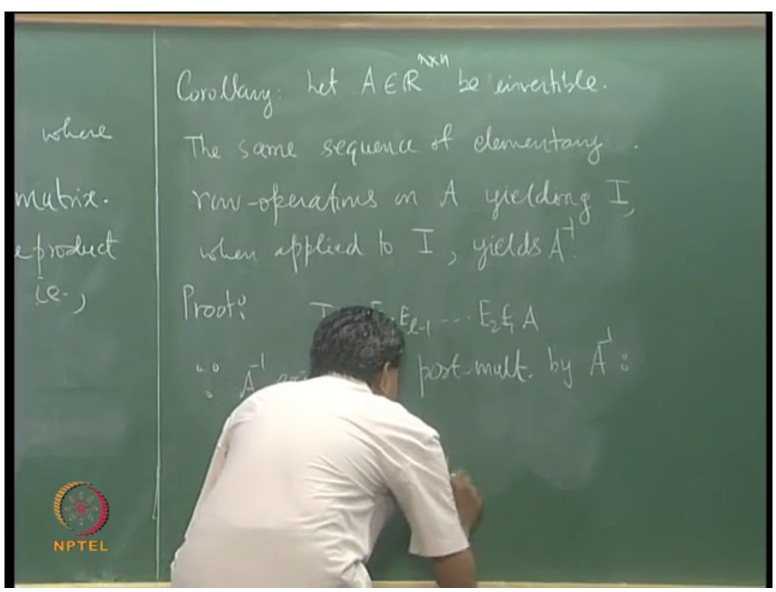

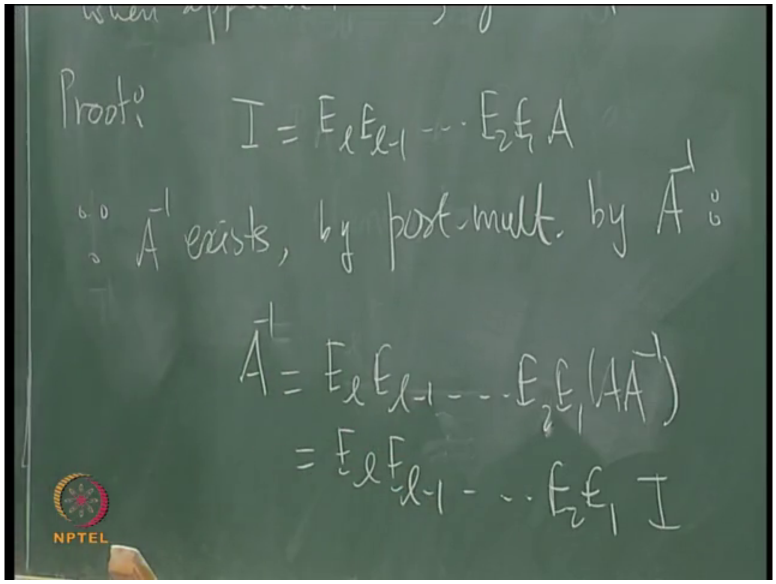

One of the corollaries of this result is the following one of the consequences I will state that as a corollary let A be invertible I have invertible matrix A, okay then I know that A is row equivalent to identity what this result says is you do the same sequence of elementary row operations that you do on A to get I on I you will get A inverse from A to I you perform a sequence of elementary row operations this corollary says you perform this sequence of elementary row operations on the n by n identity matrix you will get the inverse matrix A inverse the same sequence of elementary row operations on A yielding I the same sequence of elementary row operations performed on A yielding I then applied to identity yield A inverse same sequence yield A inverse, okay that is the statement. Proof, is the statement clear? So this gives a method to construct the inverse of a matrix if you know it is invertible by a sequence of elementary row operations. Proof, there is a sequence of elementary row operations that I do on the matrix A to get the matrix identity which means I can write identity as let us say El El minus 1 etcetera E2 E1 A first operation is E1, second operation E2 etcetera last one is El I do these operations in this sequence on the matrix A then I get the identity matrix I know that this happens because A is invertible I using the previous result if A is invertible then A is row equivalent to identity matrix.

I want to show that when I do this same sequence of operations on I I get A inverse, okay. What is given is that A is invertible, okay so I post-multiply by A inverse since A inverse exist by post-multiplying by A inverse this equation I post-multiply by A inverse I get the following I get A inverse on the left on the right El El minus 1 etcetera E1 E2 E1 A into A inverse matrix multiplication is associative so I insert a bracket in the end like this but this you see is precisely

El El minus 1 etcetera E2 E1 being operated on I times I I am emphasizing that this is done on I I could have just left this as it is but this does not tell you that you are performing these operations on I so that is why the last I I have retained here it is not necessary to include this, right? Identity is the multiplicative identity matrix is the multiplicative identity of the multiplication operation.

So I am making it a point to retain this I to emphasize the last part that you are doing this sequence in the same order on the identity matrix to get A inverse, okay. Let us consolidate do one numerical example and proceed with the theory, example where we must know that the matrix is invertible, okay let me give a numerical example.

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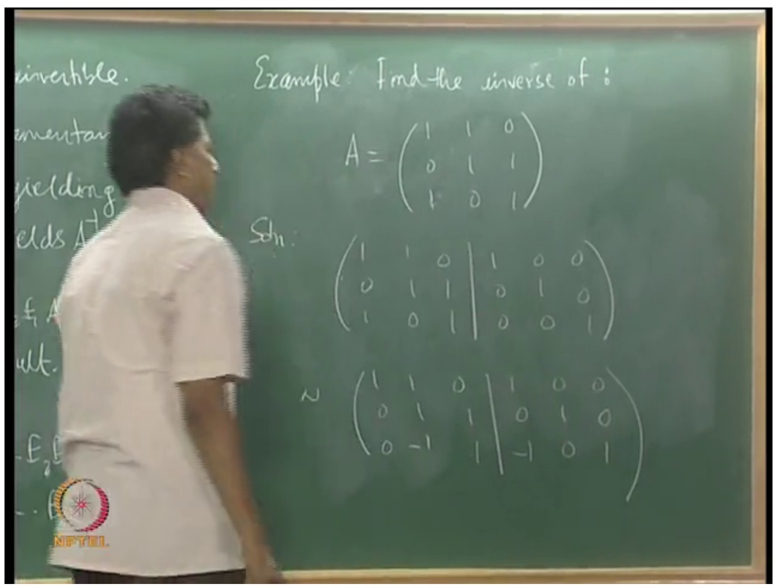

Find the inverse of the following 3 by 3 matrix 110 011 101, see when I do the elementary row operations I will eventually know if this matrix is invertible, okay but I have chosen this example in such a way that it has an inverse that is I verified that this matrix is row equivalent to identity.

I am going to use the previous result I will do the same sequence of elementary row operations on I that I do for A to get A inverse, okay for that it is convenient to append the identity matrix to the matrix A and do the operations for this A and identity together. So I will append this matrix together with the identity 110 011 101 put a vertical line and then write identity of order 3 I must reduce this part to the identity keep doing the same sequence of operations on the second part if this first part reduces the identity the second part should reduce to A inverse this time I am not going to write down the row operations tell me if the steps are correct that is the first step I am keeping the first row as the pivot row your operation elementary row operations are performed keeping the first row.

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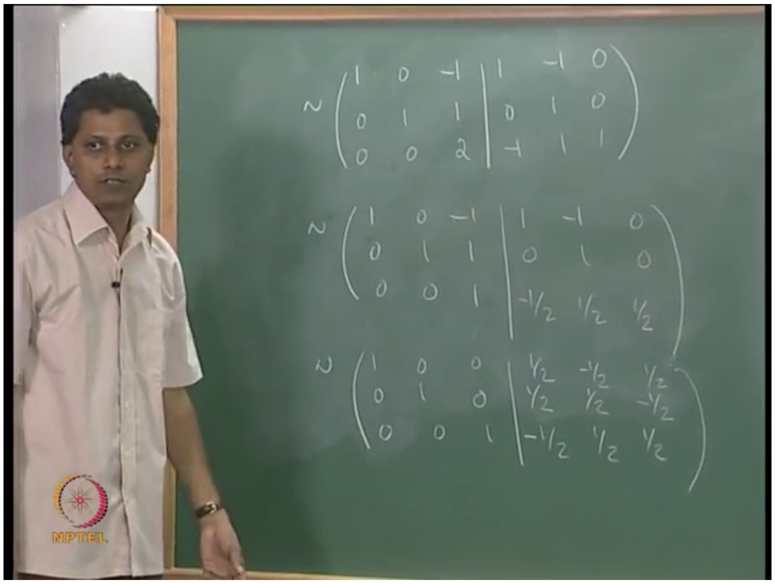

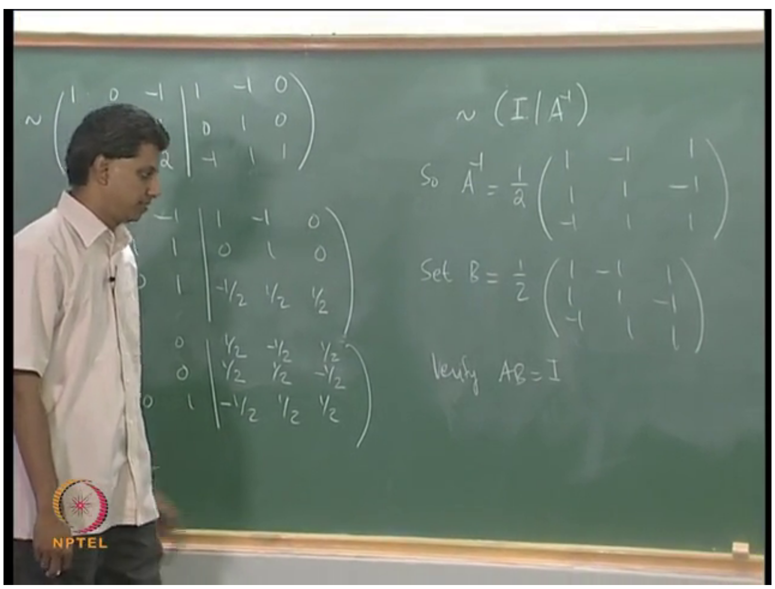

Next step is I will keep the second row as it is then I add the second and the third rows so to get 002 I add the second and the third rows minus 1 11 I add the second and the third rows. Next step is divide by 2 I will do that here itself, okay just let me write down once again 10 minus 1 1 minus 1 0 011 010 001 minus 1 by 2 1 by 2 1 by 2 I want to make these two entries 0 so the next operation I will keep the third row as it is I am keeping this as it is just add 100 second row minus this plus this, okay.

So I see that at the last step I get identity this is equivalent to A, sorry this is identity and so I know that the second part must be A inverse so A inverse I take 1 by 2 outside 1 minus 1 1, 11 minus 1, minus 1 1 1 this is the inverse of A it is always a good practice to check that this satisfies the equation AB equals identity one must actually verify AB that is you can call this B one must verify that AB equal to BA equal to identity, okay but towards the end of today's lecture we will see that it is enough if you verify one of them it is enough if you verify that either AB is identity or BA is identity we will show that a matrix which has a left inverse or a right inverse if the matrix is a square matrix then it will be invertible, okay.

So it is always a good practice to verify that you have got the inverse correctly.

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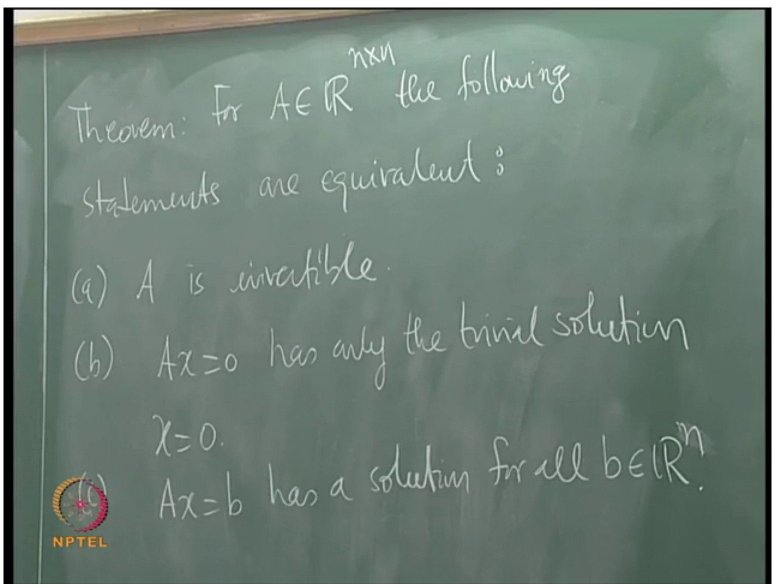

Some other consequences typically one that would relate to homogeneous equations, non-homogeneous equations is the following theorem for a square matrix A the following statements are equivalent, first statement as before A is invertible second statement the homogeneous system Ax equal to 0 has only the trivial solution x is equal to 0 that is the second statement, third statement is now about non-homogeneous systems the non-homogeneous system Ax equal to b has a solution for all b that is a third statement the non-homogeneous equation has a solution for all b these three statements are equivalent if A is invertible then any non-homogeneous system will have a solution b element of Rn, okay again we can prove A implies B, B implies C, C implies A I would follow a slightly different approach I will prove that A and B are equivalent then prove that A and C are equivalent then it could follow that these statements are equivalent, okay this is not a very efficient way of doing it but this is rather easier.

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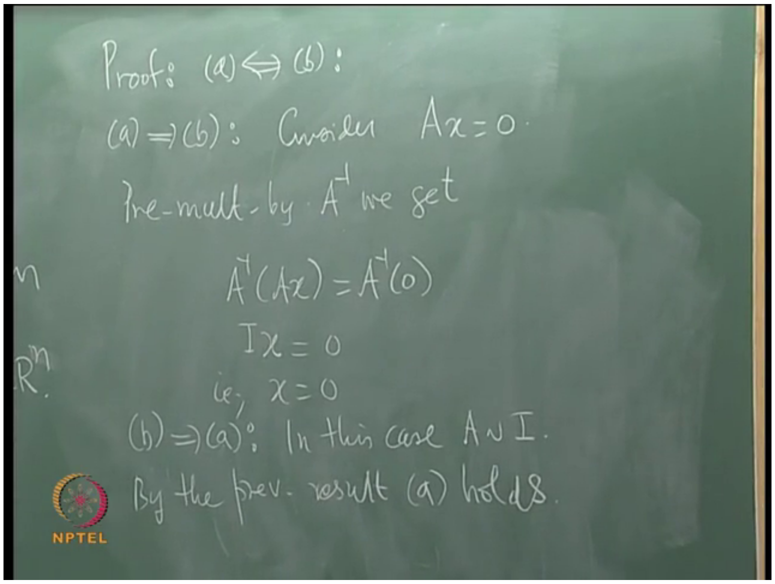

So I will prove A if and only if B first, okay proof I will as I mentioned I want to show that A and B are equivalent, okay first A implies B consider so statement A is with me capital A is invertible consider the system Ax equal to 0 I know that A inverse exists I will pre-multiply by A inverse A inverse into the 0 vector that is a 0 vector A inverse into A is identity by definition matrix multiplication is associative so this is identity x right hand side is 0 that is x equals 0.

So if A is invertible then the homogeneous system has only the 0 solution I must prove the converse B implies A B implies A is something that we have already seen in a slightly different language B implies A the homogeneous system has 0 as the only solution then we know that A is row equivalent to identity, okay this was proved some time ago A is row equivalent to, okay even the previous, no this was proved some time ago. If Ax equal to 0 has 0 as a only solution then A is row equivalent to the identity, so let me write this in this case A is row equivalent to the identity appeal to the previous theorem B implies A previous theorem statement A is A is invertible statement B is A is row equivalent to I we have proved that these two are equivalent so I will simply appeal to that.

In this case A is row is equivalent to I by the previous theorem previous result A holds, okay. So A and B are equivalent that has been demonstrated as I mentioned I will next prove that A and C are equivalent, I will first prove A implies C that is again as easy as A implies B earlier, okay the first part A implies B if A is invertible we have shown that homogeneous equation has 0 as the only solution.

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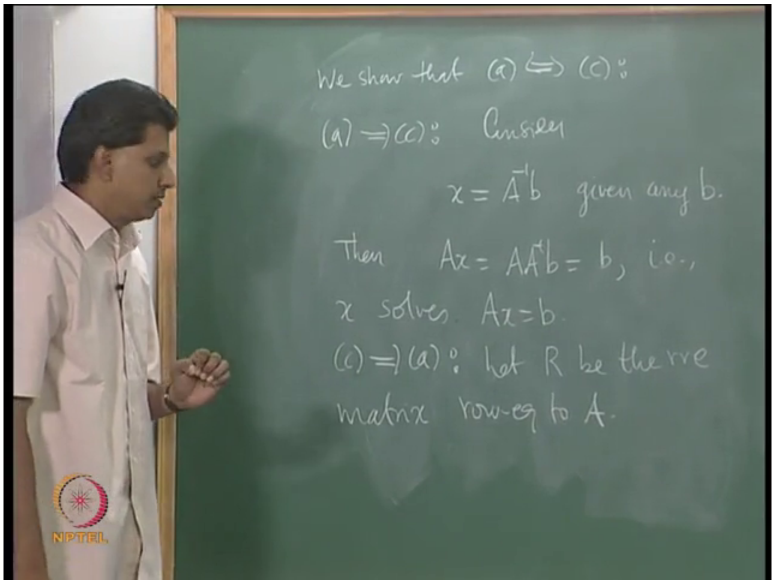

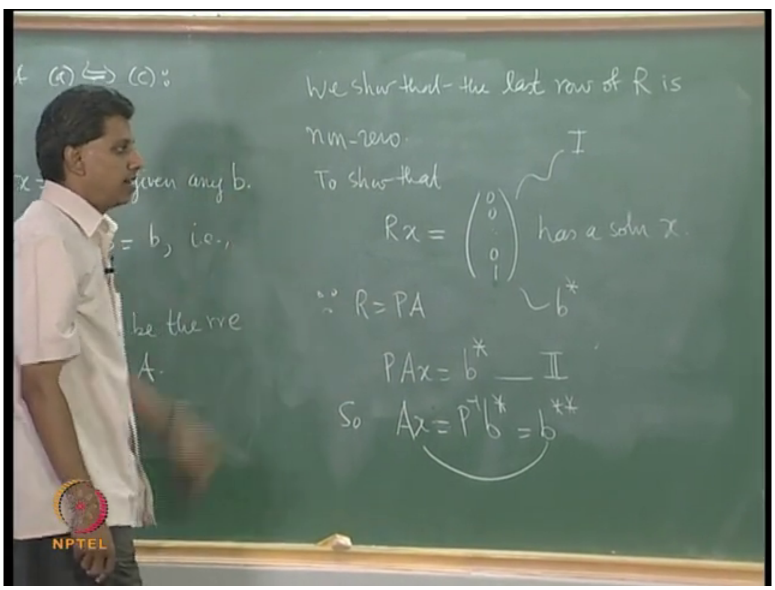

So let us prove A implies C consider x as A inverse b given any b I know that A is invertible that statement A I look at the vector x defined as A inverse b b is a given right hand side vector then Ax is A A inverse b is b that is x solves Ax equal to b. Now this I can do for any right hand side vector b all that I need to do is pre-multiply the right hand side vector b by the inverse of A which I know because statement A holds, statement A holds A inverse exists pre-multiply the right hand side vector by A inverse I get a solution.

So that is A implies C I need to show C implies A this is probably the toughest, okay so I need to show C implies A that would complete the proof of the theorem I want to show that A is invertible so as before I look at the row reduced echelon form of A let R be the row reduced echelon matrix row equivalent to A I will show that R has the last row non-zero we show that the last row of R is non-zero, is it clear that this would mean that R is identity if R is identity then it means A is row equivalent to I appeal to the previous theorem A is invertible see we are proving C implies A.

So if I show that the last row of R is non-zero then it follows that A is invertible. To show that that last row of R is non-zero I need to show that this system Rx equals I look at a specific right hand side vector all entry is 0 except the last entry which is 1 if I show that this system has a solution then it follows the last row of R cannot be 0 because if the last row of R is 0 then the last row of R into x will give me 0 on the right hand side I have 1 so that is not possible. So if I show that this system has a solution x then it follows that the last row of R is not 0, okay.

So to show that Rx equal to 0 has a solution x so this is what we will show I suppose this is clear I have taken this specific entry to be 1 this could actually be any non-zero entry, okay but R is row reduced echelon form of A so R is PA where P is a product of elementary matrices if B is row equivalent to A then B is P times A where P is a product of elementary matrices, so what happens to this system let us call this as b prime I will call this b star I am re-writing this system Rx equal to b star is P Ax equal to b star let me call this as system 1 this is system 1 system 1 has been rewritten in this form, okay if Rx equal to b star I want to show that this has a solution going back if I show that this system has a solution I should call this system 2 rather this is system 2 I am claiming that these two systems are equivalent I am claiming that if 1 is if x is a solution of system 1 then x is a solution of system 2 and conversely, that is clear because R is row reduced echelon form of A, okay.

I want to show that this has solution I will show that system 2 has a solution it then follows this has a solution if system 1 has a solution then R has to be invertible so A is invertible, okay but look at system 2 what is the property of P that I have not used P is a product of elementary matrices elementary matrices are invertible product of invertible matrix is invertible so P is invertible post-multiply by p inverse I get Ax equals P inverse b star I will call this b double star.

So Ax equals b double star does this system have a solution? I have till now not used the fact that system condition C holds so I will use that here condition C holds so whatever be the right hand side vector see this is condition C condition C whatever be the right hand side the system Ax equal to b has a solution whatever be the right hand side the system Ax equal to b this has a solution Ax equal to b double star this time this has a solution x so system 2 has a solution so system 1 has a solution so R the last row of R has to be non-zero so A is invertible then its last line of the proof.

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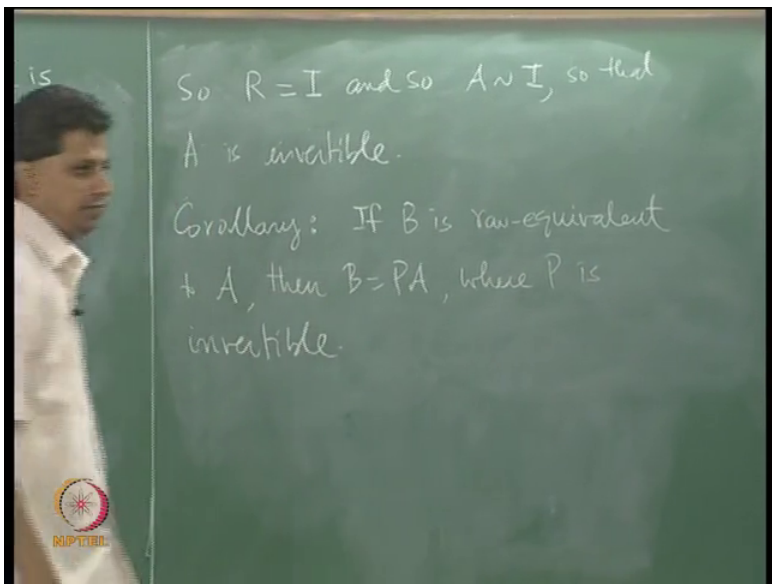

So R is equal to I and so A is row equivalent to I so that by the previous theorem A is invertible and that is the proof of this theorem, okay you have any questions? Let us move on we need to see to conclude this topic we need two more results let me complete them in today's lecture one corollary of this result it is actually a corollary of the previous result let me write down this result I am not going to prove this if B is row equivalent to A then B equal to PA where P is invertible if B is row equivalent to A then B can be written as B times A where P is invertible.

What we had proved earlier is that if b is row equivalent to A then b equal P times A where P is a product of elementary matrices we know that if you have a product of elementary matrices then that must be invertible conversely if b equal to P times A P is invertible then we know by the previous theorem that P can be written as a product of elementary matrices and so b is something like El El minus 1 etcetera E1 times A so b is row equivalent to A that is a proof, okay so please fill up the gaps write down the complete proofs yourselves that is one corollary another consequence is the following this is something that I mentioned in that numerical example if a square matrix has a right inverse or a left inverse if a square matrix is either right invertible or left invertible then it is invertible.

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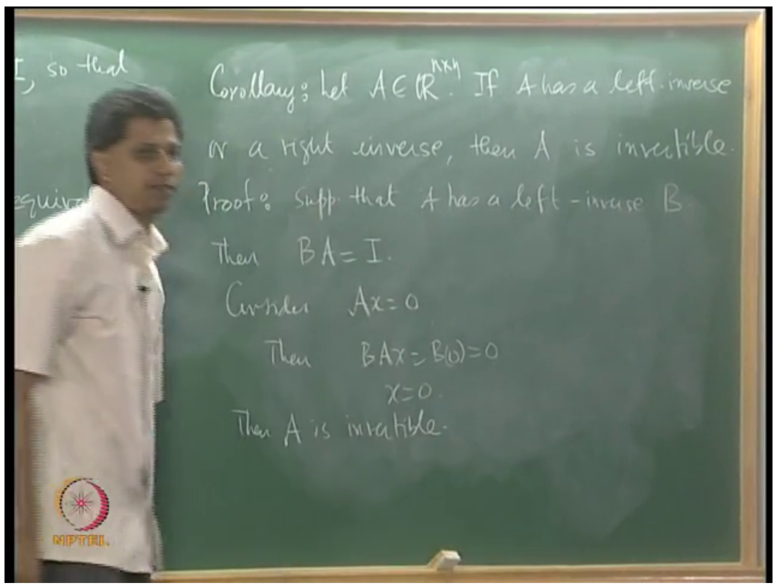

If A is left invertible, okay let me write this as if A has a left inverse or a right inverse then A is invertible so for a square matrix it is enough to verify one equation either AB equals identity or BA equals identity in order to show that it is invertible, okay quick proof let us first take the left inverse case suppose that A has a left inverse I will call it B suppose A has a left inverse I am calling that as B then BA equals identity left inverse so when I post multiplying pre-multiply A by B I get identity I want to show that A is invertible I want to appeal to the previous theorem which connects invertiblity with homogeneous systems I will show that the system Ax equal to 0 has only x equal to 0 has a solution, so consider Ax equal to b Ax equal to 0 consider Ax equal to 0 I want to show that this system has 0 as the only solution, can you guess the next step? Pre-multiply by B Bax is b0 B into 0 is 0 BA is identity x is equal to 0.

So what I have shown is that the Ax equal to 0 implies x equal to 0 that is 0 is the solution of the homogeneous system appeal to the previous theorem I know that A is invertible that is the left inverse case if it has a left inverse then we have shown that it is invertible. We must deal with the right inverse case but in mathematics there is always is reduction I would like to use what I have proved just now can I reduce this step to the previous step, can I make use of the fact that if A has a right inverse then some matrix has a left inverse?

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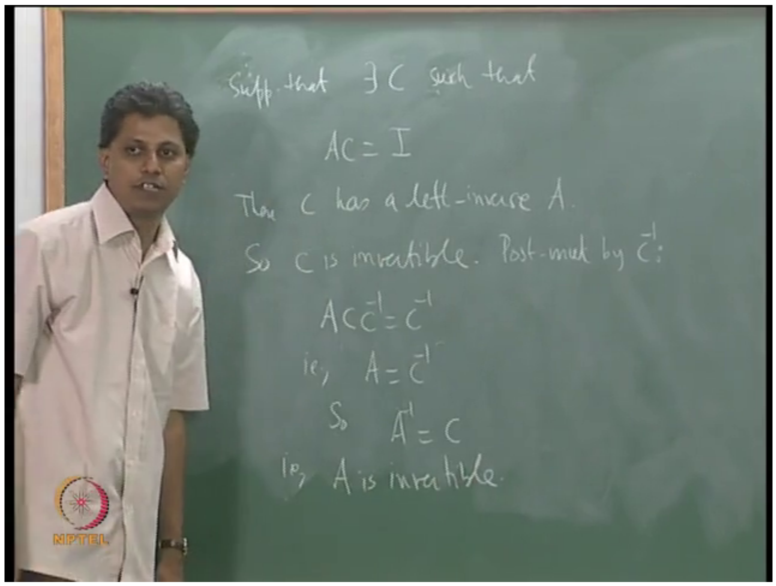

Second part let this is the second part suppose that A has a right inverse there exists C such that AC equals identity that is A has a right inverse.

Then C has a left inverse A by appealing to the first part I know that C is invertible I know that C is invertible by the first part go back to this equation post multiple by C inverse go back to this equation AC equal to identity post-multiply post-multiplying by C inverse we get AC C inverse equals C inverse that is A equals C inverse but I know that inverse of a inverse is the original matrix so A inverse is C that is C is the inverse of A that is A is invertible and its inverse is C, okay.

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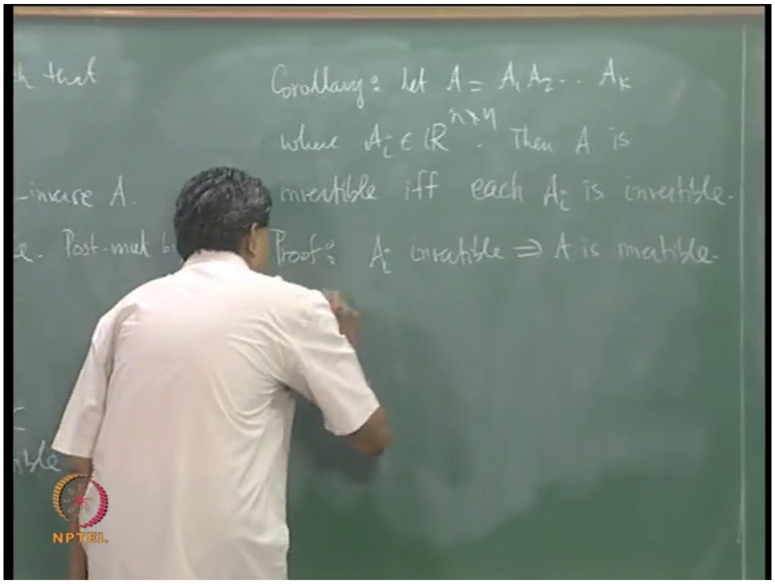

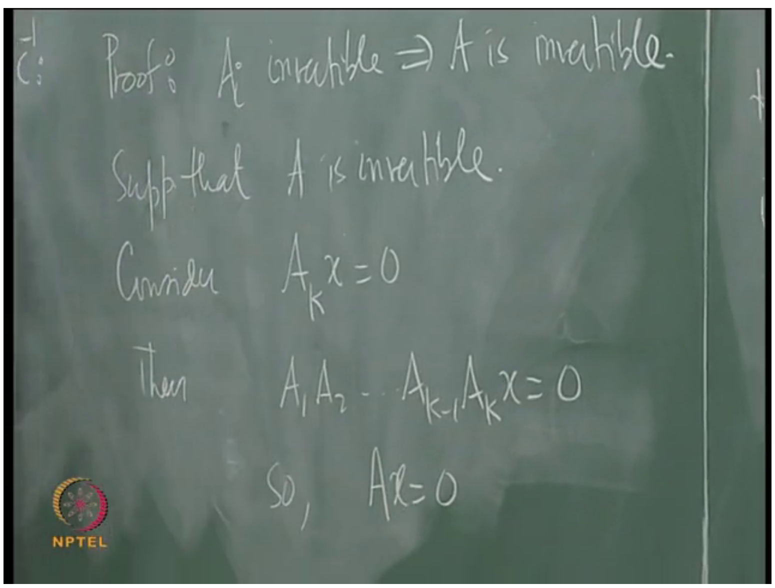

So this is another consequence once I did invertiblity for a square matrix implies invertiblity. Finally, we have the last result I have A as a product of k matrices where each Ai is square so capital A is a square matrix then A is invertible if and only if each Ai is invertible, okay now this is a last result in this topic this again follows from what we have discussed earlier one way is easy anyway if each Ai is invertible then the product is invertible so A is invertible.

So let me just write in shorthand notation, Ai invertible implies A is invertible, this is done each component if each factor is invertible then the entire product is invertible conversely we must show that if A is invertible then each factor is invertible this we have not seen before. Suppose that A is invertible we would like to show that each factor is invertible, okay I will proceed by showing a single factor is invertible is invertible at a time I will consider the system Akx equal to 0 consider the system coming from the last factor homogeneous coming from the last factor consider Akx equal to 0 I pre-multiply by Ak minus 1 etcetera A1 then I have A1, A2, etcetera Ak minus 1 Ak x equals 0 I pre-multiply the right hand side also by this but I am having the right hand side vector as 0 so the resultant vector is also 0.

But this entire product is A so I have Ax equal to 0 but I know A is invertible so by the previous theorem x equal to 0 is the only solution.

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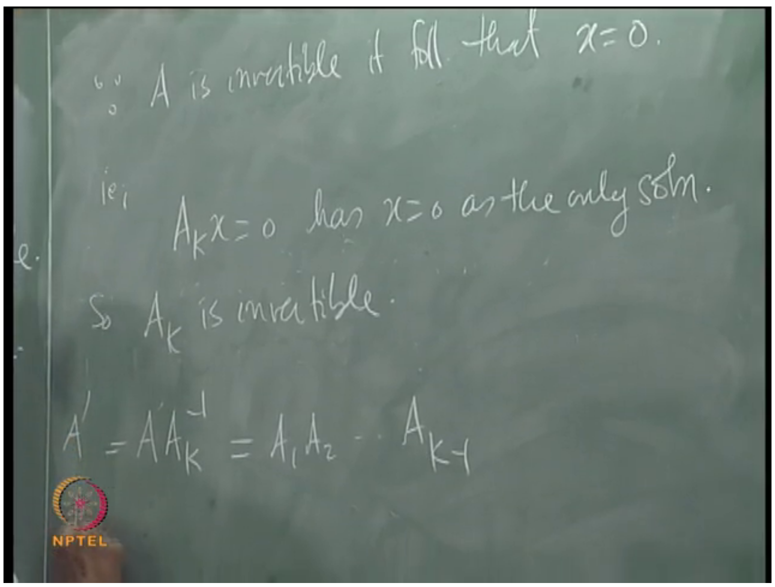

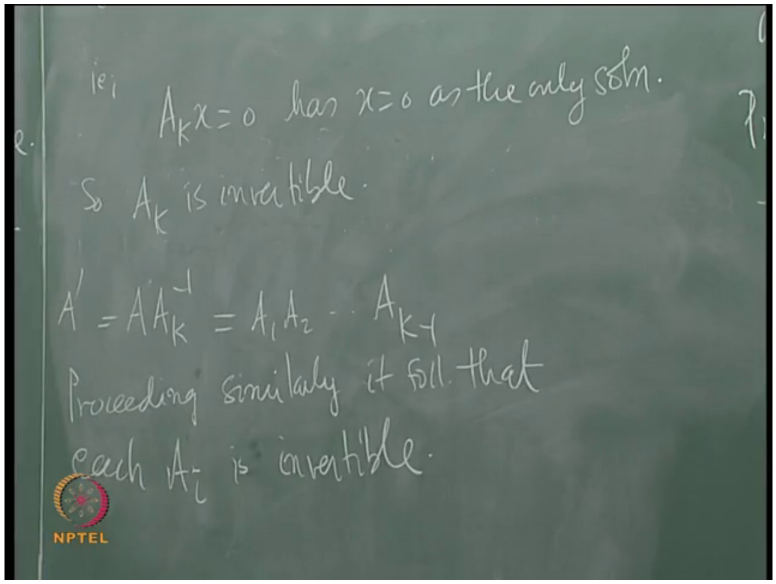

Since A is invertible it follows that x is equal to 0 that is what we have shown is that Akx equal to 0 has x equal to 0 as the only solution Akx equal to 0 has x equal to 0 as the solution that is what we shown remember we have started with Akx equal to 0 that implies Ax equal to 0 since A is invertible x is equal to 0.

So this implies Ak is invertible its square so Ak is invertible I will go back to this equation this formula for A becomes A into Ak inverse I will call it A prime A into Ak inverse this time it is A1, A2, etcetera Ak minus 1 I call it A prime this time it is A1, A2, etcetera Ak minus 1 I am post-multiplying by Ak inverse which I know exists. Now, yes A is invertible Ak is invertible inverse is invertible so A prime is invertible so Ak minus 1 must be invertible by the same argument as we have seen just now it follows that each factor is invertible.

Proceeding similarly it follows that each Ai is invertible, okay so you see that again it is a idea of homogeneous equations having 0 as the only solution that we have made use of in proving that a certain matrix is invertible, okay. So that completes our discussion on matrices, elementary rows operations formalizing Gaussian elimination then discussing the notation of elementary matrix, discussing the notion of an invertible matrix finally the summary, summary connecting invertibility with product of elementary matrices the invertible matrix being necessarily row equivalent to identity matrix homogeneous equations necessarily having 0 as the only solution non-homogeneous equation always having a solution whatever be the right hand side vector, okay.

So that is the summary we would from the next lecture onwards we would discuss the notions of vector spaces, examples of vector spaces, linear transformations, matrix representations, properties of linear transformations and other things, okay I will stop here.