Linear Algebra
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Lecture 6
Elementary Matrices, Homogeneous Equaions and Non-homogeneous Equations

See the next objective is to introduce a systematic scheme by which we can formalize elementary row operations, okay. Towards that I will discuss a notion of elementary matrices and some of their properties you will see that in the next lecture this will culminate in a complete theoretical understanding of the solution of a system of linear equations homogeneous or otherwise but before that I will make a quick review of matrix multiplication I suppose all of you know what matrix multiplication is how it is to be done I will make a very quick review and then move on to the topic of today's discussion, okay discussing elementary these are called elementary matrices and where do they come in this discussion on elementary row operations, okay.

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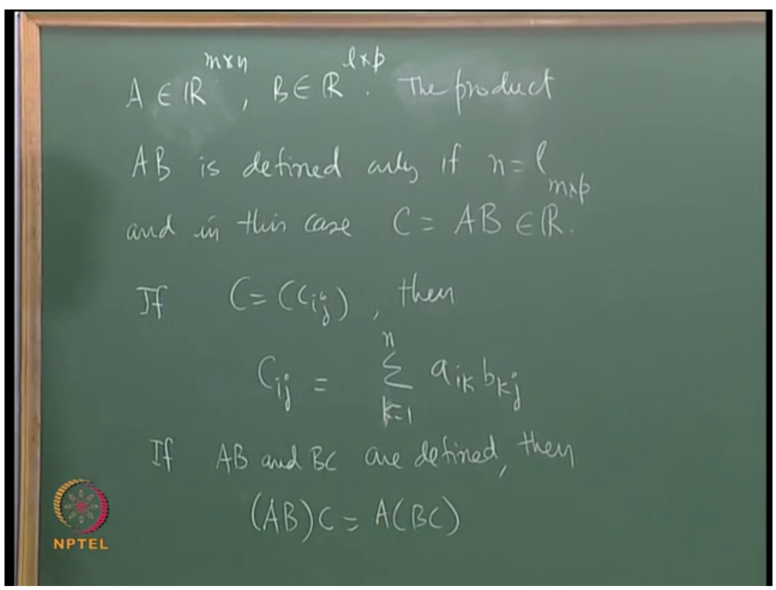

So matrix multiplication is about I would like to discuss first very quickly, let us say I have two real matrices A and B I have two matrices let me give these orders like this A is R m cross n B is R l cross p, okay these are given to us then the product is defined the product AB is defined only if n equal to l, okay and in this case if you call the product as C, C equal to AB this matrix is of order m cross p, okay this all of us are aware of I am sure these are rather peculiar way of multiplying two objects, okay what I would like to do is to write down the formula for the ijth entry of the product that is what we will need today, if C(ij) this ijth entry of C then what is the formula for Cij is there a volunteer formula for Cij, see I asking you not the determinant I am asking you the ijth entry of the product summation k equals 1 to what, okay this is the how the product is defined.

That is if you write down the matrices fully then you will know that the ijth entry of the product is the product dot product dot product of two vectors dot product of the ith row of A and then jth column of B dot product of the ith row of A and then the jth column of B that is Cij, okay. Now this is you can take this as a definition of the product then what can be verified I am going to leave this as an exercise for you what can be verified is that product is associative to whenever it is defined, okay that is if AB and BC are defined then AB into C is A into BC, okay product is associative to matrix product is associative.

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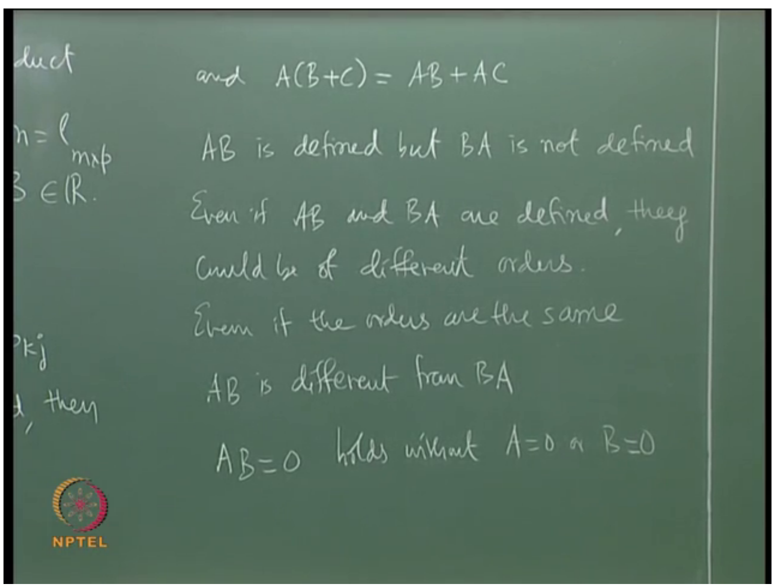

It is also distributive over addition that is if you have A into B plus C if this is defined then this will be equal to AB plus AC, okay product is distributive over addition the product is peculiar I mentioned let me give you one or two instances of its peculiarity it is quite possible that the product AB is defined but BA is not defined you can construct simple examples show that the product AB is defined but BA is not defined even if AB and BA are defined they could be of different orders, for example you take A 2 by 3, B 3 by 2 then AB is 2 by 2 BA is 3 by 3 they could be of different orders even if the orders are the same AB is in general different from BA even if the orders are the same, okay you can give an example for each of these illustrations.

Finally, it could also happen that AB is the 0 matrix it could happen that the product is 0 without either of the factors being 0, okay it could happen that the product of two matrices is 0 without any of the factors being equal to 0 you must studied in algebra 0 devisers, okay product there are 0 devisers if you look at the binary operation as the product of matrices then there are 0 devisers that is there are matrices A and B such that A not equal to 0 B not equal to 0 but the product AB is 0, okay.

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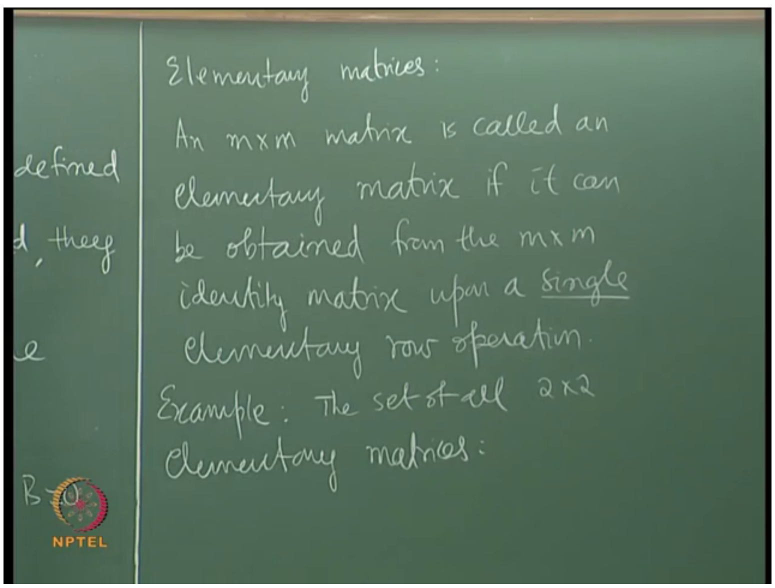

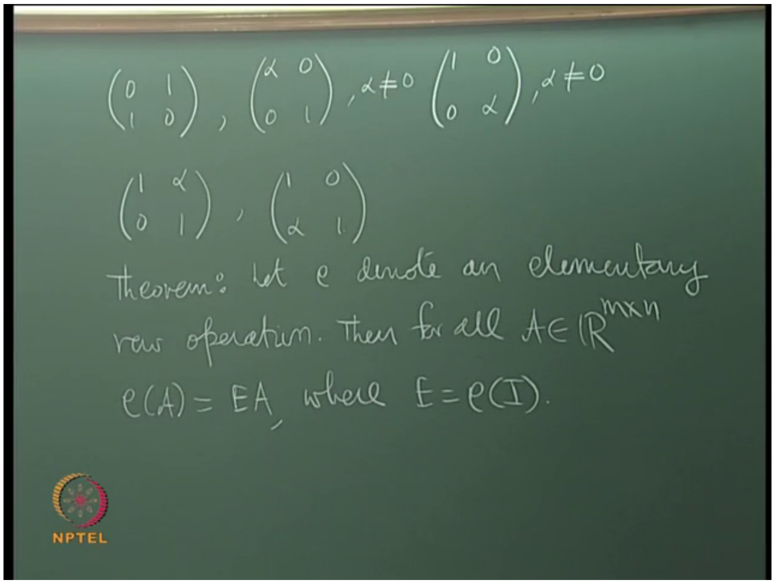

So I have listed some of the peculiarities of matrix multiplication I think this is enough for me I will move on to the topic of today's discussion that is the concept of elementary matrices using the concept of elementary matrices I am going to further formalize the elementary row operations and then bring about systematic scheme of analyzing systems of linear equations, okay what is an elementary matrix? It is a square matrix an m by m real matrix is called an elementary matrix if it can be obtained it is a square matrix if it can be obtained from the m by m identity matrix, okay I start with the m by m identity matrix and then do a single elementary row operation if it can be obtained from the m by m identity matrix upon a single elementary row operation, okay take the m by m identity matrix do a single elementary row operation on it the resulting matrix will be called an elementary matrix, okay.

Let me list all the 2 by 2 elementary matrices you will see that there are only five possibilities, so you remember this is upon a single elementary row operation not a sequence, okay example the set of all 2 by 2 elementary matrices I want to write down all these there are 5 in number, okay you will see that these are the only five 0110 interchange of the two rows of the identity matrix take the first row that is 1, 0 alpha 0, 0, 1 multiplying a row by a non-zero constant multiplying the first row by a non-zero constant, similarly multiplying the second row by a non-zero constant, okay three possibilities replace a row by constant times another row plus the row that I started with, yes I have replaced the first row by first row plus alpha times the second row a similar operation for the second row the first row remains as it is second row will be alpha 1, okay so these are all the elementary matrices of order 2 precisely five of them in number, okay.

Elementary matrices have (())(12:08) that they are invertible, okay so before I discuss this particular property of an elementary matrix let me go to the notion of inverse of a matrix, okay maybe I should do something else before that, what is the effect of elementary matrix has been obtained by a single elementary row operation on the identity matrix, okay how is this related to an elementary row operation being performed on a single matrix A let me first discuss that I want to discuss the following, this is the importance of introducing an elementary matrix take an elementary row operation perform these elementary row operation on the matrix A, okay I am given a matrix A and an elementary row operation I perform these elementary row operation on the matrix A.

The effect of this the resultant matrix which is row equivalent to A is the same as pre-multiplying the elementary matrix corresponding to this elementary operation by A, is that clear? E capital E equals small e of I means that I am doing this elementary row operation on the identity matrix of order m I get an elementary matrix I calling that as e pre-multiply this matrix E with A then I get another matrix, this matrix is the same as performing the particular elementary row operation e on the matrix A, okay.

So performing elementary row operations is effectively pre-multiplying the matrix A by elementary matrices, this is an important observation, okay. Let us first prove this and the look at the consequences again out of the three operations I will discuss only one operation the other two are simple so I will discuss the case when you replace a row by a row plus the constant times another row and show that this formula holds in that case the other two you could treat them as exercises so I need to prove this, okay is that clear what this theorem says, pre-multiplying for example this could be used in a program matrix multiplication is easily done by computers so one could think of sequence of elementary row operations as multiplying pre-multiplying the matrix A by a sequence of elementary matrices, okay.

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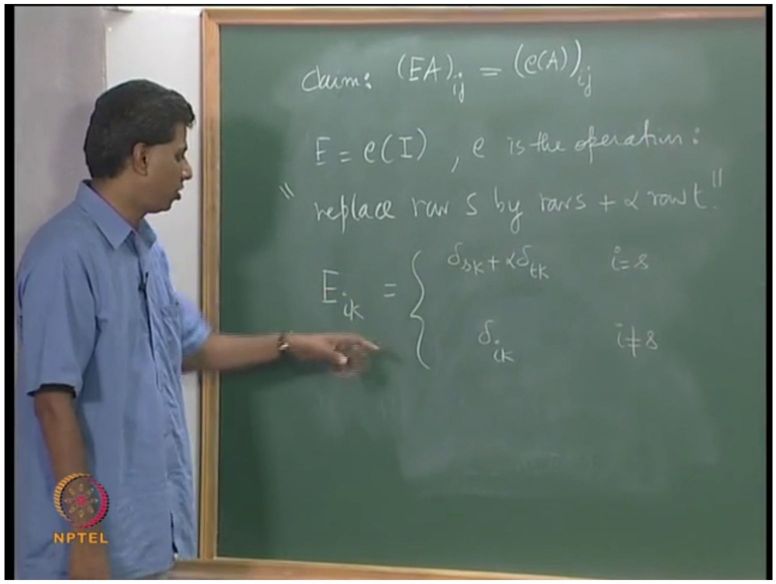

So let us see how the proof goes I will move in here for the proof, okay so I have two matrices one on the left one on the right I need to show that these two coincide I will show that the corresponding entries coincide two matrices are equal if there corresponding entries are the same I will show that so this is the claim then to show that EA ij ijth entry of EA is the same as small e of A ijth entry then it would follow the these two matrices are the same, okay.

For this I need the formula for the product and also what capital E is, what is capital E? Capital E is see for me E is the operation replace row S by row S plus alpha times row t this is my operation, okay E corresponds to this operation I will prove this result for this case the other two are similar simpler. So I want to write down the matrix E I will write Eik because that is what I will need here I want the formula for Eik just tell me if this formula is correct for i not equal to s see I am doing this particular elementary row operation on the identity matrix so take the identity matrix look at the sth row replace sth row by sth row plus alpha times tth row all other entries are left as they are.

So if i is not equal to s it is the same as the entries of the identity matrix, for the identity matrix there are special notation the chronicle delta notation delta ij, delta ij this is equal to 1 if i is equal to j 0 if i is not equal to j I am looking at the entries other than the sth row, so can you see that this is Eik so that is delta ik when i is not equal to s it is the same entry as the identity matrix that

I started with that entry is delta ik, if i is equal to s what is a formula sth row delta sk plus alpha times delta tk is that, no there is, yeah this is fine Eik is this, okay.

So I will use this definition of Eik and then use the definition of matrix multiplication and then see what this left hand side is it will turn out to be what you have on the right hand side.

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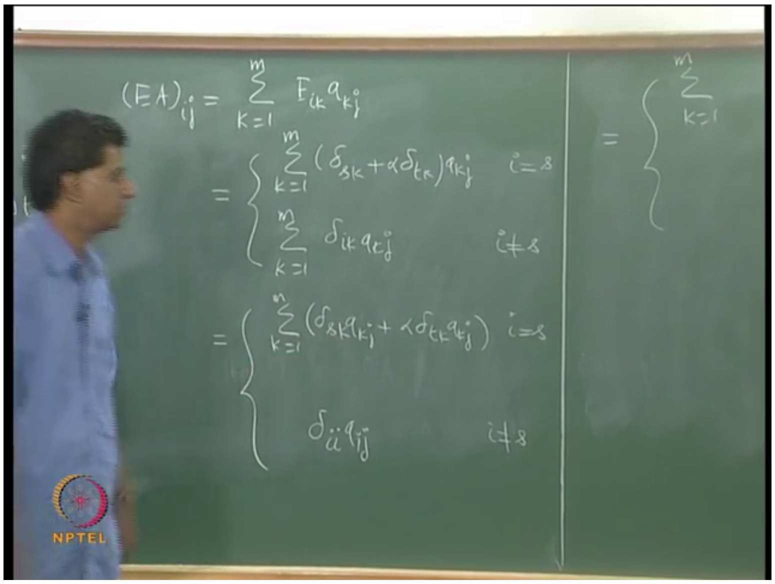

So let us do matrix multiplication to verify this equality, so I want the ijth entry of EA, okay that is I am writing down the formula summation k equals 1 to m see capital E has is of order m cross m remember this must be a square matrix and in order to pre-multiplying with A the order of E must be m by n A is m cross n k equal to 1 to m Eik Akj this is the formula for the product the ijth entry of the product.

Now since Eik has this split formula I will write like this summation k equals 1 to m for i not, sorry for i equals s summation k equal 1 to m I will have to substitute this formula delta sk plus alpha times delta tk into akj if i is equal to s it is summation k equals 1 to m delta ik akj if i is not equal to s I have just substitute the formula for capital Eik the ikth entry of capital E expand and simplify this is summation k equals 1 to m delta sk akj plus alpha delta tk akj if i is equal to s and look at this now this is delta ik akj k is the a running index i and j have been fixed ith row jth column entry is what I am trying to calculate i and j are fixed k is a running index k takes the value 1 to m i varies between 1 to between 1 and m so when k takes a value i this will be 1 all other entries are 0, okay. So remember that k is our running index i and j are fixed so when k equals i so this has only one term that corresponds to k equals i when k equals i it is delta ii aij i not equal to s, okay just complete the bracket here, okay.

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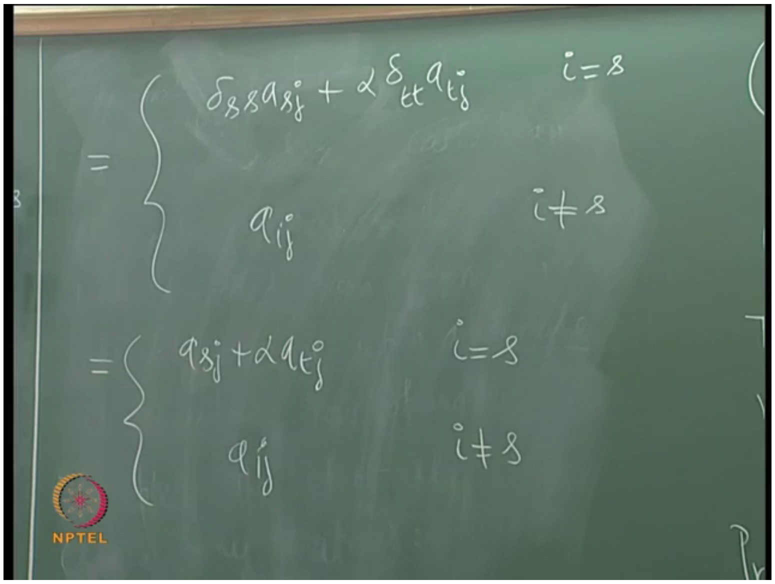

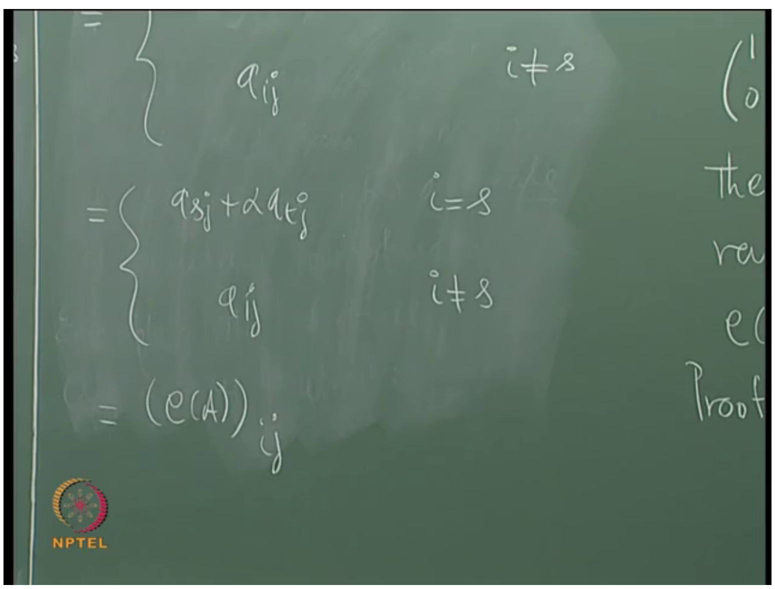

A similar thing we need to apply for the first part so let me give that here for the first part it is summation k equals 1 to m I will right away simplify delta sk akj k is a running index when k takes a values s it is 1 all the other terms are 0 corresponding to the first term. So when k takes the value s it is delta ss, so the first term here reduces to delta ss asj plus the second term when k takes a value t it is 1 all the other terms are 0 alpha times k takes a value t delta tt atj this is when i is equal to s and delta ii is 1 so this is aij if i is not equal to s, I hope this is clear.

This simplifies to delta ss as 1 asj plus delta ttt tt is 1 alpha atj this when I is equal to s it is aij when i is not equal to s but you see that this is precisely the definition of E of A where e is this operation the sth row is replaced by sth row plus alpha times tth row all the other entries are the same. So this is the ijth entry of e of A, okay so these two matrices are the same. So performing an elementary row operation on a matrix has the same effect as pre-multiplying the given matrix by a particular elementary matrix this elementary matrix is obtained from the identity matrix by applying this particular elementary row operation on the identity matrix single elementary row operation, okay.

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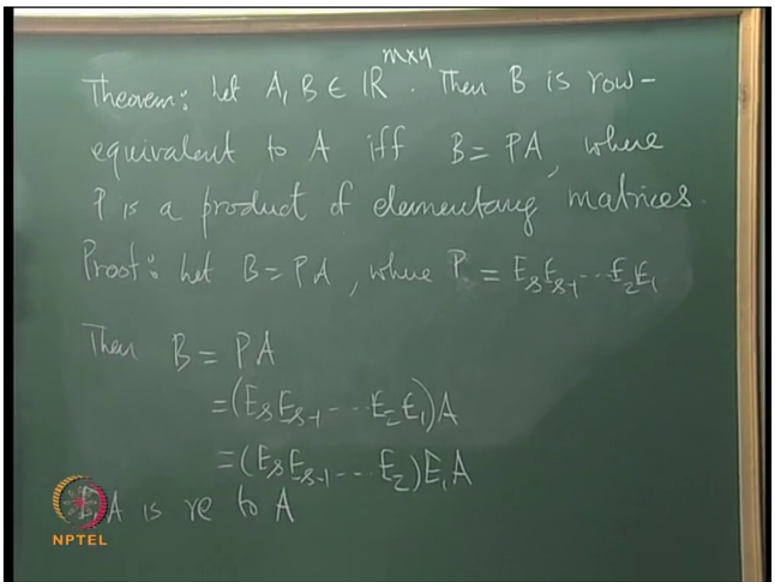

So what does this tell you about row equivalent matrices that is what we will see next what we will see is the following given two matrices A and B what we would like to show is that then B is row equivalent to A we will see that this is a consequence of the previous theorem, B is row equivalent to A if and only if B equals P times A where this time P is not a single matrix it is a sequence, okay so this P is a product by which I am in a finite product where P is a product of elementary matrices if I do a single elementary row operation and get B from A then this P will be an elementary matrix otherwise it is a product of a finite product of elementary matrices, okay.

Now you will see that this will be useful later also there are certain things that are preserved by elementary row operations in the case of a square matrix for instance we will look at certain numbers the determinant, the rank, the inverse etcetera one could calculate the inverse we will show by using elementary row operations we will show that certain numbers like the determinant the rank will remain the same if A and B are row equivalent, okay. In proving those results we will make use of this that B is a B can be written as P times A where P is a product of elementary matrices, okay.

Now there are two parts, okay for this result so let us take the first suppose B is P times A I must show that B is row equivalent to A, okay suppose B is P times A where P is what I am given is that P is a product of elementary matrices so let me write like this Es Es minus 1 etcetera E2E1 B is equal to P times A where P is a product of elementary matrices, we would like to show that B is now equivalent to A, okay. What do we have B is P times A use the formula for P Es Es minus 1 etcetera E2E1 A let me introduce a bracket here matrix multiplication is associative so I can write this as Es Es minus 1 etcetera E2 times E1 A matrix multiplication associative that is what I have used here. Now look at E1 A E1 A by virtue of the previous theorem is row equivalent to A because E1 is an elementary matrix and so it is an elementary row operation being performed on the a single elementary row operation being performed on the identity matrix by the previous theorem E1 A is row equivalent to A, okay.

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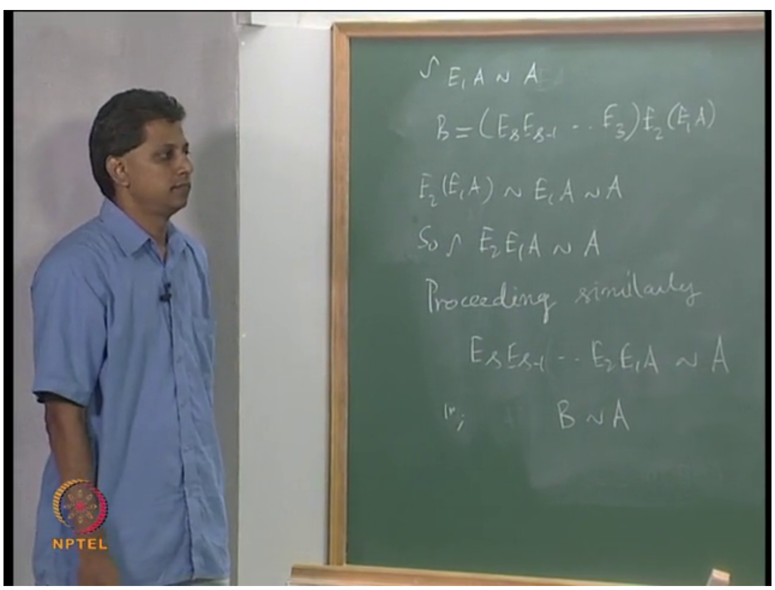

E1 is row equivalent to A then the next step you push E2 again consider B equals P times A Es Es minus 1 etcetera E3 E2 E1 A E2 E1 A is row equivalent to E1 A by the previous theorem row equivalence is a an equivalence relation in particular transitive it is row equivalent to A so I have E2 E1 A row equivalent to A, okay being row equivalent to A this is the first step, this is the second step proceed by induction proceeding similarly what follows is that Es Es minus 1 etcetera E2 E1 A is row equivalent to A taking one elementary matrix at a time but this left hand side Es Es minus 1 etcetera that is precisely B, so B is row equivalent to A, if B is P times A where P is a product of elementary matrices then we have shown that B is row equivalent to A, I hope it is clear you must prove the converse, is the first part clear?

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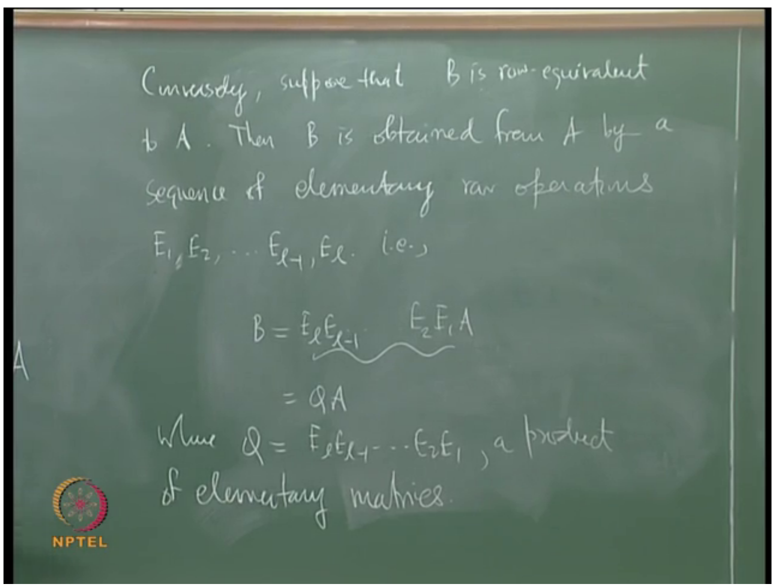

Conversely suppose that B is row equivalent to A, okay then B is obtained from A by a sequence of elementary row operations B is obtained from A by a sequence of elementary row operations I will call it E1, E2 etcetera El minus 1 El that is I am doing first E1 on A then E2 on E1 A etcetera that is that is the first operation E2 on E1 A etcetera El El minus 1 here again I am making fact that every elementary row operation corresponds to pre-multiplying the matrix A by an elementary matrix. An elementary row operation on A has the same effect as pre-multiplying the matrix A by an elementary matrix that is why I get this sequence E1 E2 etcetera El, okay so first operation is E1 I do E1 on A so that is pre-multiplying A by E1, second is E2 pre-multiplying E2 pre-multiplying E1 A by E2 etcetera.

So this sequence is El El minus 1 E2 E1 A I can write this as this whole thing I will call it as P or Q, so I have B equals QA and Q is the product of elementary matrices Q is El El minus 1 etcetera

E2 E1 which is a product of elementary matrices, okay that is the second part, okay so the important step to observe is that an elementary row operation on A is equivalent to pre-multiplying the matrix A by an elementary matrix and I am taking this sequence of elementary matrices, okay.

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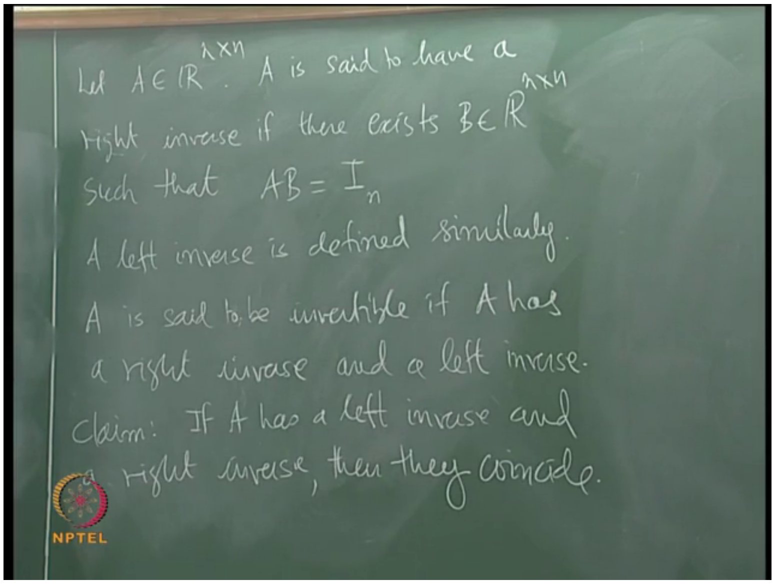

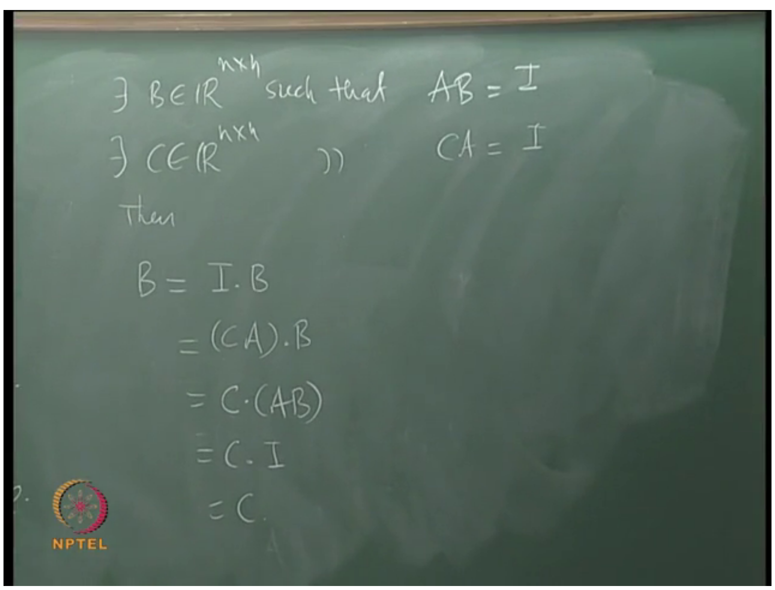

Let us now move on to the inevitability properties of elementary matrices I need the notion of the inverse of a matrix, okay let me give the definition some of you might be aware of it I have an n cross a square matrix with real entries then this matrix A is said to have a right inverse if there exists another matrix B of the same order as A such that right inverse so I have AB equal I identity is the identity matrix of order n I on the right is the identity matrix of order n, okay if there is a B that satisfies this equation then A is said to have a right inverse a left inverse is defined.

Similarly if there exists a matrix C such that CA equals I then A is said to have a left inverse, okay together if A is said to be invertible if A has a right inverse and a left inverse A is said to be invertible if A has a right inverse as well as a left inverse what we will first show is that if A has a right inverse and a left inverse then these two coincide, okay A has a left inverse and a right inverse then they must be the same, so let us prove this this is easy let us write down the definition there exists B such that AB equals I that is I am considering B since A has a right inverse there exist a B that must satisfy this equation there exist C such that CA equals I let us now start with B, B can be written as identity times B identity is the identity matrix is the identity operation of matrix multiplication for square matrices identity times B.

Now for this I I will borrow this equation this is CA times B this is matrix multiplication is associative C times AB but AB is identity from that equation so this is C times identity that is C, okay so I have made use of both the equations to conclude that if it has a right inverse and a left inverse then they must coincide.

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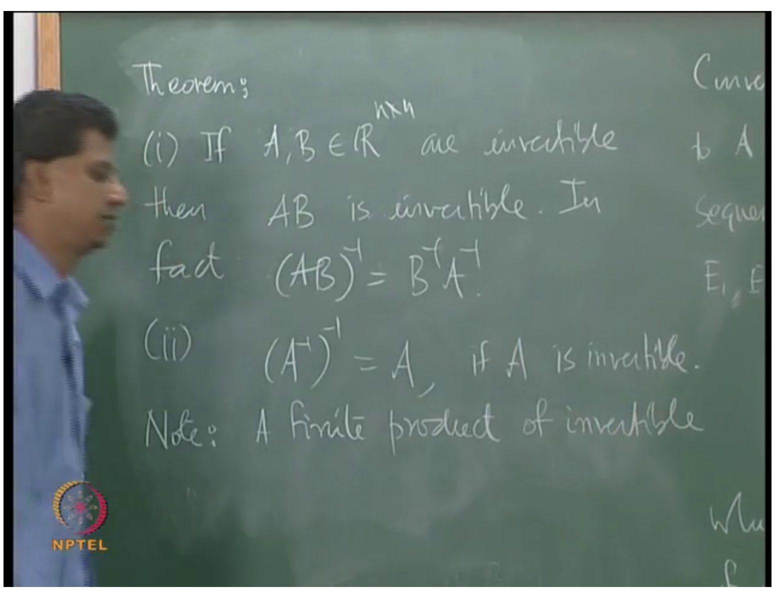

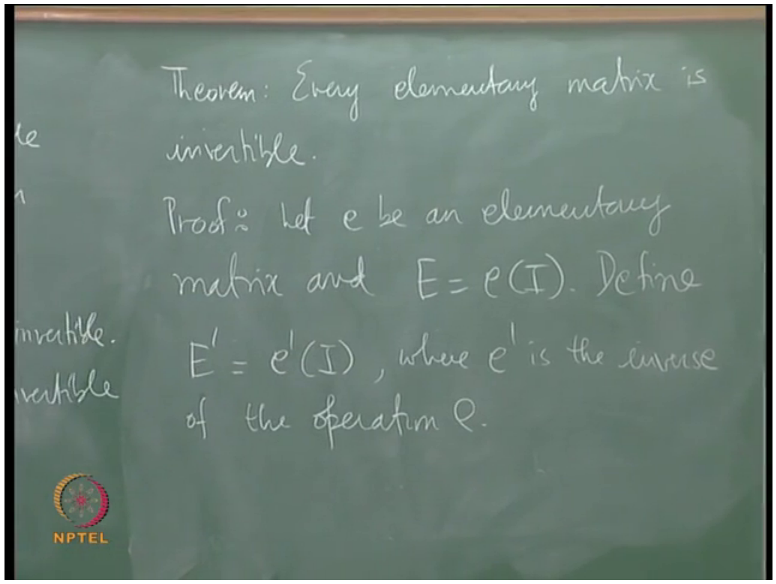

Let me also poof two more properties of invertible matrices if two matrices are invertible then there product is also invertible and a formula for the product can be given immediately the inverse of the product is the a product of the inverse is taken in the reverse order, okay I am going to leave this as an exercise it is easy to verify just look at the defining equations for the inverse of a matrix.

For a single matrix what must be true is that you do the inverse operation twice you get back the original matrix A inverse invers is A, okay if A is invertible, okay and the first property can be extended first property I can also say that a finite product of invertible matrices is invertible finite product of invertible matrices is invertible these are easy exercises, okay what is important more to what we are discussing is that every elementary matrix is invertible every elementary matrix is invertible, okay that is what we will prove do you have a choice for the inverse of an elementary matrix E I want you to make a guess of E prime E prime will be the inverse of E, yeah what is, okay we are using prime for that, okay so let us define capital E prime as E prime of I E prime is a notation that I used for the inverse of E so let me complete where E prime is the inverse of the operation E E prime is the inverse of E.

So when I am given E I know E prime explicitly in fact I have written down those three formulas for E prime, so I know E prime explicitly so the claim is that this E prime satisfies E into E prime equals I E prime into E equals I so it is both a left inverse and a right inverse and so capital E is invertible that is every elementary matrix is invertible. The proof uses composition of functions and the fact that E prime is the inverse of E.

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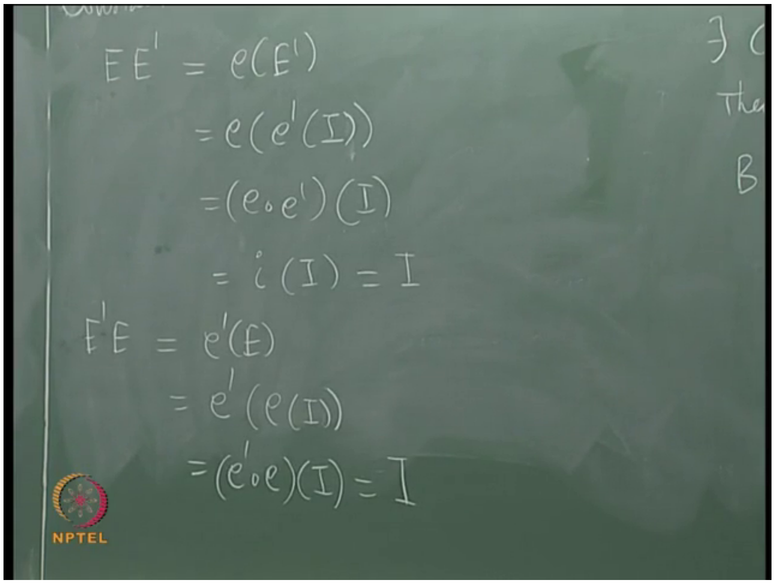

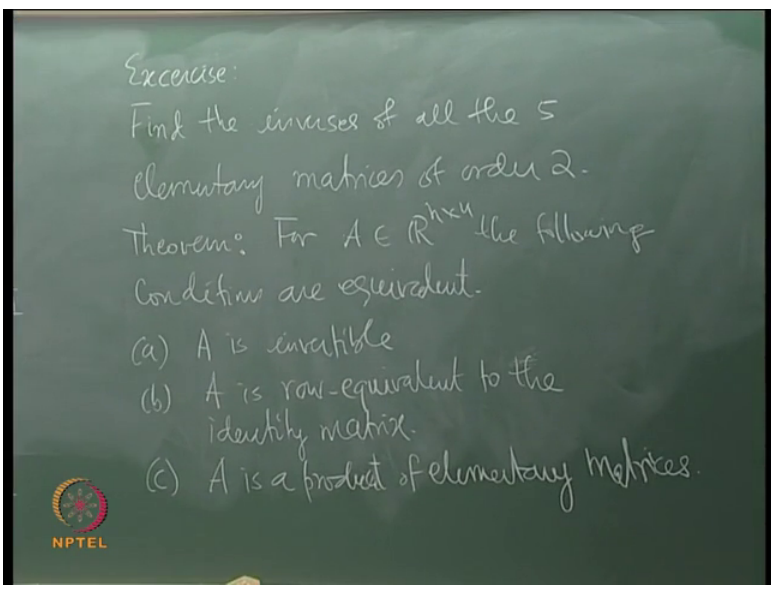

So proof proof is already there I have to just verify this, so consider E times E prime I want to show that this is equal to identity E into E prime is what this is pre-multiplying the matrix E prime by the matrix E, E is an elementary matrix so this corresponds to an elementary row operation so this is E of E prime, do you agree? Pre-multiplying the matrix E prime by E is performing the elementary row operation little e on capital E prime this is e of E prime definition is E prime of I small e prime of I, okay this is composition of functions, this is e circle e prime operating on I, okay but e prime is the inverse of E so E circle E prime is the identity function I will call that i i of I identity function on i leaves it as it is so I have e into E prime as the identity matrix identity operation on i I get back to i E prime E can be worked out similarly, okay let me do that also quickly consider E prime E this is pre-multiplying the matrix capital E by E prime that is performing the elementary row operation small e prime on the matrix E which is performing E prime on E the definition is e of I, okay this is again composition this is e prime circle e operating on identity as before this is the identity matrix, okay.

So what we have shown is that E prime is a right inverse as well as a left inverse so capital E is invertible that is each elementary matrix is invertible, a little exercise for you at this stage find the inverses of all the 5 elementary matrices of order 2 I had written down the elementary matrices of order 2 all the 5 elementary matrices I have written down find the inverses of all these operations, you need to only find the inverses of the elementary row operations, okay what we would like to do next is in the next lecture is to discuss properties of elementary matrices, inevitability, connection to systems of equations, okay I will just give the result that I would like to start with from the next on the next class.

Next class I want to prove this theorem, for a square matrix A for A in R n cross n the following conditions are equivalent, okay this gives the connection between row equivalence, elementary matrices, invertible matrices condition A A is invertible, condition B A is row equivalent to identity that is condition 2 A is row equivalent to the identity matrix of order n, condition C A is a product of elementary matrices, okay that is A is invertible if and only if A is row equivalent to I if and only if A is a product of elementary matrices, okay.

In proving this I would make use of a little result which I will ask you now, the question I will ask you you try to come up with a an answer suppose I have a row reduced echelon matrix a square row reduced echelon matrix which is invertible I have a square row reduced echelon matrix which is invertible, what can you say about that matrix, capital R is a row reduced echelon matrix its square its invertible what is R, R is equal to identity try to prove this we will make use of this result next time, I will stop here.