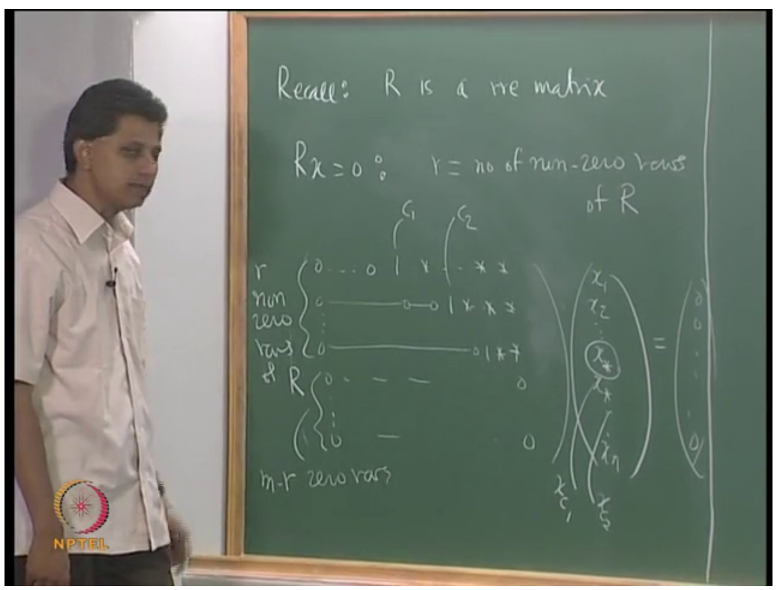
Linear Algebra
By Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 5
Row-reduced Echelon Matrices and Non-homogeneous Equations

See a little preamble before I move on, this course is about linear transformations on vector spaces, okay. Now concrete realizations of linear transformations on vector spaces are matrices that explains why this indulgence on matrices, okay for so many lectures we will have another two lectures probably including today's lecture on matrices, elementary row operations, properties etcetera, okay today I would like to look at certain qualitative aspects of the elementary row operations row equivalence row reduced echelon form etcetera, okay but before that I will recollect what we discussed last time, okay.

(Refer Slide Time: 1:02)

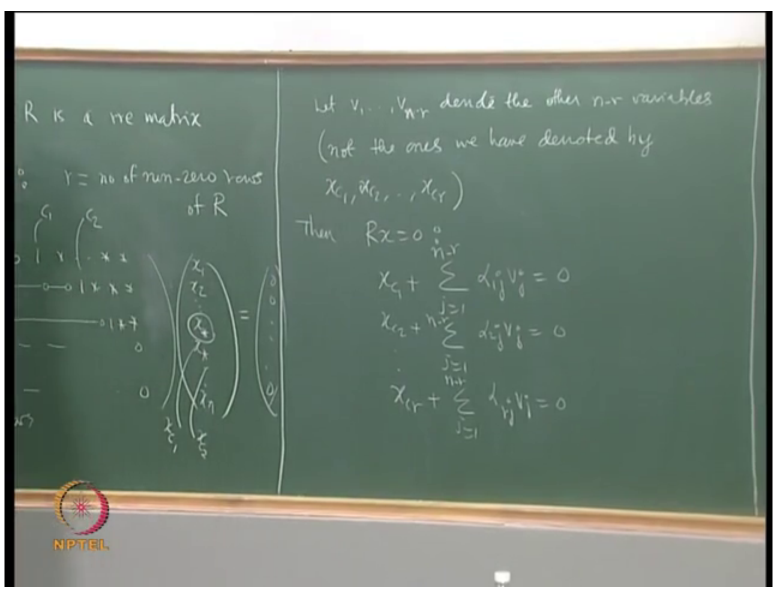


Towards end of last lecture I remember having discussed this problem R is row reduced echelon form, okay so this is to make a quick reminder R is a row reduced echelon matrix I am writing down the expanded of Rx equal to 0 I was writing down the expanded form of Rx equal to 0 what I know is that small r is the number of non-zero rows of capital R, okay this is my notation. So let us look at this equation Rx equal to 0 I will have something like this R is a row reduced echelon form so there are certain 0's I have the first non-zero entry that is 1 and then certain other entries then the second row will be 0 etcetera let us say up to here and then a 1 here etcetera I have 0's here 1 somewhere here, okay so this is the first r rows let me write this is first r rows let me write on the left r non-zero rows of capital R that corresponds to this the rest of the rows are 0, okay this is my r m minus R totally there are m equations (())(2:45) m minus r 0 rows that is a last part, okay.

Now this R into x is equal to 0, so I will do matrix multiplication that into x, x1, x2, etcetera I will call this some x star this is another x start etcetera this is xn this is the 0 vector, okay I am already assuming you know matrix multiplication but in any case we will review it a little later. So what is matrix multiplication tell us? The right hand side vector 0 the first 0 the right hand side number this 0 is the dot product of this row with this column, okay. Now this 1 appears in the column number C1 this corresponds to certain x I am calling that xc1 this is column C1 this 1 appears in column C2 this corresponds to certain other unknown I am calling that xc2 etcetera.

So I have what I have done is to effect a relabeling of the unknowns as those corresponding to the leading non-zero entry of the non-zero rows of R those are xc1, xc2, etcetera xcr the other unknowns I am calling then v1, v2, etcetera v n minus r I am just relabeling them for the purpose that I will write down the r equations coming out of this homogeneous system and then it gives us a qualitative information.

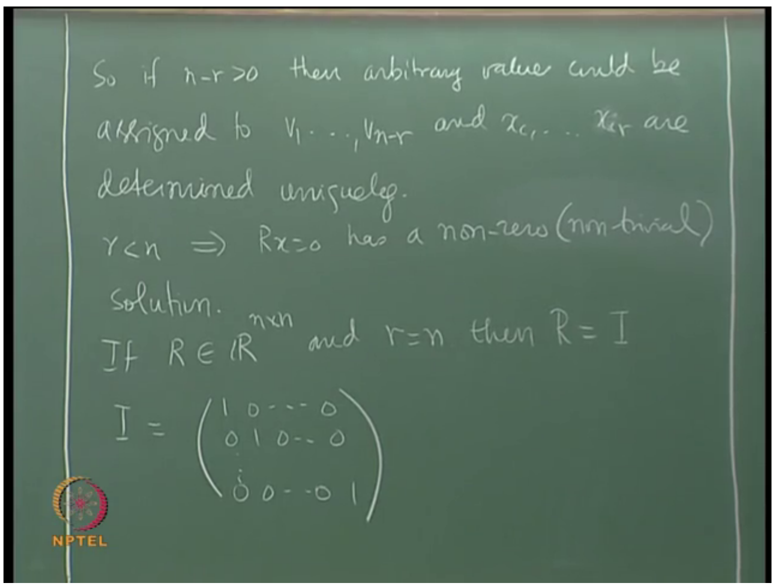
(Refer Slide Time: 4:26)



So based on this what I had written is that let v1, v2, etcetera v n minus r denote the other m minus r, I am sorry n minus r denote the other n minus r variables, okay means not the 1's we have denoted by xc1, xc2, etcetera xcr just to make it a little more precise.

Then these R equations can be written using a summation formula Rx equal to 0 is the same as xc1 plus summation j equals 1 to n minus r alpha 1j vj equals 0 that is I am now denoting the first row entries after this 1 as alpha 1 C1 plus 1 alpha 1 C1 plus 2 etcetera this is alpha 1 C this is alpha 1 n etcetera that is my notation this is the equation then, second equation is written similarly alpha 2j vj etcetera xcr the last non-zero row gives me this equation xcr plus summation j equal to 1 to n minus r alpha rj vj equals 0 the last non-zero row gives me this equation.

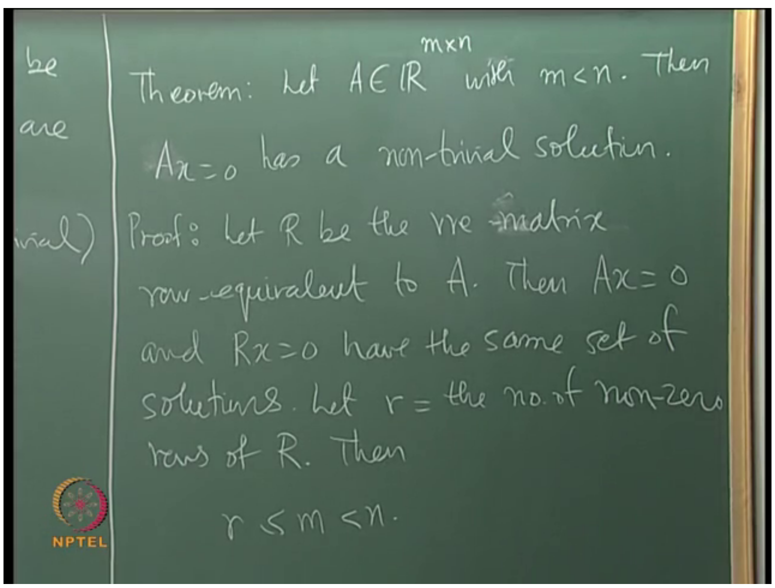
(Refer Slide Time: 6:18)

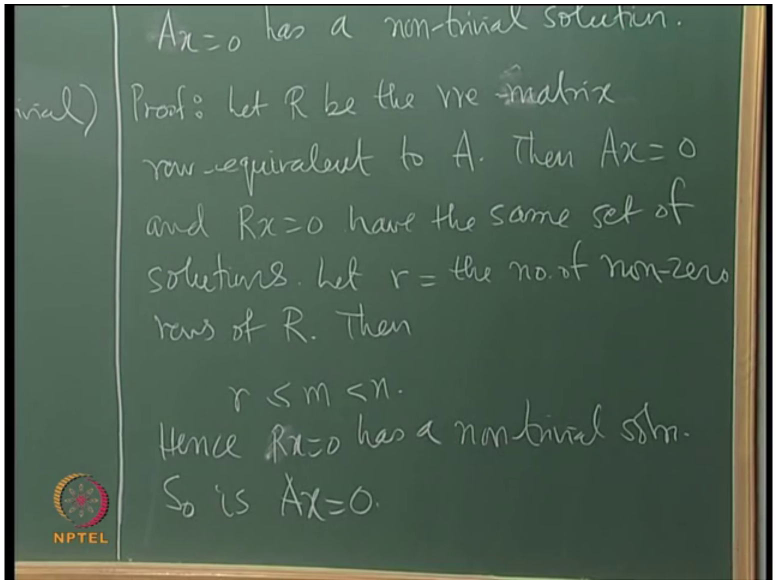


So what this means is that so if n minus r is positive then arbitrary values could be assigned to v1 etcetera v n minus r and xc1 etcetera xcr are determined uniquely for every choice of v1, v2, etcetera v n minus r I will have unique xc1, xc2, etcetera xcr coming from this set of r equations for every such choice of v1, etcetera v n minus r I get xc1, xc2, etcetera xcr that gives me the set of all solutions, okay and we had also observed towards the end of last class that if r is strictly less than n like what I have assumed here, if r is strictly less than n then Rx equal to 0 has a non-trivial solution a non-zero we call it a non-trivial solution, a non-zero solution means at least one coordinate at least one unknown is not 0 at least one unknown is not a zero, okay.

Remember this is important because this is a homogeneous system we know it is consistent it always has 0 as a solution so one is interested in a non-zero solution. So there is a guarantee if r is less than n then the homogeneous system has a non-trivial solution I also left you with a little problem if r is a square matrix and the small r is equal to n I asked you the structure of r, so what is the structure of r? Is that clear to all of you, so r is equal to identity of order n identity matrix of order n so just for the sake of completeness I write the identity matrix, okay it has only that the non-zero entries the of diagonal entries are 0 the principle diagonal entries are all 1 this is the identity matrix of order n that is what happens if r is so the only row reduced echelon matrix whose number of non-zero rows equal to the order of the matrix is the identity matrix, okay we will need this a little later, okay.

(Refer Slide Time: 9:28)





Let us now proceed with today's lecture a little qualitative information, so in particular I want to prove two or three results let me the start with the first one, I consider a rectangular homogeneous system where the number of equations is strictly less than the number of unknowns a homogeneous system is what I am going to consider a number of equations strictly less than the number of unknowns, okay this is a fundamental result of linear algebra that a rectangular system of homogeneous linear equations where the number of equations is strictly less than the number of unknowns always has a non-trivial solution, okay.

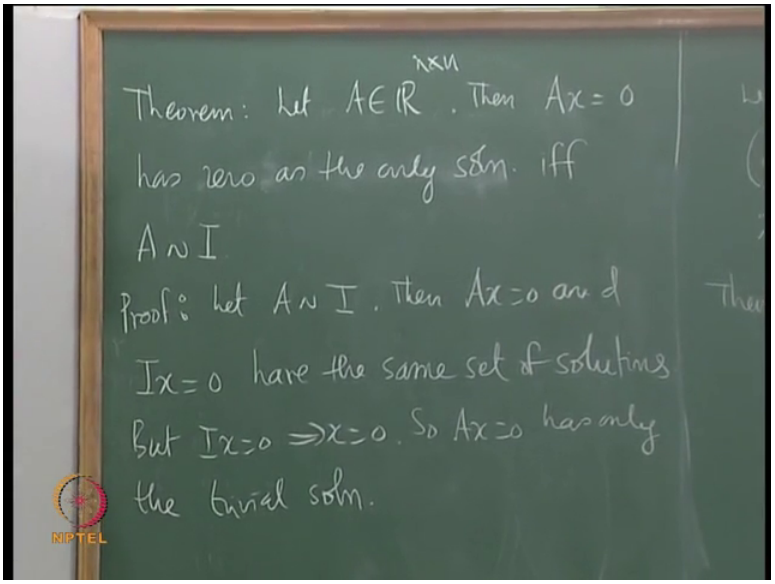
The proof will essentially make use of what we have proved what we have stated just now let R be the row reduced echelon form echelon matrix row equivalent to the matrix A, okay standard notation capital R is a row reduced echelon form of A that is the row reduced echelon matrix row equivalent to A then the systems Ax equal to 0 and Rx equal to 0 we call them as equivalent systems that is they have the same set of solutions Ax equal to 0 and Rx equal to 0 have the same set of solutions.

As before let us keep r as a variable that is the number of non-zero rows of r let r equal the number of non-zero rows of capital R, the number of non-zero rows of R cannot exceed the number of rows of R, so r is less than or equal to m, m is the number of rows and what is given as that m is less than n as part of the data of the problem m is less than n, okay. So r is strictly less than n by the observation that we made just now it follows that there is a non-trivial solution, okay. Hence Rx equal to 0 has a non-trivial solution so is x so is Ax equal to 0 so is the system

Ax equal to 0 because in set of solutions of Rx equal to 0 coincide the set of solutions of Ax equal to 0, okay.

So this is based on the observation that if r is strictly less than n then Rx equal to 0 has a non-trivial solution at least one non-trivial solution, okay.

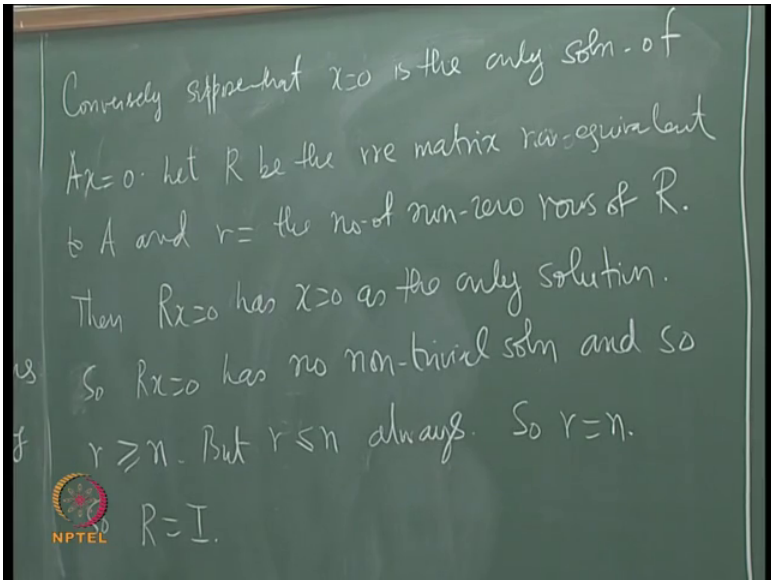
(Refer Slide Time: 13:24)



For this square case we can do a little better square homogeneous system when does it have a non-zero non-trivial solution that will be my next result I have a square matrix with real entries when the system Ax equal to 0 has a non-trivial solution if and only if A is row equivalent to the identity matrix of order n has 0 as the only solution has 0 as the only solution if and only of A is row equivalent to I, okay this denotes A is row equivalent to I identity is the row reduced we know it is a row reduced echelon equivalent matrix, okay.

So now there are two parts unlike the first the previous theorem there are two parts if and only if let us proof the easy one first, suppose that A is row equivalent to I then the system Ax equal to 0 and Ix equal to 0 by definition have the same set of solutions Ax equal to 0 and Ix equal to 0 have the same set of solution that is I am applying this fact that we had observed previously that if r is row reduced echelon form of A then Ax equal to 0 and Rx equal to 0 have the same set of solution I is the row reduced echelon form of A so these two systems have the same set of solution but Ix equal to 0 has only x as a solution this implies x equal to 0, okay that is Ax equal to 0 has only the trivial solution Ix equal to 0 implies x equal to 0 so this is the easy part of the proof the homogeneous system Ax equal to 0 has only 0 as a solution if A is row equivalent to the identity matrix we need to prove the converse.

(Refer Slide Time: 16:22)

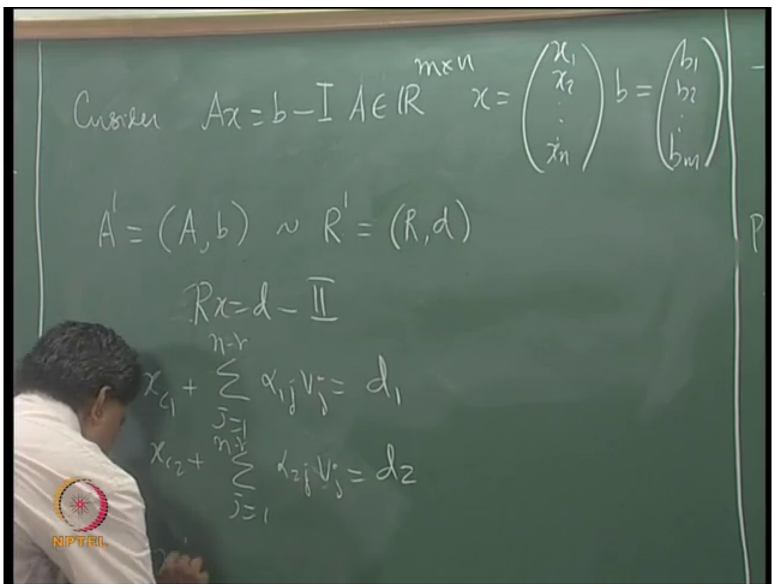


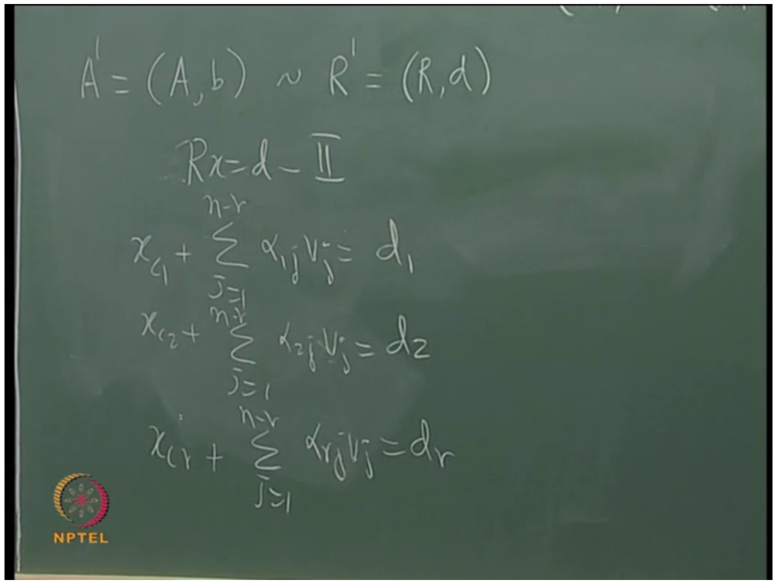
We next prove the converse so again we need to look at the row reduced echelon form of A and then show that this row reduced echelon form is equal to identity, so conversely suppose that x equal to 0 is the only solution of the system Ax equal to 0 we will look at Rx equal to 0 let R be the row reduced echelon matrix row equivalent to A and small r is equal to the number of non-zero rows of capital R, okay then what we have is that then Rx equal to 0 has x equal to 0 as the only solution because Ax equal to 0 has x equal to 0 as the only solution which means the system Rx equal to 0 does not have a non-trivial solution appeal to the previous result we are appealing to the previous result what follows is that R is greater than or equal to n, okay.

So let me write down this state so Rx equal to 0 has no non-trivial solution, so r is at least n, okay but r is a number of non-zero rows of capital r it cannot exceed n number of rows number of non-zero rows of R cannot exceed the number of rows of R and so r is equal to n that is I have row reduced echelon matrix which a square row reduced echelon matrix with the property that the number of non-zero rows is equal to the order of the matrix, so R is equal to I as we have observed before, okay. So what we have shown is that if Ax equal to 0 has x equal to 0 as the only solution that is if this homogeneous system of a homogeneous square system does not have a non-trivial solution then the matrix A is row equivalent to the identity matrix, okay.

So much for the homogeneous case let us now move on to the non-homogeneous case that is when the right hand side is the not the 0 vector, so let us consider the non-homogeneous case and remember that the essential difference between a homogeneous system and a non-homogeneous system is that a non-homogeneous system need not have a solution, okay. The homogeneous system always has a solution so this information also should be obtainable when we do the elementary row operations ideally when you do the elementary row operations on a system Ax equal to b we must also know whether at the end of the algorithm for instance whether it can tell us if the system has a solution, okay. Now that information if you have a row reduced echelon matrix is immediate so that is what I want to illustrate next.

(Refer Slide Time: 20:18)





So I am considering the non-homogeneous system Ax equal to b, okay so let us recall that A is an m cross n matrix m rows and n columns x is the unknown vector x1, x2, etcetera xn b is the requirement vector b1, b2, etcetera bm I am writing them as column vectors. What I will do now is I not only have the matrix A I also have the right hand side vector, so I will form a new matrix call it A prime I will adjoin b as a last column of A prime.

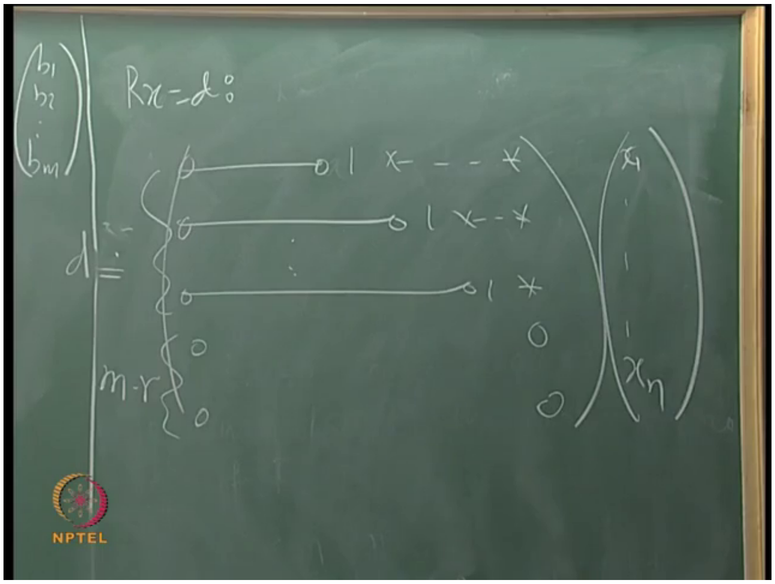
So my new matrix A prime is A, b I will apply elementary row operations on this then I will get R prime row reduced echelon matrix, now this R prime is R comma d, okay what I do is I apply elementary row operations on A I get R and the apply the same set of elementary row operations in the same sequence to the matrix that is a column vector b and then I get the vector d, okay that matrix I am calling that as R prime, okay.

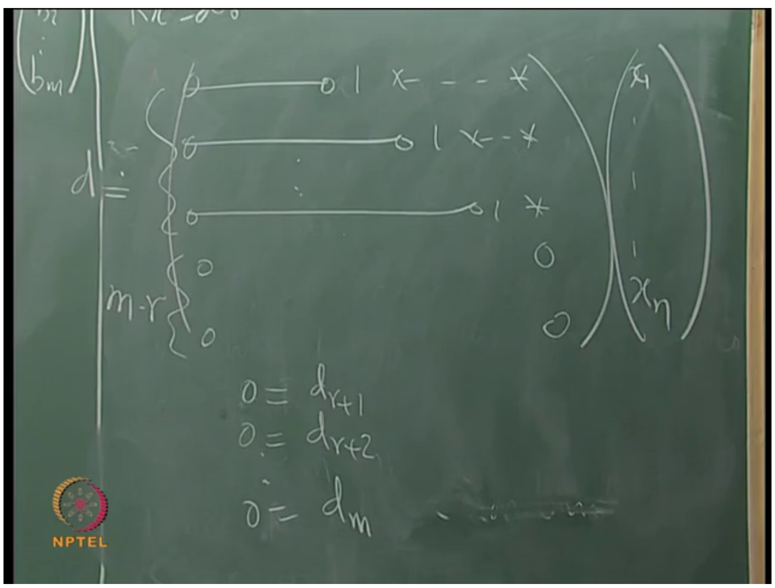
Now what is not clear but can be shown is that if you take this matrix as it is and then do the elementary row operations on this treating it as a single matrix then you will eventually end up with R comma d, okay that is there are two ways of proceeding one way is to apply elementary row operations on A to get the row reduced echelon matrix R apply the same elementary row operations on b you will get the matrix d you will get the vector d that is one way of proceeding, the other way is to take this as a matrix as a single matrix apply elementary row operations write down you will get it of this form, the first part will be row equivalent to the first part, second part will be row equivalent to the second part, okay.

So this is what we have done, from what we have discussed earlier you have row equivalence which means if you look at the system Ax equal to b I am calling this usually as I I will also look at Rx equal to d I am calling that system 2, if R comma d is obtained from A comma b by these elementary row operations then what know is that any solution of Ax equal to b is a solution of Rx equal to d and conversely any solution of Rx equal to d is a solution of Ax equal to b because elementary row operations correspond to linear combinations equations of that system any equation of system 2 is a linear combination of the equations of system 1 and conversely, okay and so we know that this if this happens then these two systems have the same solution which we observed in the homogeneous case that holds in the non-homogeneous case also.

In the non-homogeneous case what do these equations represents, so I will look at the expanded version of this something like Rx equal to 0 that I had written down in the beginning let me writing down Rx equal to d as before small r is a number of non-zero rows of R cx1, xc2, etcetera they correspond to the first leading non-zero entries of the R non-zero rows v1, v2, etcetera v n minus r they correspond to the other variables then I get the following xc1 plus these are very similar to the previous the homogeneous case alpha 1j vj this is now d1, okay it is 0 earlier xc2 plus summation j equals 1 to n minus r alpha 2j vj equals d2 etcetera xcr plus summation j equal to 1d to n minus r alpha rj vj equals dr there is nothing much in these equations what happens to the other m minus r equations, what are the other m minus r equations? These m minus r equations in the homogeneous case do not matter they do not contribute anything they do not give any they do not lead to any constraints they do not give any information but the last m minus r in the non-homogeneous case in fact have the essence of the fact that the system has a solution or does not have a solution.

(Refer Slide Time: 25:26)

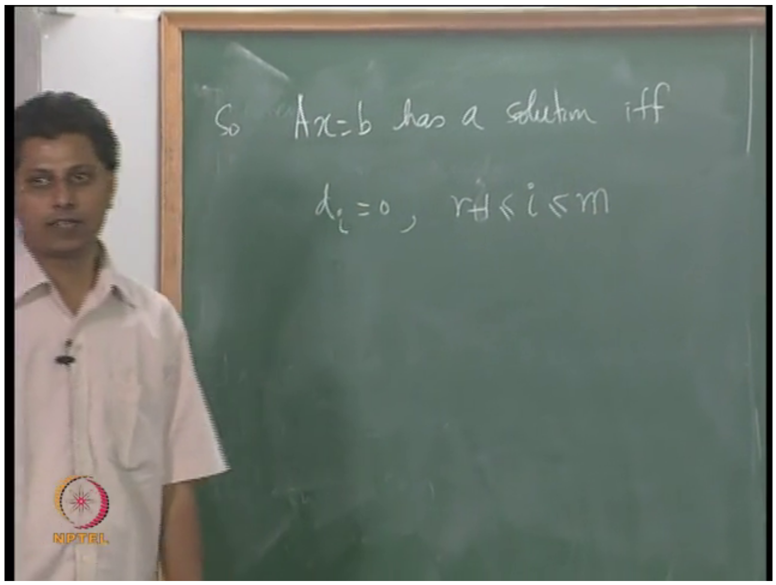




So what are the last m minus r equations in the non-homogeneous case remember on the left I have r say I am writing down Rx equal to d, okay let me just do it again so that it is transparent r is there are certain zeros 1 here certain entries here some zeros 1 here certain other entries here etcetera 0 here 1 etcetera these correspond to the r non-zero rows the other entries are all 0 this is m minus r this is r this into x that is my d my d is equal to this I have written down the first r non-zero equations what happens to the last equations, the last equations give me 0 times x1, etcetera xn so it is just 0 on the right hand side I have dr plus 1 next equation 0 equals d r plus 2 etcetera dm this is what I must have.

Now these I told you do not appear in the homogeneous case because d is 0 in the homogeneous case, d is 0 in the homogeneous case because b is 0 you have a 0 column you have a 0 matrix you do elementary row operations it will remain 0 there will be no change. So b is 0 in the homogeneous case so is 0 so these equation do not figure in the homogeneous case these things have to be satisfied in order that the system has a solution in the non-homogeneous case, okay.

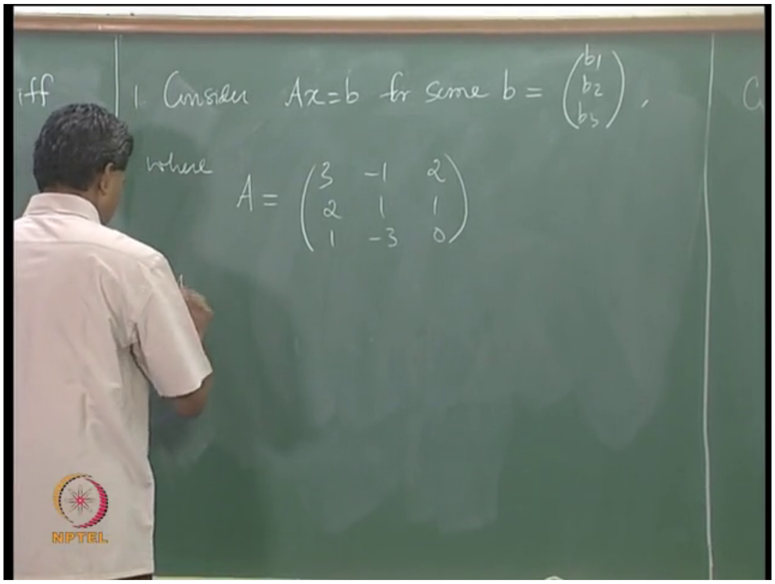
(Refer Slide Time: 27:05)

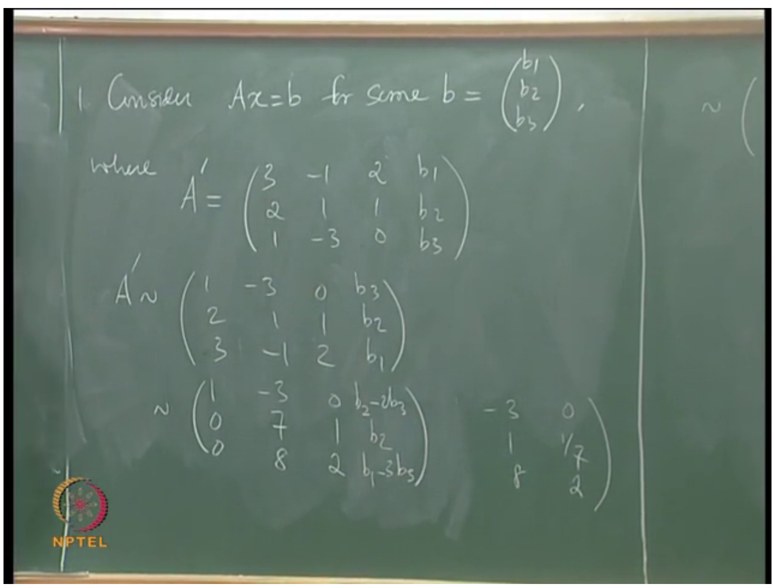


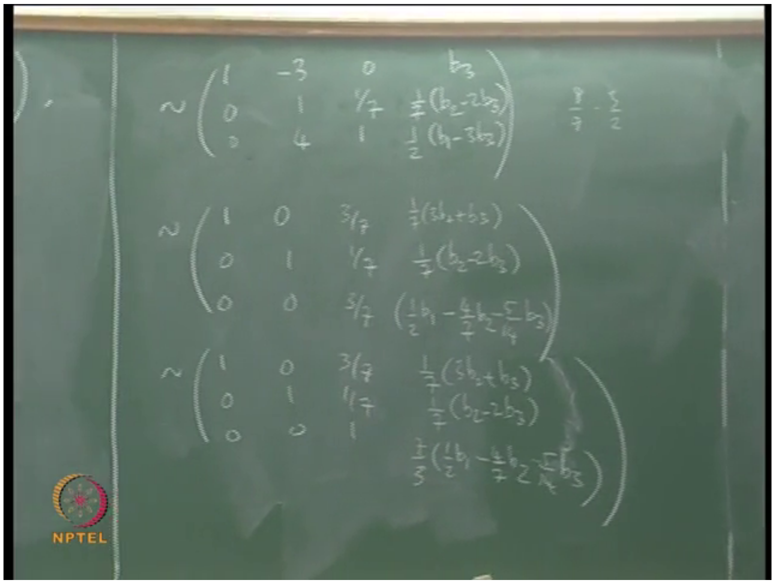
Now that is the condition, what is a condition that Ax equal to b has solution this is the condition let me write down Ax equal to b has a solution if and only if di is 0 for all i greater than r this is m for all i greater than r these numbers must be 0 dr plus 1 dr plus 2 these are numbers constituting the vector d these must be 0 this is the condition that must be satisfied for the system to have a solution.

Again, if these conditions are satisfied then one will fix arbitrary values to v1, v2, etcetera vn minus r fix determine the other variables xc1, etcetera xcr and one can write down the set of all solutions, okay. So these are the conditions that are necessary and sufficient for the system Ax equal to b to have a solution, okay. So let us consolidate by looking out working out one or two examples numerical examples, you have any questions? What illustrate two examples one where the system has a solution one where the system does not have a solution, if you do not have any questions let me proceed.

(Refer Slide Time: 28:42)







So I want you to consider Ax equal to b for some vector b where my matrix A is this 3, minus 1, 2, 2, 1, 1, 1, minus 3, 0, okay this is my matrix I would like to verify by using elementary row operations to verify if this system is consistent, if it is consistent can I write down all the solutions, okay let us do the elementary row operations we are so familiar now I do not want to write down what operations I am performing.

So I know that A is equivalent to let us say I interchange these two, so 1 minus 3 0 2, 1, 1, 3, minus 1, 2 this is equivalent to 1 minus 3, 0 minus 2 times this this is 0 minus 3 6 plus 1 7 this will remain as it is so please check my calculations, minus 2 times this plus this minus to that 6 plus 1 7 this will remain as it is minus 3 times this is 0 minus 3 9 9 minus 1 8 minus this remains the same, the next operation is to divide by 7, okay.

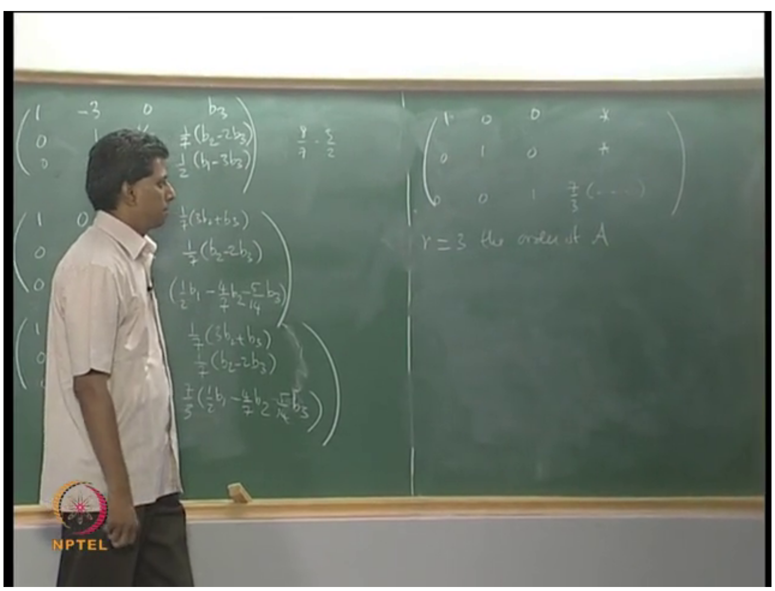
So maybe I will do it here itself 1 minus 3 0, 0 1 1 by 7, 0 8 2 that is my next operation dividing the second row by 7 then I know this is equivalent to I will keep the second row this entry remains the same 3 times this is 0 3 by 7 I should have done it along with b no problem I just append it, okay this is my A, b A prime this will be interchange interchange if these two b3, b2, b1 this time I am keeping this as the pivot row minus 2 times this minus 2 times this plus this b2 minus 2 b 3 minus 3 times this plus this, okay that is equivalent to let me do that here this is fine.

Then I have divide by 7 I can do one more operation so that I will divide by 8 or 2 is better 0 4 1 I am dividing by 2 this becomes 4 this is 1 1 by 2 times that then yeah I need to use this as the pivot row so that is equivalent to I will do these two simultaneously the second row I keep it as it is the first row is three times this, this will remain 1 this will remain 0 this is 3 equivalent times this 3 by 7 3 times this plus this tell me this is fine, 1 by 7 3 b 2 plus b3 minus 6 by 7 plus 7 by 7 that is one by 7 3 times the second row, please the check the calculations here and this one will be minus 4 times this plus this minus 4 1 minus 4 by 7 is 3 by 7 minus 4 times this plus this minus 4 times this plus this I want a volunteer for that entry minus 4 times this plus this 8 by 7 minus 3 by 2 16 minus 21 minus 5 by 14, do agree with this?

Minus 4 times this plus this, so 1 by 2 b1 remains the same b2 will be minus 4 by 7 b2 b3 only will change minus 4 by 7 that makes it 8 by 7 8 by 7 minus 3 by 2 essentially that is 16 minus 21 minus 5 by 14 b3, agreed? Okay, one last step then I must make this 1 and then reduce this so this is equivalent to I divide this by 1 to get 1 0 3 by 7, 1 by 7 3b2 plus b3, 0 1 1 by 7 1 by 7 b2 minus 2 b3 divide by 7 by 3 I divide by 3 by 7, 0 0 1 then this last entry is 7 by 3 1 by 2 b1 minus 43 by 7 b2 minus 5 by 14 b3, actually I can solve the system as it is from here but I want this row reduced echelon form so I must do two more operations which will make sure that these two entries are 0, okay.

So let me complete that let me do these two operations also write down the solution, is it clear even at this stage that the system has a solution for whatever be, okay.

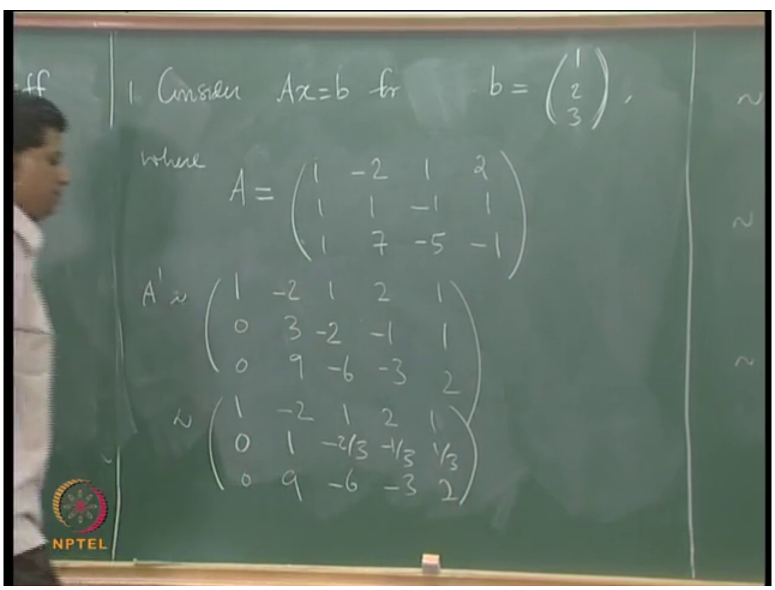
(Refer Slide Time: 37:13)

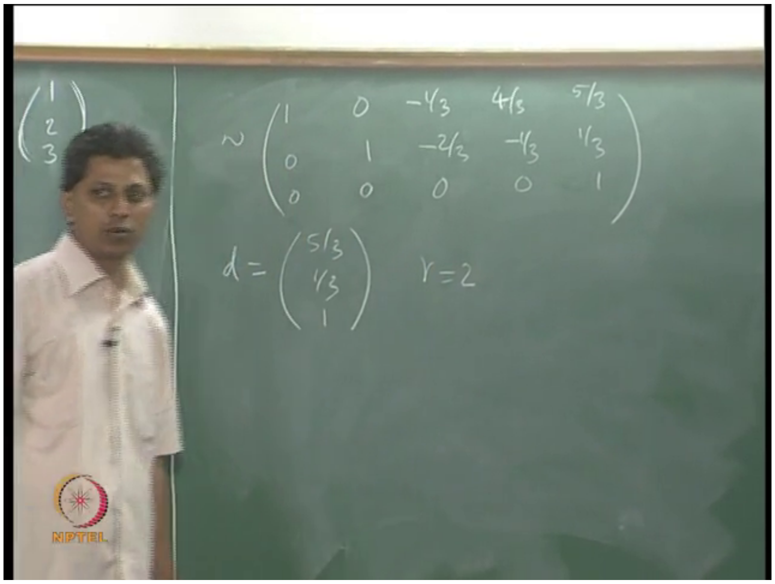


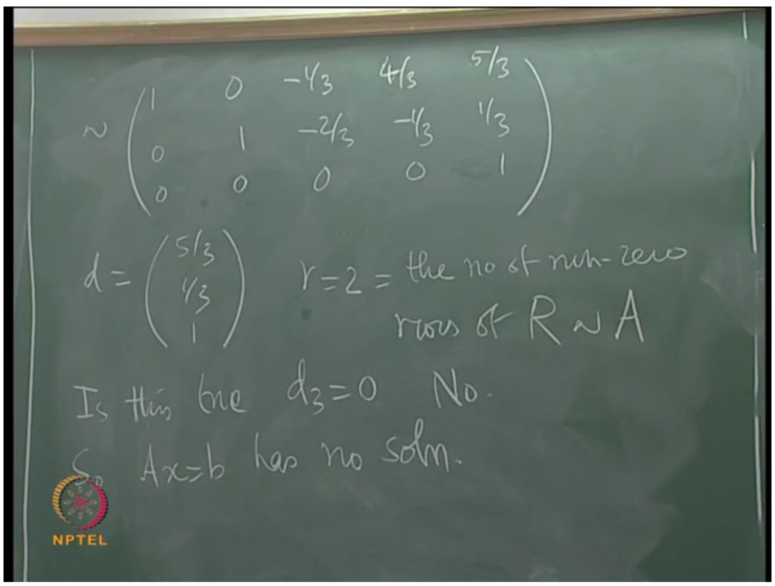
Let me proceed the last row is kept as it is 0 0 1 7 by 3 etcetera please I am leaving this I want to make these two 0 so I am going to leave the other calculations also determine those two stars lots of fractions involved so I am going to leave that, okay but what is clear is that this system has a solution for all b because the number of see what I have done in the last step is keep the last row as it is second row I am using this I am multiplying this by minus 1 by 7 I am adding it this makes this entry 0 multiply by minus 3 by 7 to add to this this entry will be 0 there will be appropriate changes in the first two rows I am leaving them for you to calculate, but what is clear from this example is that the number of, yeah what is clear is that r is equal to 3 the order of the matrix A, okay and so you do not have to verify these di equal to 0 for this this is vacuously true, okay.

And so for this system for this coefficient matrix A whatever be the right hand side vector b this will always have a solution, so Ax equal to b has a solution for all right hand side vectors b, okay this has a solution for all right hand side vectors b, okay please remember that we could make this conclusion based on the fact that these equations are not present if these equations are present then you must verify that these are satisfied but they are not present here the number of non-zero rows is 3 the order of A.

(Refer Slide Time: 39:40)





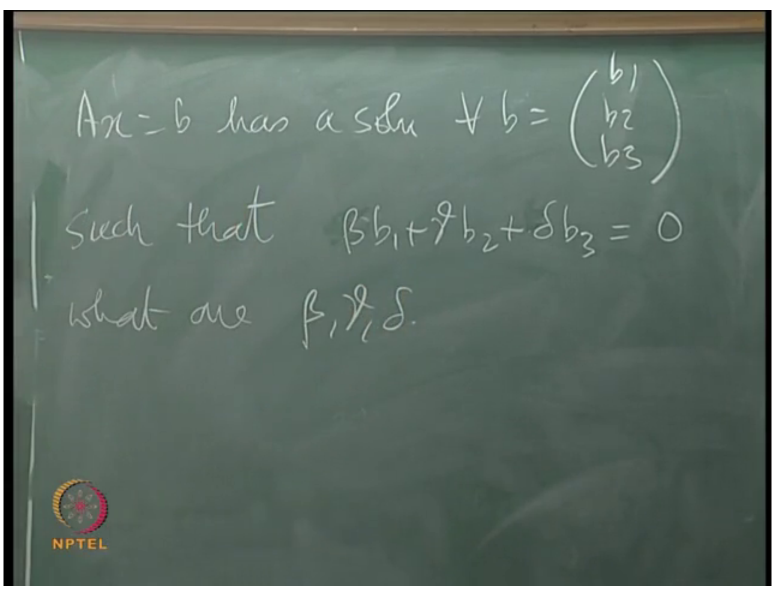


So I have only d1, d2, d3 there is no d4, okay so that is a first example this always has a solution, let us look at another example A this is my A this time I have the vector b, okay b is this vector the number of unknowns is 4 the number of equations is 3, okay so let us quickly do the elementary row operations see what happens, so I am starting now with A prime, okay let me do the first operation at once because this entry is 1 for me has made matter simple 1 minus 2 1 2 the right hand side vector is 1 minus this plus this, okay that is 0 3 2 minus 1 1 minus this plus this 0 9 minus 6 minus 3 2, what is going on?

This I know is row equivalent to the second row I will divide by 3, okay that is a next set of operations, this is not yet in the row reduced echelon form, okay but I can make the last entry 1, okay this is still not in the row reduced echelon the last non-zero row is not this, okay but I can multiply by minus 1 and then remove this, okay but this is a final row reduced echelon form, okay this is a final row reduced echelon form. Now this is my d my vector d in this problem is 5 by 3 1 by 3 1 r is 2 the number of non-zero rows of capital R which is row equivalent to A please remember not this this has three non-zero rows let me confirm r equal to 2 is equal to the number of non-zero rows of R which is row equivalent to A, okay which means I must verify is this true? r is 2, so I must verify that d3 is 0, it is not true.

So Ax equal to b has no solution, is that okay? Number of non-zero rows of capital R which is row equivalent to A what we have done here is A prime this is what I am calling as A prime right from the beginning this A prime has three non-zero rows A has only this A prime is row equivalent to a matrix which has three non-zero rows A is row equivalent to a matrix which has only two non-zero rows, when there is a difference of this that will lead to inconsistency, okay. So this problem this system does not have a solution that is clear, okay if I change this b to b1, b2, b3 we can also determine the condition under which this system will have a solution.

(Refer Slide Time: 45:45)



So I am not going to give the details I am sure you can see this happening in the previous problem this system will have a solution for all b which satisfy the extra condition, can you tell me the extra condition that must be satisfied, see I want the answer is terms of b, I know that d3 must be 0, okay you please I will leave this as an exercise then I have something like let us say r, okay beta times b1 plus gamma times b2 plus delta times b3 equal to 0 my question is what are these values beta, gamma, delta and why is that I have written the single equation not several equations, okay with this I will stop.