Linear Algebra
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Module 14-Self-Adjoint, Normal and Unitary Operators
Lecture 50
Self-Adjoint Operators 2, Spectral Theorem

Ok last time we discussed this result I want to make an emphasis the result is self adjoint operator on a finite dimensional inner product space has an eigenvalue.

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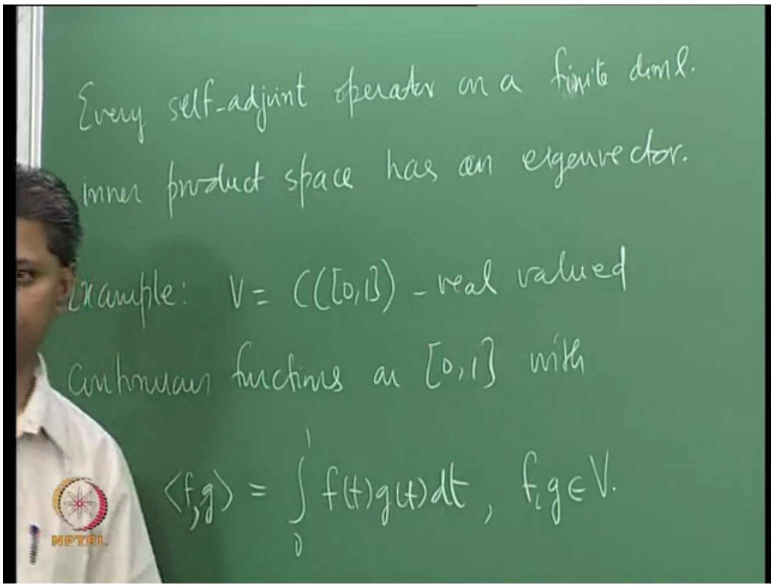

I want to just mention that every self adjoint operator on a finite dimensional inner product space has an eigenvector ok, it is the same thing showing that something an operator has an eigenvalue is the same as saying there exists a vector x knot equal to zero such that if the operator is T, T x equals lambda x. What is important is in this result is that, this is, see for, this is for a finite dimensional inner product space.

For a finite dimensional vector space we have already proved that for a finite dimensional complex vector space, we have already proved that any operator has an eigenvalue ok. But if it is a real vector space there are operators which do not have eigenvalues ok. For example the rotation matrix ok rotation matrix does not have eigenvalues, if the rotation is not 90 or 270 ok. So this result have been proved earlier that is what I want to emphasize for a complex vector space an operator T having an eigenvalue is a simple application of the fundamental theorem of algebra. Fundamental theorem of algebra says that the roots of that polynomial are the roots exists they are either real or complex ok.

So there is no guarantee that the roots are real, so we have this general result for a complex vector space. So this is more result for the real finite dimensional inner product space than for complex finite dimensional inner product space ok. This result is more for the real case more important for the real case than for the complex case. Complex case has been settled already ok. There are also one or two comments I need to make. One is this says if you have a complex finite dimensional inner product space and a self adjoint operator on it.

(Now) for a self adjoint operator you can look at the matrix corresponding to that operator relative to some orthonormal basis, then that matrix is a Hermitian matrix if A is the if T is operator and if A is the matrix of T relative to some orthonormal basis then this A is equal to A star ok. I am still in the complex finite dimensional inner product space so the entries of A could all be complex ok but this theorem says that the characteristic polynomial has only real coefficients because it has only real roots. If it has only real roots then it can be written as the characteristic (equa) characteristic polynomial can be factorized with linear factors lambda minus lambda 1, lambda minus lambda 2 etc lambda minus lambda n where each of these lambda 1, lambda 2, lambda n are real ok.

So it maybe a completely complex matrix but if it is self adjoint then its characteristic polynomial is real, no this is not a trivial observation, this is the consequence of the previous the proof of the theorem and finally finite dimensionality is important. If the space is not finite dimensional and if the operator is self adjoint then we could we need not have eigenvalues so I will give that example. So I am saying that this is not true in the case of an infinite dimensional inner product space so again for us the familiar infinite dimensional space will be C 0, 1 this time I will take real value (so) it need not be complex value.

Real valued continuous functions on 0, 1 with the inner product f, g being 0, 1 f of t, g of t dt, this an inner product space. Let's ok, real inner product space I am not taking the complex conjugate.

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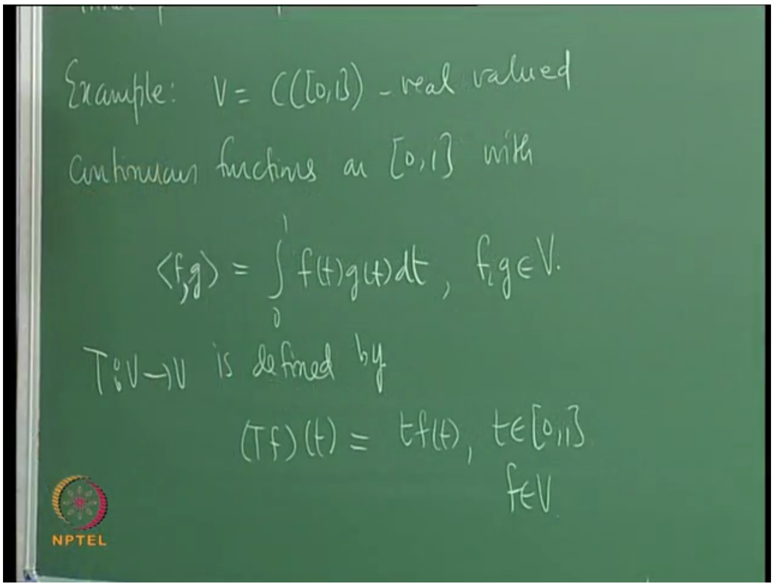

Let's look at the operator T on V defined by T f this must be a continuous function so T f acting at T is T times f of t, multiplication operator we have encountered this before. Obviously it is continuous because it is product of two continuous functions. So this is well defined T is an operator on V, T is linear that can be verified.

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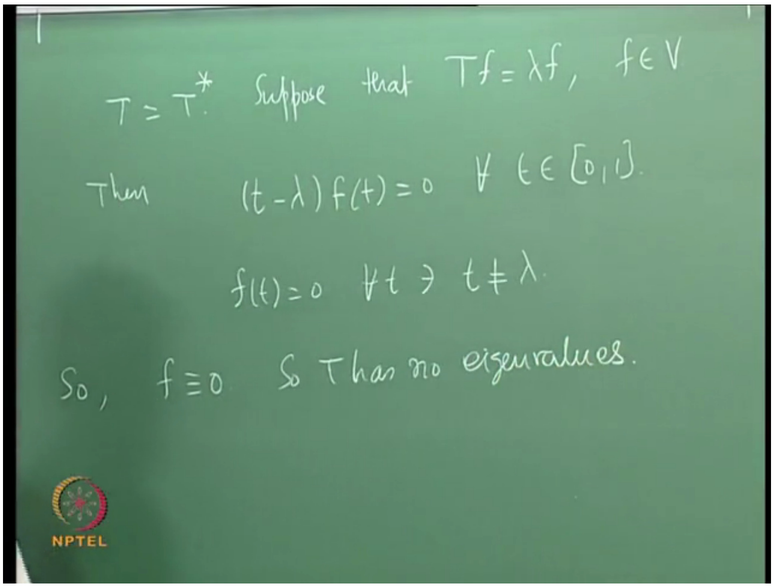

T is also self adjoint that is an exercise, simple exercise. T is a self adjoint operator ok. Suppose that I want to show that this T does not have an eigenvalue, suppose that there exist an f such that T f equals lambda f ok just look at the definition of T f then it means T f minus lambda f is zero I can write this as T minus lambda f of t this must be zero for all t in 0, 1.

If this equation holds for some lambda then that lambda must satisfy this equation for all t ok. Lambda is if this equation holds for some fixed lambda so lambda is fixed when t is not equal to lambda this means F t is zero, lambda is just one number provided offcourse lambda belongs to 0,1 ok. But for if a continuous function it is zero at all points except at one point in 0,1 then what must be the value of the function at that point? Also be zero.

You take either a left limit or the right limit depending on the situation, depending on whether you are to the left of lambda or to the right of lambda so it simply follows that F must be identically zero so F cannot be a Eigen function, eigenvector it is a function here continuous function that we are seeking so there is remember the condition on the eigenvector is that x knot equal to zero. T f equals lambda f T x equals lambda x, x knot zero, F is zero is the only function that satisfy this equation so T does not have an eigenvalue ok. So T has no eigenvalues.

But we have proved that in the finite dimensional real inner product space also if it is self adjoint then it has eigenvalues. So finite dimensionality is important ok. The next result is how is given an invariant subspace of corresponding to a linear transformation how does the orthogonal compliment of that subspace behave. This question comes for the following reasons.

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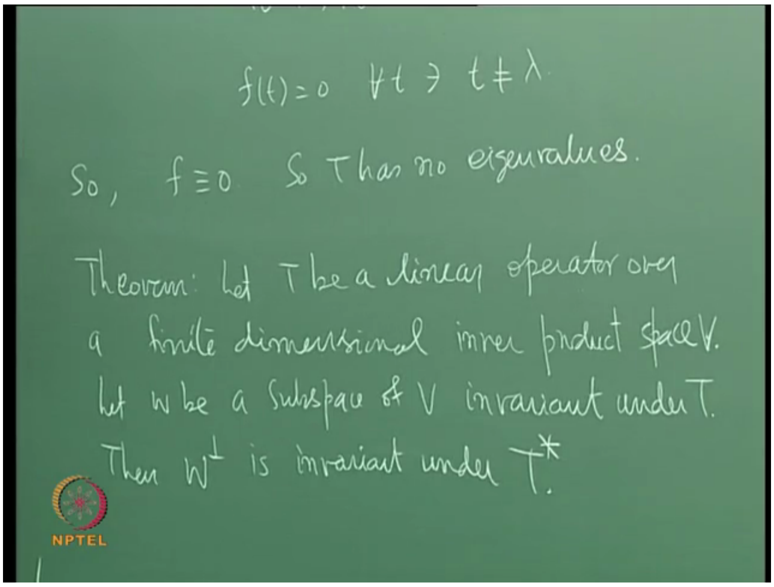

If see all Eigen spaces corresponding to a given eigenvalue are invariant subspaces ok we have seen this before. If you are in an inner product space what more can be said. If w is a subspace invariant under a linear transformation T then W perpendicular will be invariant under T star ok. This result will prove useful and only for finite dimensional spaces.

So Let be a linear operator over a finite dimensional inner product space, let W be finite dimensional inner product space I will call it V, let W be a subspace of V invariant under T. For instance you could take the Eigen spaces. Then W perpendicular is invariant under T star, the proof is really straight forward.

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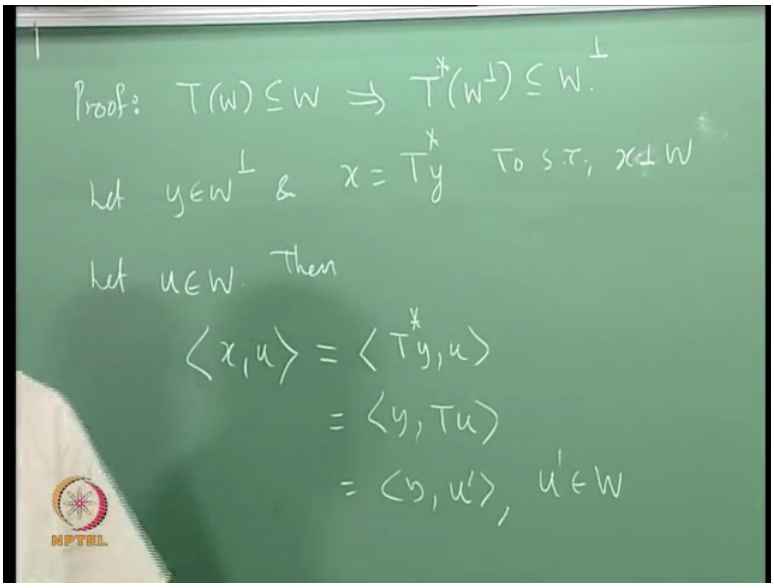

Proof is as follows so all that I want to show is given T W contained in W it follows that T star W perpendicular contained in W perpendicular, this what we want to show ok. W is invariant under T, W perpendicular invariant under T star.

So lets take Y in W perpendicular and T star y to be x, so this x belongs to this left hand side subset I must show that, that is perpendicular this vector x is perpendicular to W. I will rewrite it as x perpendicular to W. Ok so take an arbitrary W ok lets say U let U belong to W and consider the inner product of x with U, I must show that this is zero, I want to show x is perpendicular to W, x is taken from the left hand side subset, x is star y, y belongs to W (perpendicular). So look at inner product of x with U, it is T star y with U and this is y with T U, the proof is through right.

See this T U, U is in W, T of U must be in W so this is in W so I can write this as y, U prime where U prime belongs to W. But y has been taken from W perpendicular. So this a dot product of a vector in W perpendicular and a vector in W which is zero by definition. So x is perpendicular to U and so x belongs to W perpendicular.

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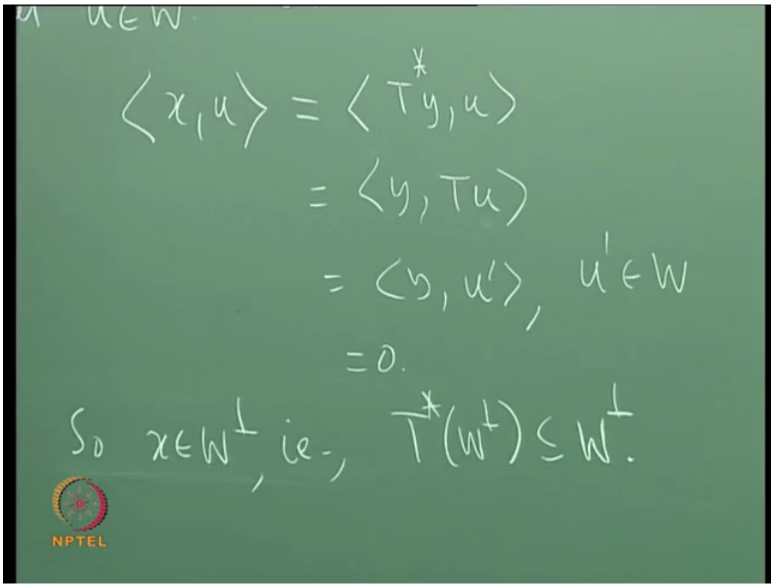

Ok in particular will apply this result for the case of a self adjoint operator. So for a self adjoint operator if W is invariant under T then W perpendicular is invariant under T ok, will make use of this, that is our next result. So the next result is an important corner stone.

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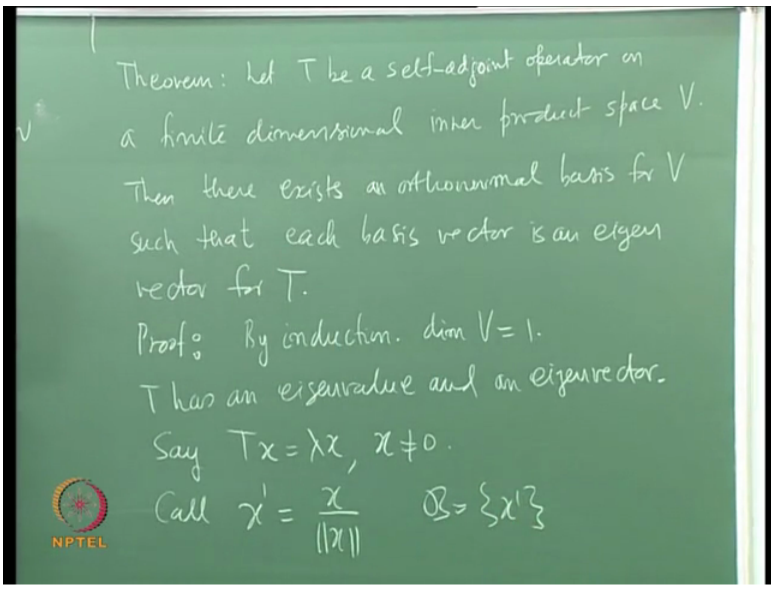

Let T be a self adjoint operator on a finite dimensional inner product space. See we have shown that V has sorry T has real Eigen we have shown that all eigenvalues of T are real ok. What we want to mention further is that, there exists an orthogonal basis self adjoint on a finite dimensional inner product space there exists an orthonormal basis for V such that each basis vector is an eigenvector. Remember that we proved already the converse of this result, that is how we started the section.

If T is a, is that agreeable? We started with the following assumption, let T be a linear operator on a finite dimensional lets say real or a complex inner product space lets say T is a finite T is a linear operator on a finite dimensional inner product space with a property that there exists an orthonormal basis B such that the matrix of T relative to this B is a diagonal matrix ok, then we have seen the T must be (())(15:27) in the real case we have seen T must be self adjoint, in the complex case we have seen that T must be normal T-T star equals T star T ok.

In the real case normality is not possible in the real case only self adjointness is possible that is only for self adjoint, so all that I am saying is this is the converse of that result the question that one could ask is in the complex case there is normality of the transformation T, in the real case there is self adjointness of T. So I am saying that the self adjoint case the answer is yes. Can you see that the matrix of T relative to this basis must be diagonal? If this happens, there exists a basis V each of whose vector is an eigenvector. So the matrix of T relative to that basis is a diagonal matrix, so this is a converse of that result ok.

We, ok let's take up the complex case a little latter but let me mention presently that the equation similar to normality, that is if T-T transpose lets say A-A transpose equals A transpose A does not necessarily imply that A is diagonalizable ok. This is the real case for the definition of normality. Definition of normality over complex is A-A star equals A star A ok, the claim is that if you have a complex matrix that satisfy if you have a normal complex matrix then it can be diagonalized ok that is the claim.

That is the claim that I am making now, I told you that this is the converse of the question that we stared with which we will see is true. We are only look at the real case for real case remember that normality when you replace star by transpose does not hold ok. Example is again the rotation operator. The rotation operator for theta not equal to pie by 2 or 3 pie by 2 (trans) satisfies the equation A-A transpose equals A transpose A identity infact ok but the rotation operator we know that for these two values does have an eigenvalues ok so no question of even asking for eigenvectors.

Ok so lets prove, so this is the result both fore real and complex case right, I have not mention anything about the underlying field. You have a self adjoint operator then it is diagonalizable by means of a unitary matrix or an orthogonal matrix depending on whether it is a complex space or a real space ok that is what this theorem says. So the proof will make use of the two results that we proved earlier, for a self adjoint operator we have shown that there are all eigenvalues are real we have shown that a self adjoint operator has eigenvalues ok, these two results are important offcourse I will also make use of this result.

The proof is by induction, so lets take the case proof is by induction lets take the case when dimension of V is 1, I know that T has an eigenvalue and so an eigenvector T has an eigenvalue value and offcourse an eigenvector ok, what I mean by this is that if you are in the complex case offcourse is this make sense if you are in the real case let us just remember once again that we have shown for a self adjoint operator that there exists a real eigenvalue and which actually means the corresponding eigenvector can be taken to be real ok.

So T has an eigenvalue and an eigenvector, lets take see if dimension V is 1, so let me call it ok let us say T x equals lambda x, lambda is eigenvalue x is eigenvector. In this case lets call x1 as x by norm x, x is not zero so norm is not zero call x1 as x by norm x then just look at the basis B consisting of this vector alone, the matrix of T this is the basis for V and this is an eigenvector by construction. So the induction the first step of induction principle that is satisfied ok. V is one dimensional this is a basis, this vector by construction is an eigenvector.

So lets assume that the result is true for all finite dimensional vector spaces of dimension less than n ok that is I have a self, whenever there is a self adjoint operator on a finite dimensional vector space of dimension less than the dimension of V the there is an orthonormal basis each of whose vector is an eigenvector ok.

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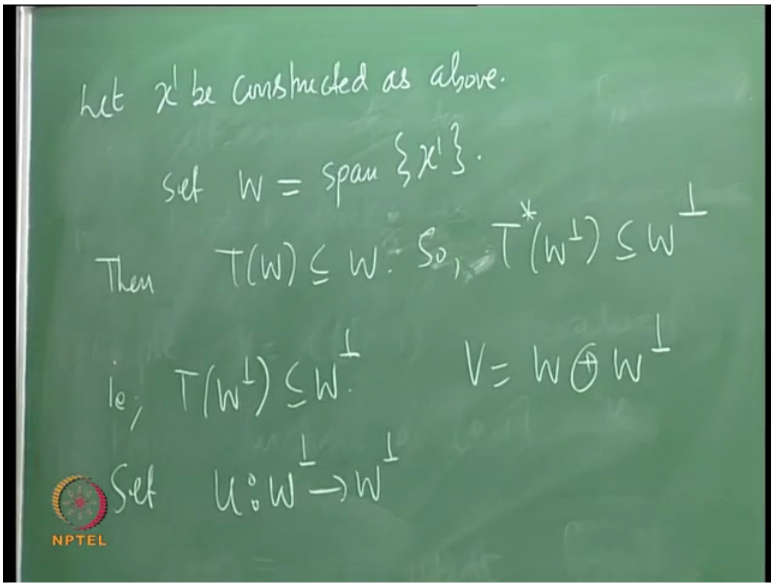

Ok so lets now look at this construction can be done in any case T has an eigenvalue real eigenvalue in the real case x is a real eigenvector so this construction can be done. What I will do is to look at W as the subspace span by this vector x1 ok then this is this eigenspace an eigenvector so obviously T of W is contained in W and W is invariant under T by the previous theorem.

So T of sorry T star of W perpendicular in contained in W perpendicular but T start is T self adjoint operator. So T of W perpendicular is contained in W perpendicular. The dimension of ok T is self adjoint. The dimension of W perpendicular is one less than the dimension of V because the vector remember V is equal to W plus W perpendicular finite dimensional vector space V is W plus W perpendicular the dimension of W is one so dimension W perpendicular is one less than the dimension of V. So now I will define an operator U on W perpendicular using the operator T.

Lets set U from W perpendicular to W perpendicular, so U must be a linear operator, the spaces must be the same,

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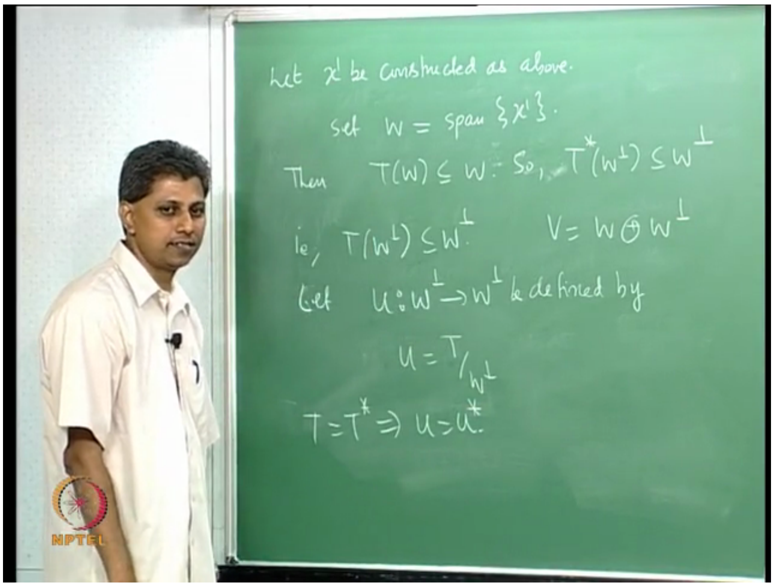

set this defined by not set now it is let U be defined by U is T restricted to W perpendicular, the restriction of T to W perpendicular that is my operator U, and remember you need to verify that see when you look at U as T restricted to W perpendicular it means you are restrict your attention in the domain, the domain is W perpendicular you are making sure but what is the guarantee that the co-domain is W perpendicular?

Because I am saying U is an operator from W perpendicular to W perpendicular, that comes from this, see this comes from this will tell you that T takes that element in x, that element x in W perpendicular to W perpendicular again it won't go to W and so this is well defined ok this that U is an operator on W perpendicular is well defined because of this ok. Now U is an operator on ok T is self adjoint implies U is self adjoint, I am going to leave that as an exercise. T equal to T star implies U equals U star ok, this an easy exercise you have to again use the fact that V is W plus W perpendicular that is all ok.

So U is a self adjoint operator on a finite dimensional vector space W perpendicular whose dimension is less than dimension (W) dimension V so by induction hypothesis see this is another induction principle that I am using ok so U corresponding to this U there is a orthonormal basis.

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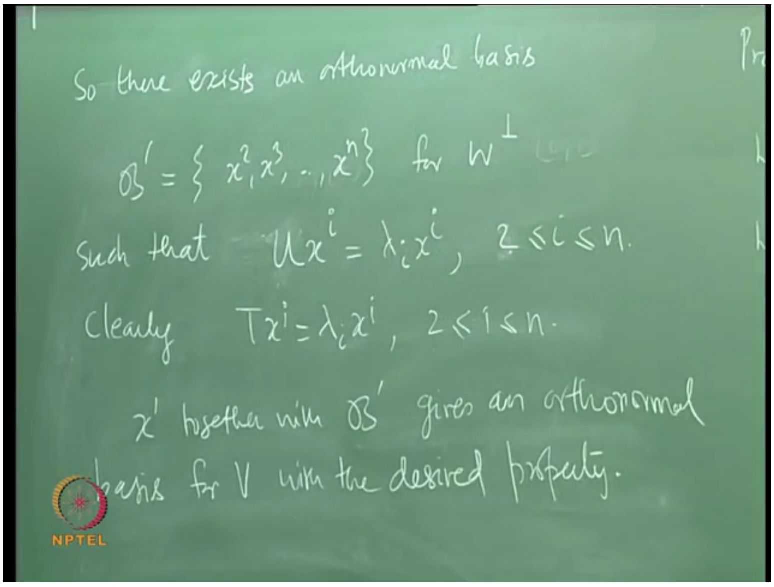

So I am sure you will agree when I write that there exists an orthonormal basis I will call it B prime because I have already used B the reason orthonormal basis B prime I will call the elements x2 x3 etc x n there exists an orthonormal basis B prime for W perpendicular it's a spaced W perpendicular that we are concerned about.

For W perpendicular which also has the extra property that such that each such that ok you tell me if this is ok such that T xi equals some lambda i xi for (one) sorry 2 less or equal to i less or equal to n.

X2 I have used for the first vector, this is an orthonormal basis, so they are mutually perpendicular and norm of each of these vectors is one, each vector must also be an eigenvector sorry corresponding to U, (())(26:24) have objected corresponding to U, U is a operator that we are talking about, such that U xi equals lambda i xi for each of this vectors. So I varies from 2 to n. So the natural thing is to ask whether this vectors are also eigenvectors for T. If they are eigenvectors for T then I am through, there is one eigenvector x1 already these are n minus 1 eigenvectors, the dimension must add.

Dimension 1 there, the dimension of this is n minus 1 this must add to the dimension of V so that this union will give me a an orthonormal basis for V and the matrix of T with respect to this basis will be a diagonal matrix. Each of each vector of this basis is an eigenvector ok. So does it follow that each xi is an eigenvector for T also from this that is by definition. See these xi’s belong to W perpendicular and U is T restricted to W perpendicular. So it follows immediately that T xi equals lambda i xi.

Some of these lambda may repeat but doesn’t matter to us. What we are interested in is, the vectors. Do I have a basis? Orthonormal basis ok. So I have these vectors together let me say x1 together with B prime gives an orthonormal basis, basis for V with the desire property I have repeated this too many times ok. So the story stops for the real inner product space because you must take this theorem along with the rotation operator to conclude that you need self adjointness in order to conclude that there is an orthonormal basis, each of whose vector is an eigenvector.

For the rotation operator there are no eigenvalues, it is normal with regard to a real inner product space. The rotation operator T satisfies T-T transpose equals T transpose T equals identity but T cannot be diagonalized in I mean it fails in the worst possible case in the sense that it does not even have real eigenvalues ok. So T as a rotation operator on or to on real space does not have eigenvalues so for real space this is the result and remember the question of diagonalizablity has been specialized here. See the original question of diagonalization is for a finite dimensional real vector space there you are interested only in a general basis.

But if it is an inner product space it is only natural to require something extra from the basis which is orthonormality ok. So for orthonormality you need A equals A star ok for if you want orthonormality then the operator must be self adjoint especially if it is a real inner product space the matrix version as we always do.

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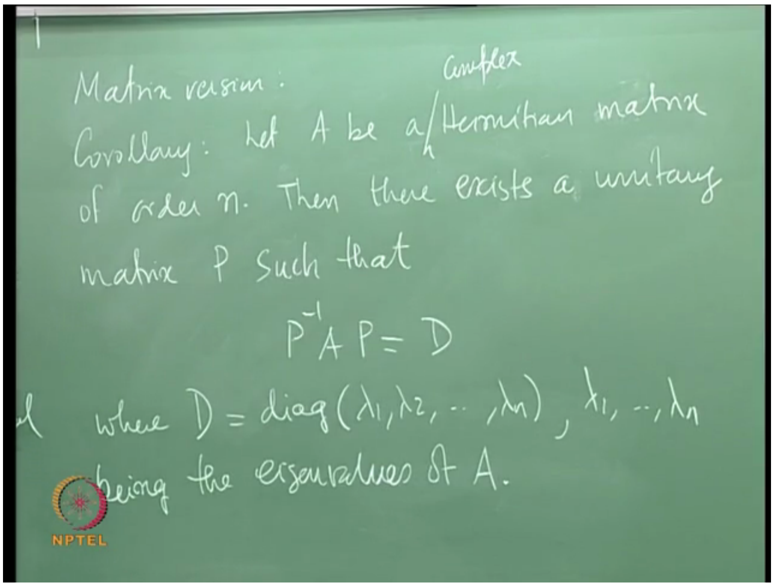

The matrix is the following, the matrix version is a corollary of this result. Let A be see in the case of complex self adjoint the word Hermitian is used.

Let A be a Hermitian operator Hermitian matrix of order n, then ok let me also emphasize that it is complex, be a complex Hermitian matrix of order n then there exists a unitary matrix I will call it P such that P inverse A P equals D where D equals diagonal lambda 1, lambda 2, etc lambda n, lambda 1 etc lambda n being the eigenvalues of A.

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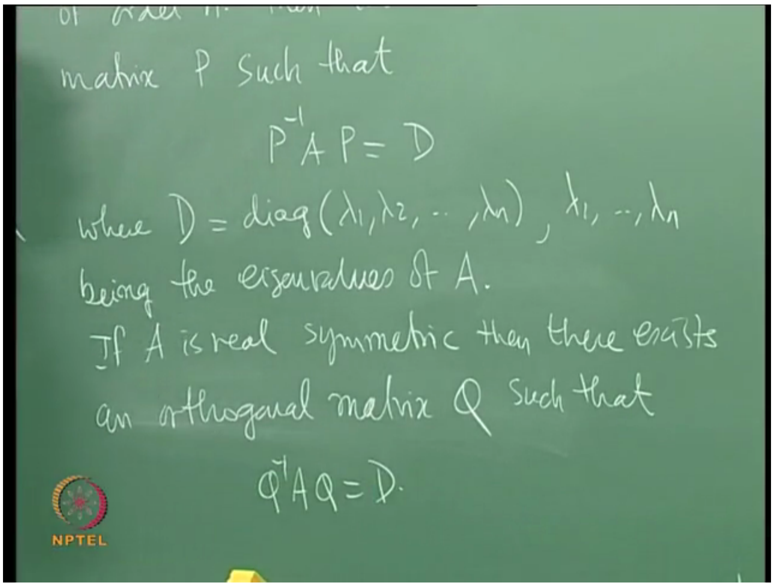

If A is real symmetric then there exists an orthogonal matrix I will call it Q different from P there exists an orthogonal matrix Q when I say orthogonal matrix it is a real orthogonal matrix because if it complex then will call it a unitary so there exists an orthogonal matrix Q such that Q inverse A Q equals D where D is diagonal as before diagonal entries of D being the eigenvalues of A.

Ok so here I need to only emphasize that P inverse is equal to P star because P is unitary. Similarly here P inverse is P transpose ok, what is the proof? Is the corollary of the previous result Q, Q inverse Q transpose ok this is the corollary of previous one so we can appeal to the previous result.

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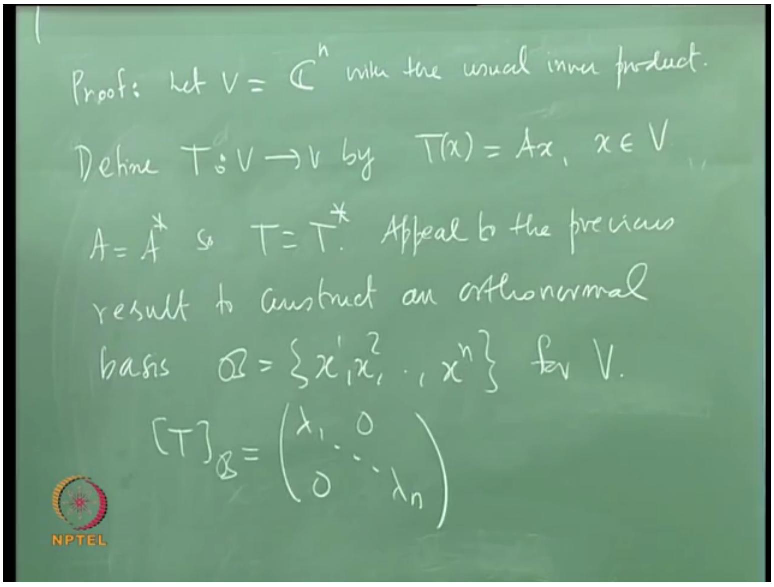

You are given a complex Hermitian matrix A so you can define linear transformation through this so you define C n with a usual inner product. Define T on V by T of x equals A x. You have a matrix through which you can define a linear transformation then this definition means that the matrix of T relative to standard basis is A.

The matrix of T relative to the standard orthonormal basis is the matrix A. A is complex Hermitian so A is A star so T is T star. So I have a self adjoint operator on a complex inner product space then I know that by the previous theorem there is an orthonormal basis for C n satisfying the property that each vector in that orthonormal basis is an eigenvector for T. Eigenvector for T means T x equals lambda x but T x equal to A x so A x equals lambda x. Collect all this eigenvalues collect all the yeah, collect all these eigenvalues, arrange them as a diagonal matrix then we know that this is the same as writing down matrix of T relative to the new orthonormal basis that we have construct ok.

So I will simply say appeal to the previous theorem, appeal to the previous result to construct an orthonormal basis this time I will call it B so I have x1, x2, etc x n for C n for V. What I know is that each of these vectors is an eigenvector for the operator T so if I look at the matrix of T relative to this basis then I know that that's a diagonal matrix lambda 1 etc lambda n ok. The proof is complete if I tell you what must be P, just give one choice for P ok.

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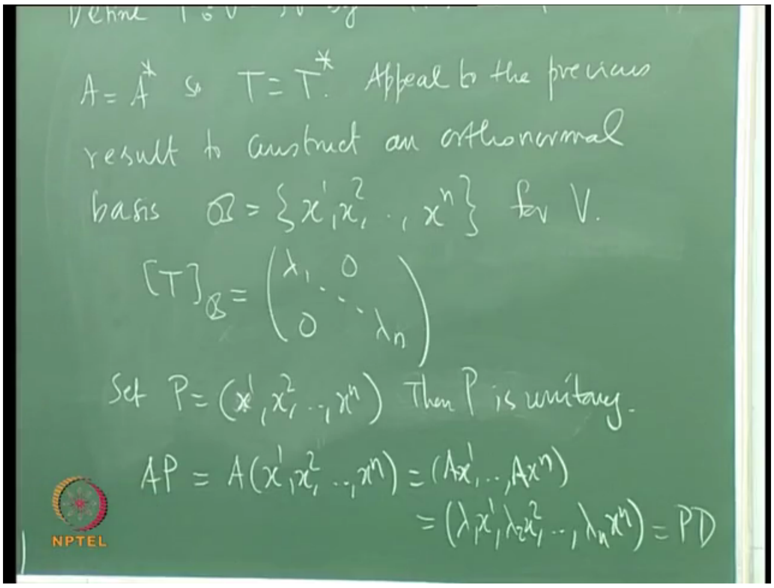

Let us call P as matrix whose first column is x1 second column x2 etc x n. You have these vectors constructed by the previous theorem existence not construction.

So collect those vectors so this is something that we have done even in the ordinary case without the inner product space say P equal to this then this P this matrix P has a property that its columns are mutually orthogonal and the norm of each column is one. So this is a unitary matrix that is P star equal to P transpose. So then P is unitary. Finally this equation must be verified but as before this equation we have seen before. Look at A P, A P by definition is A into x1, x2 etc x n, we know that this A can be brought inside to write A x1 etc A x n.

Each of these is an eigenvector so I have the eigenvalues coming now, lambda 1 x1, lambda 2 x2 etc. Let me just write down the last step which is a little exercise for you, verify that this is equal to P times D. Yes, which is almost obvious, you first write P and then D ok. So A P equals P D you know that P is invertible so you can pre-multiply by P inverse and the you get this equation ok. Real case is similar, in the real case you know that the eigenvalues are real corresponding vectors can be taken to be real so this will be basis consisting of real vectors.

Now real vectors giving you Q for instance then it is an orthogonal matrix right it will be an orthogonal matrix and the rest of the proof is as before ok. So this is just version matrix version of this important theorem. The last part is really for normal operators that I will do in the next class ok. So what is means is that, an operator is diagonalizable by means of an operator on a complex vector space this time, just complex vector space is diagonalizable by means of a orthonormal transformation by means of a unitary matrix if and only if it is normal ok.

So there is a significant difference between the question of a real symmetric matrix and the complex symmetric matrix that is if A is real and if A is equal to A transpose then this theorem says A can be diagonalized ok. Take A to be complex, A equal to A transpose there is no theorem which can guarantee that A is diagonalizable ok. Whereas you take A complex A equal to A star the conjugate transpose then A is diagonalizable ok. So the question is really about what is the corresponding operation for transpose in the complex case?

The corresponding operation for transpose in the complex case is conjugate transpose ok. So remember that the statement is wrong a complex symmetric matrix is diagonalizable is wrong ok. A real symmetric matrix is diagonalizable, a complex Hermitian matrix is diagonalizable ok, so let me stop here.