Linear Algebra
Professor K.C Shivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Module 14-Self-Adjoint, Normal and Unitary Operators
Lecture 49
Unitary Operators 2, Self-Adjoint Operators 1

We are discussing unitary operators more generally will discuss what are called as normal operators but before that there is one last result, I told you briefly about this last time I will prove this result. What is a relationship between the matrices of a linear transformation corresponding to two orthonormal basis ok.

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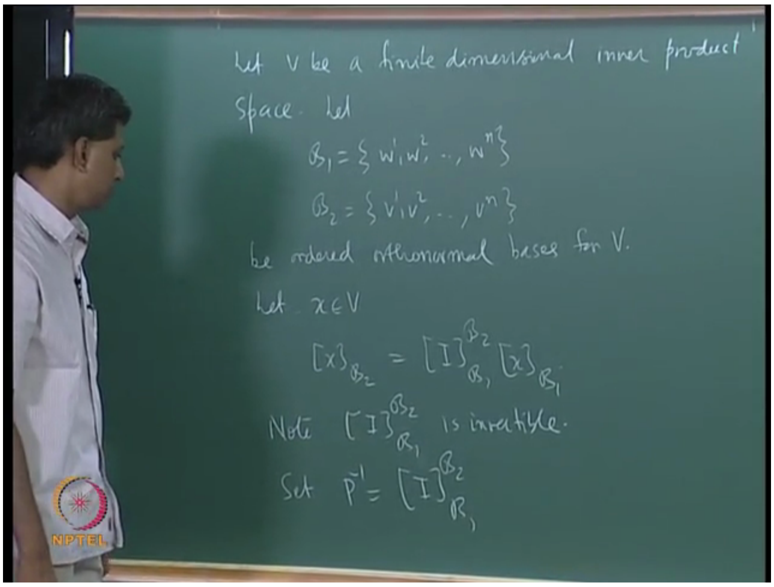

So this is the framework, let V be a finite dimensional inner product space. Lets take two orthonormal basis lets say B1 I will use W1 W2 etc W n this is an orthonormal basis, I have another one. I will use V1 V2. Suppose these are ordered orthonormal basis for V ok. Now for any two basis we have derived one or two relationships, for instance if you look at a fixed arbitrary vector x, then if you look at the matrix of x relative to B2 this is related to the matrix of x relative to B1 by means of the following formula ok.

The matrix of x relative to B2 is equal to identity matrix between the basis B1 B2 that is I write down the elements of the basis vector B1 that is W1, W2 etc W n lets W1 that is a linear combination of these vectors that will be the first column of this matrix etc, right down W n in terms of V1 etc V n that will be the last column of this, in general if B1 is not equal to B2 then the matrix of the identity transformation is not the identity matrix ok.

It is never the identity matrix unless B1 is equal to B2, but in any case this matrix is invertible ok. Note that this matrix that is a matrix of the identity operator with respect to this two basis, this is invertible ok I will call it P inverse. Set P inverse to be the matrix of the identity transformation I know that P is invertible I know that this is invertible so I am just calling it P inverse.

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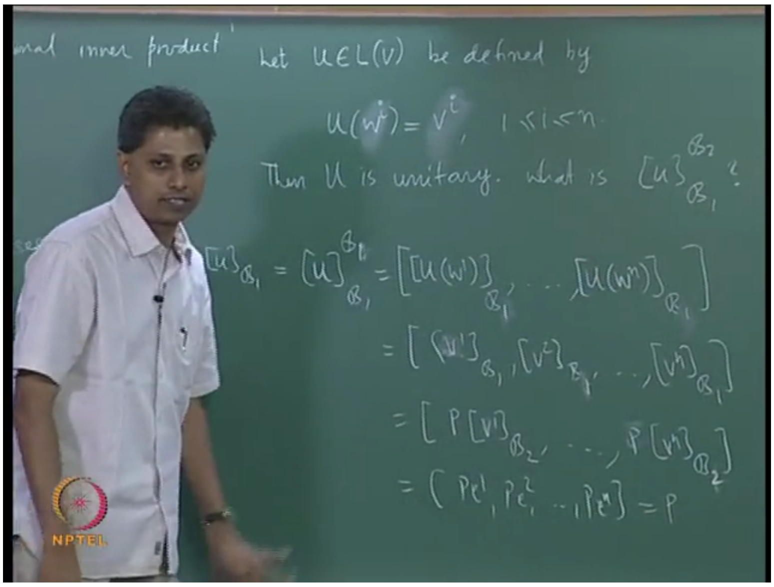

Let us now define, yes, what is the reason? (You) should tell me, ok suppose you take this map lets call it Q ok. Suppose Q into x is equal to zero, Q into x is equal to, x is a vector in V ok I am talking about a matrix, so lets take x to be in R n or in R Cn ok for simplicity lets take R n, call this matrix Q ok.

X belongs to R n Q x equal to zero, does it mean x is zero? Yes? Over, this is square matrix, homogeneous system Q x equal to zero has the unique solution zero so it must be invertible ok. See all this we have seen before I am just trying to quickly recall this things so that we will be able to go to an inner product space, what we know all this things for ordinary finite dimension vector space. We are trying to specialize this result for the inner product ok. I will define now a linear map U, let U belong to L of V be defined by defining a linear transformation U using this two basis.

The definition is U of W i equals V I, define a linear map between two basis one define a linear map then it is completely determined, so this is the map U. Now remember that the basis B, B2 are orthonormal so this U is unitary from the results that we have seen before. U is offcourse linear, this U is unitary. It takes one it takes a particular basis particular orthonormal basis to another orthonormal basis so this must be unitary. Say I am defining a linear operator from B1 to B2 that is U W i is V i and I am trying to look at the matrix of U relative to just B, that is this is by definition the matrix of U relative to B1 just this basis ok same basis not different.

Then this will be B1 etc B n, which is by definition V1 relative to B1, V2 relative to B2 etc V n relative to B1 there is only one basis ok. Ok what is V B 1 from this equation? V1 B1, V2 B1 etc I can write this as V1 B1 from this equation V1 B n from this equation is this inverse into V B2 ok but this inverse is P. So tell me if this is correct? The first vector is P, P is matrix remember into the matrix of V1 relative to B2. See I am using this equation from this equation V B1 is what I want to write V1 B1, V2 B1 etc V B1 is what I want to write V B1 is the inverse of this matrix into V B2. But the inverse of this matrix is P because P inverse is this.

So V B1 is P times V B2 that is what I have written etc the matrix P times V n B1 sorry this is B2, now it is B2 from B1 I have move to B2 by making use of this equation and the notation that the inverse of this matrix is P. Now you see what is going on V1 V2 etc they are the second basis elements so the representation in terms of B2 is just E1 E2 etc so this is P E1, P E2 etc P E n but you know that for matrices if this is what I have for a single matrix P then this can be written I can take the P outside so this is P into E1, E2 etc E n but that is identity so this is just P, this is just the matrix P.

So if you look at the matrix of U relative to B1, B1- B1 then it is P. Does it follow that P is a unitary matrix? Yes because U is a unitary matrix, the matrix of a unitary (mat) the matrix of a unitary operator relative to any two orthonormal basis that is a unitary matrix, so this P is a unitary matrix.

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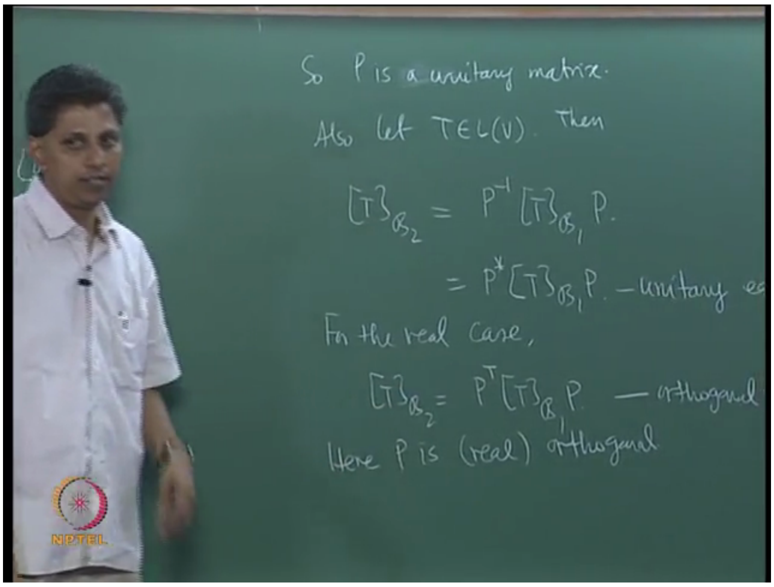

P is unitary, P is a unitary matrix use a unitary operator and finally recall how the matrix of a linear transformation corresponding to two basis look like. Take a general matrix general operator let T element of L V then the matrix of T relative to B2 can be written as a matrix P inverse into the matrix of T relative to B1 P ok.

I remember we used the notation M T, M inverse but M T, M inverse but M is M inverse is equal to this matrix ok. So you can go back to that notation and verify that this is what we have got instead of M we have P inverse. It must be M T, M inverse instead of M we have P inverse ok. But P is unitary, so this is P star T B1 P, so this is the relationship then.

The matrix of linear operator corresponding to one basis one orthonormal basis related to the matrix of the same linear operator with respect to another orthonormal basis in this manner so this matrix P is unitary this is for the complex case for the real case, for the real inner product case the matrix of T relative to B2 is P transpose the matrix of T relative to B1 into P, real orthogonal matrix.

So in this case P will be an orthogonal matrix. Ok so this is a relationship there is a definition coming out of this relationship, this is called unitary equivalence this is called orthogonal equivalence that si two matrices B and A are said to be unitary equivalent if there exists a unitary matrix P such that B can be written as P star A P ok. Similarly two matrices two real matrices B and A are said to be orthogonally equivalent if there exists an orthogonal matrix P such that B equal to P transpose A P ok, this is more specialized than the usual equivalence of (())(15:59) which comes from the similarity transformation.

Matrices A and B are said to be similar if B can be written as P inverse A P for some invertible matrix P. In that case this is what leads to the definition (())(13:14) if there (x) if A is a matrix such that A can be written as P inverse D P where D is a diagonal matrix then A is said to be (()(13:23) ok. Similarly here, if a matrix A is said to be diagonalizable by means of an orthogonal transformation or by means of a unitary transformation if there exists either an orthogonal matrix P or a unitary matrix P such that A can be written as either P transpose D P or P star D P, second case when A is complex ok.

Will be interested in this question (())(13:56) was discussed when we discussed operators on finite dimensional vector spaces, will discuss unitary equivalence ok, given a complex matrix when does it happen that there exists a diagonal matrix D such that A can be written as P star D P, where P is unitary ok. We will address this question this is for unitary equivalence. Let’s look at now more general operators called normal operators ok. But little more particular is a self adjoint operator. I will discuss self adjoint operators and then move to normal operators.

So for this are the operators which will give you the correct answer for this question. Given a complex matrix A when is it unitary equivalent to a diagonal matrix ok but before that lets look at this problems. See this is really the final problem in finite dimensional inner product spaces.

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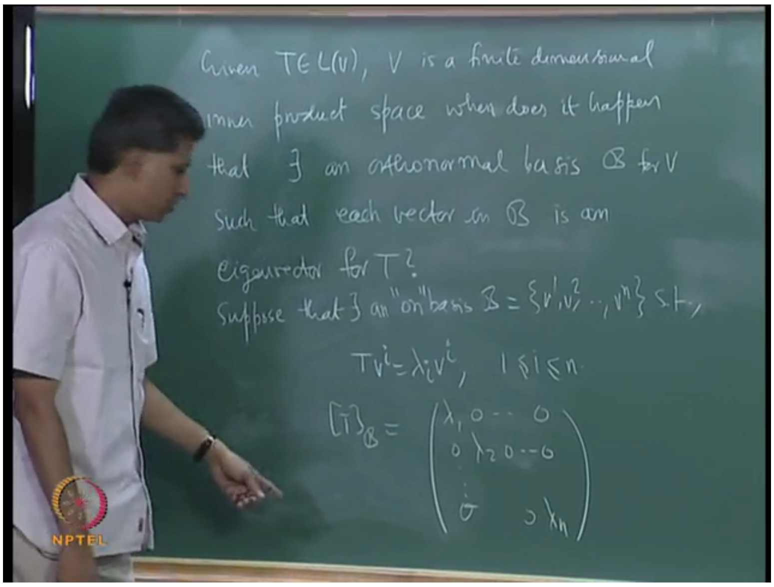

The problem is given a linear transformation T, given T element of L V where the framework is V is finite dimensional inner product space. Given a linear operator T on a finite dimensional inner product space V when does it happen that there exists an orthonormal basis B such that orthonormal basis V for V such that each vector in B is an eigenvector for T.

This question we asked for the usual vector space not the (finite) not the inner product space will try to answer this for the inner product space where you will see that the notions of self adjoint normal operators come naturally. Before answering this question lets see what happens if T satisfies such a property. Suppose there is a an orthonormal basis B such that each vector of that basis is an eigenvector for T ok. Then what happens? So we are really looking at the necessary condition. If this happens what? This question is really sufficient, when does it happen that this holds.

Suppose this happens, suppose that there exists a basis let me call the elements V1, V2, V n suppose there exists an orthonormal basis. Orthonormal basis, the such that each vector in the basis is a eigenvector so that means T V i equals lambda I Vi this holds. What is the meaning of this? The meaning of this is, this is related to the problem of diagonalization. So if you look at the matrix of T relative to the basis B for which this happens T must be a diagonal matrix, just look at the right hand side. T V1 is lambda 1 V1 so the first column is lambda 1 all other entry is 0, etc. So T B is lambda 1, 0 0 0, 0 lambda to etc all entry is 0 the last entry is lambda n, that is you get a diagonal matrix whose diagonal entries are the eigenvalues.

Here some of the eigenvalues may repeat I am just assuming that they are lambda 1 etc lambda n some of this may repeat that is possible. So the matrix of T relative to the basis for which this happens that matrix is a diagonal matrix. What about the matrix of T star relative to B? This question we can ask because we are inner product.

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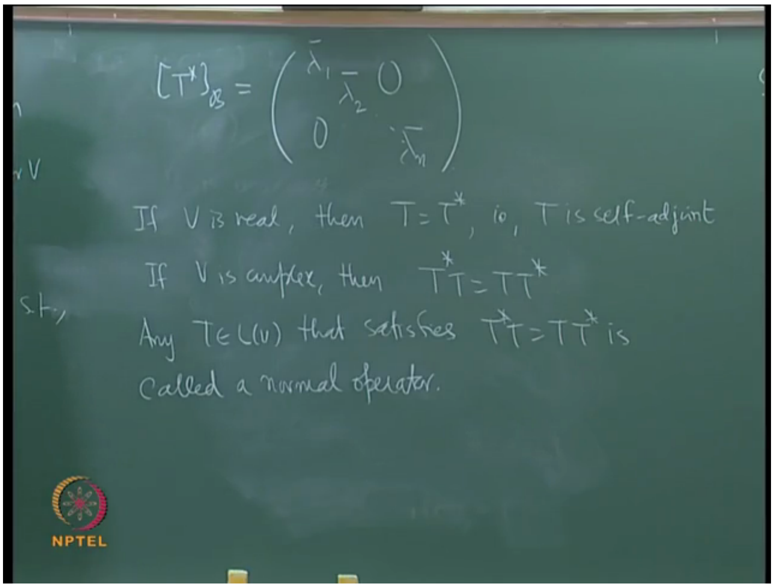

The matrix of T star relative to B is lambda 1 bar see in general it is complex vector space all other entry is zero ok. This is from one of the results that we have discussed. If A is a matrix of T relative to an orthonormal basis B, B is a matrix of T relative to another normal sorry the same orthonormal basis, matrix of T star then A is equal to B star.

So T star has this representation if V is real then T is equal to T star, that is T is self adjoint. If V is real then T is equal to T star I can even write T transpose ok. If V is complex then T is equal to T star is not true but T-T star equal to T star T ok. Two diagonal matrices any two diagonal matrices come out. T star T equal to T-T star. An operator T that satisfies such an equation is called a normal operator. So if I have a complex finite dimensional vector space V which has an orthonormal basis satisfying this property that is each of the vector coming from the basis is an eigenvector for the operator T then T must be a normal operator ok.

Now what is interesting is that see, this is only a necessary condition I told you but interestingly this condition is also sufficient. That is what will show is that if I have a complex finite dimensional inner product space V and if T is a normal operator then T is diagonalizable by means of a unitary matrix ok. By the way can you see the unitary matrix? See I have not written down the unitary matrix here explicitly. Can you see what that unitary matrix must be? I will just written the diagonal form. Can you see that unitary matrices are hidden in this? Ok just think about it.

Whenever we say that a matrix of a linear transformation relative to a basis is a diagonal matrix, it means there is a matrix P such that A equals P inverse D P. That is what I have written down here. There is also see if you look at the (mat) see this matrix A this is the diagonal matrix D then A will be equal to P star D P that is hidden in this equation ok. So this is unitary equivalence of this matrix with this diagonal matrix ok. What we will see is that the converse is also true that is if T is a normal operator then there is a diagonal matrix D such that this equation holds for some orthonormal basis B ok.

That is I wanted to first write down this implication the necessity part. This is also a sufficient condition and for the real case will show that if T is equal to T star then T is unitary I am sorry T is orthogonally equivalent to I mean the matrix of T relative to a basis an orthonormal basis is orthogonally equivalent to a diagonal matrix ok. But before that lets look at some properties. Properties of self adjoint operators ok, so just to summarize we have seen that for the real space real inner product space case if an operator is self adjoint then there exists an orthonormal I am sorry, for the real case what we have seen is if there is a basis B it has a property that each vector from the basis is an eigenvector for the operator T then T is self adjoint.

In the complex case we have shown that if this happens then T must be normal, we will prove the converse also ok. But as I told you some properties before we prove this results.

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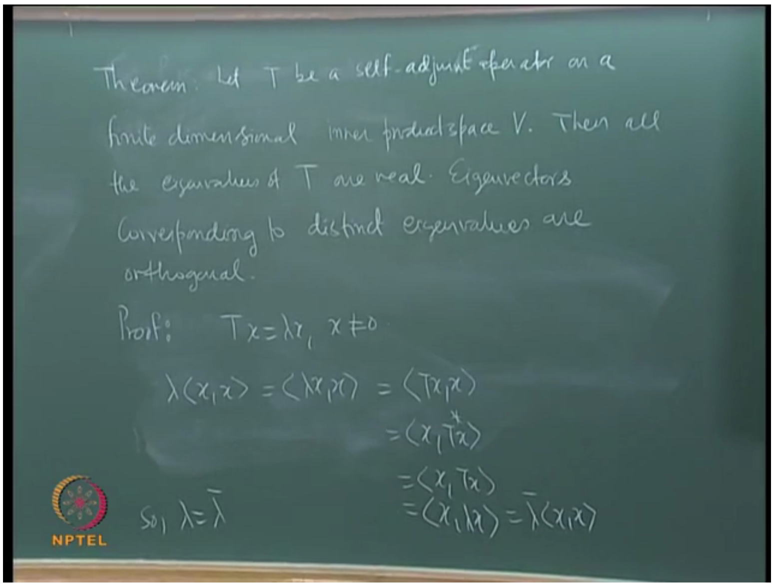

Lets look at the self adjoint case. For the self adjoint case we have the following, so this is in general this is a dimension free results. So T be a self adjoint operator on a finite dimensional inner product space. Then you have the following, the eigen all the eigenvalues of T are real numbers, eigenvectors corresponding to distinct eigenvalues are, can you make a guess here? We have seen the general case we have studied the general case earlier T is a linear operator on a finite dimensional vector space V then eigenvectors corresponding to distinct eigenvalues are remember the property that we proved?

For any operator eigenvectors corresponded to distinct eigenvalues they are linearly independent for a general vector space. If they have a inner product space they are orthonormal, something more ok. If you have a self adjoint operator and the inner product space then you can say something more. Eigenvalues are real in the first place, eigenvectors corresponding to distinct eigenvalues are orthogonal. Proof, first I want to show that the eigenvalues are real so lets take T x equals lambda x, that is x is an eigenvector corresponding to the eigenvalue lambda.

Remember an eigenvector by definition is a non-zero vector ok. I want to show that lambda is real. So I will look at the inner product lambda x, x this is lambda x, x I have taken lambda into the first argument I will use this lambda x equal to T x so there is inner product T x, x. T x, x can be written as x, T star x that is a definition of the adjoint but T star is T so this is x, T x again use T x equals lambda x this is inner product x, lambda x, lambda comes in the second argument so it goes out with a complex conjugate so this is lambda bar x, x.

So lambda x, x is equal to lambda bar x, x. X is a non-zero eigenvector so the dot product of x with itself cannot be zero infact that is a positive number. So lambda must be equal to lambda bar ok. So lambda is real all eigenvalues of a self adjoint operator must be real numbers. Lambda non-zero

Student: (())(28:52)

See look at T x equals lambda x that can be written as T minus (())(29:17) look at the matrix case, A minus lambda I of x is equal to 0. This has a non-trivial solution if and only if the determinant of A minus lambda I is not zero ok. But we want non-trivial solutions that happens only if determinant of A minus lambda I is 0. So lambda could be zero that is not a problem, why? What's a problem? If A is a singular matrix lambda could be a zero. It is a condition on x, x must be non-zero, lambda can be zero there is no problem ok, we are seeking non-trivial solutions of a homogeneous equation, that happens only if and only if the determinant of that coefficient matrix is zero.

That gives rise to this definition, that is A minus lambda I is singular for a matrix for an operator T minus lambda I is singular for an operator T minus lambda I is singular you can show that this happens if and only if the matrix of T minus lambda I with respect to any basis is singular. That is why this question of T x equals lambda x just gets passed onto the equation A x equals lambda x where A is a matrix of T ok. Ok so this proves lambda is real.

Eigenvalues corresponding to distinct eigenvalues are orthogonal.

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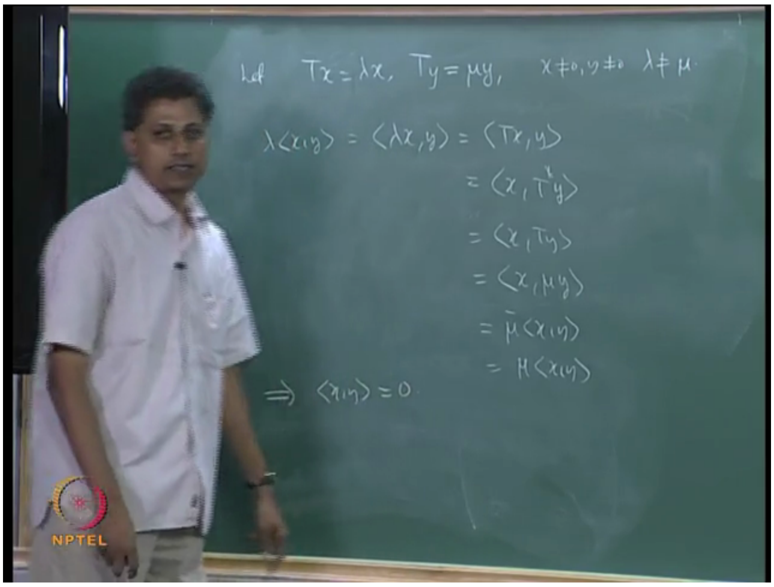

So lets take T x equals lambda x, T y equals Mu y where offcourse x is not zero y is not zero, lambda is not Mu. I have eigenvectors corresponding to distinct eigenvalues, I must show that they are orthogonal. So consider again lambda x, y as before this is lambda x, y that is lambda x is T x, T x, y this can be written as x, T star y, T is self adjoint x, T y, x, Ty is x, Mu y entirely similar to the previous proof, Mu bar comes out Mu bar x y but Mu is real because of what we proved in the first part so this is Mu x y, lambda x y equals Mu x y.

Lambda knot equal to Mu, these are numbers lambda knot equal to Mu, so this means inner product x y is zero. This something more than saying that they are independent. This offcourse doesn't say anything about the existence of eigenvalues, it says if the eigenvalues exists then they must be real for a self adjoint operator and they must be orthogonal if they correspond to distinct eigenvalues.

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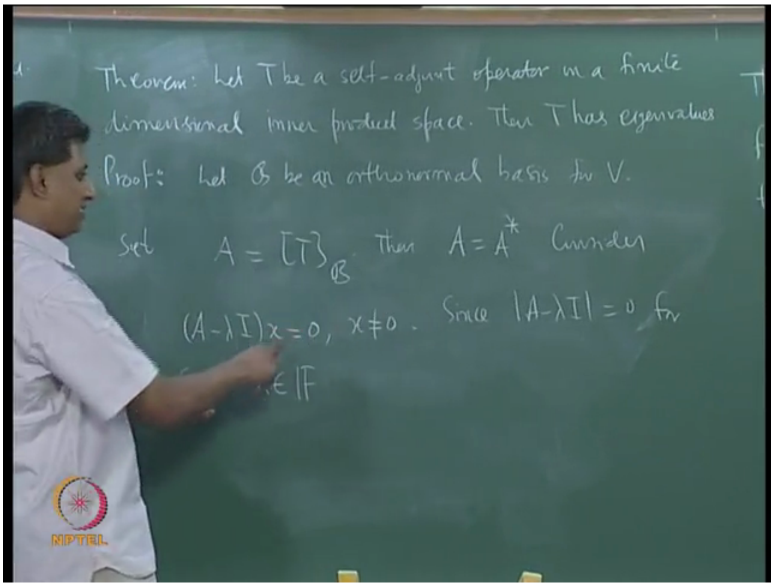

Lets prove that for a self adjoint operator first eigenvalues exists then T has eigenvalues. The proof will use fundamental theorem of algebra, any polynomial of degree n must have atleast one root real or complex ok.

So lets take let B be a basis, an orthonormal basis, ordered orthonormal basis for V and lets call A as the matrix of T relative to B then A is self adjoint, A is a Hermitian matrix, A is equal to A star orthonormal basis. So A is equal to A star. Consider the equation A minus lambda I of x is equal to zero, I wanted to show that this equation has a non-trivial solution ok. Since determinant of A minus lambda I equal to 0, so consider this equation determinant A minus lambda equal to 0. See in general this happens in a complex vector space ok. Can, ok I have made this I have already made this assertion that determinant of A minus lambda I is equal to 0.

So I will use this explanation to show that this equation has a non-trivial solution. But then can you justify the statement? Why is determinant of A minus lambda I zero? I will just specify for some lambda element of F, F is R or C depending on whether it is a complex or where it is real or a complex space for some lambda (())(36:11) Can you justify this? Fundamental theorem algebra again, see this equation has a polynomial equation of degree n ok, it will have atleast one root real or complex.

But can you see that ok, suppose there is a lambda equals lambda knot for which determinant M minus lambda knot I is zero, consider the general complex case, lambda knot is a complex number. If I am in the complex space setting then this equation will have a solution for that lambda knot, non-trivial solution x for that lambda knot ok. But forget about it, what we want is more importantly there exists a lambda knot such that A minus lambda knot I x is equal to 0 for x not equal to zero. So we have a complex number in that case.

Ok, but in general it is a complex number in the real case also that will be a complex number ok. See if it is a complex space there is no problem the proof is there, proof is over if it is a real space then this number lambda knot could be complex but if the number lambda knot is complex then can you see that, that needs a little explanation but you can write it down. If the number lambda knot is complex the vector x will also be complex ok. First can you see that lambda knot cannot be complex? Ok I think that needs an explanation ok.

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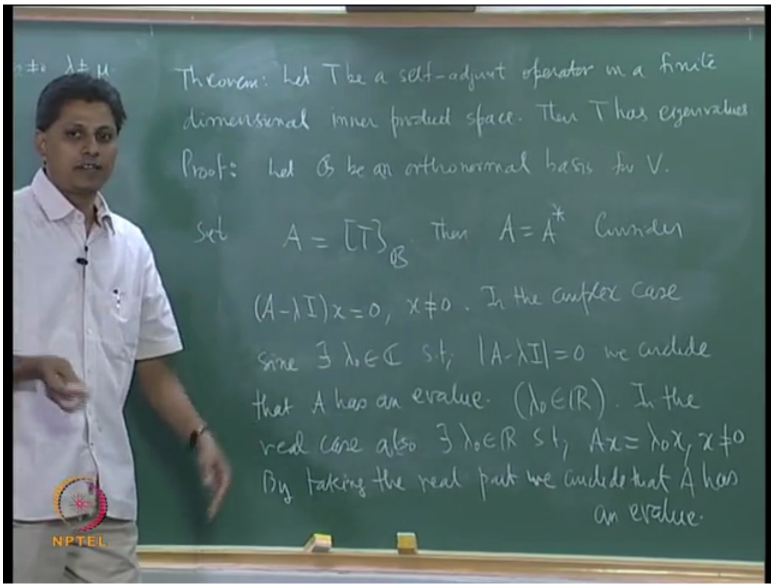

What I want to say is then the following, in the complex case that is the underlined field is complex this means in the complex case since there exists lambda knot element of C such that determinant of A minus lambda I is equal to 0, by fundamental theorem of algebra we conclude that A has an eigenvalue. By the way this has to be real because we have just now shown that if you have an self adjoint operator the eigenvalues must be real ok. So just to specify I will write lambda knot belongs to R also ok this is a complex case.

In the real case again lambda knot is real, in the real case lambda knot, in the real case also there exists lambda knot in R such that A x equal lambda knot x, x knot equal to 0, then what is the problem? The only problem is that, x could be complex but remember A is real lambda is real if x is complex the for each coordinate of x I take the real part, imaginary part. Real part forms a vector, imaginary (form) part forms another vector. So I will have something like A into Z plus I y equals lambda knot into Z plus I y. Where Z and y will have real entries equate real and imaginary parts you will get real eigenvalues real eigenvectors ok.

The only thing in this case is, remember the fact that lambda knot is real comes from the previous result ok but before that I have the equation A x equals lambda knot x, this x could be complex. But then take the real and imaginary parts of x and make use of the fact that the entries of A are real and the fact that lambda knot is real, you can take out the real parts of x, imaginary part of x infact each one will be an eigenvector, corresponding to the eigenvalue lambda ok.

So I will just conclude by saying by taking the real part for instance we conclude that A has an eigenvalue stick the real part of x that will be an eigenvector, either the real part or the imaginary part will be non-zero both cannot be zero, either the real part or the imaginary part will be non-zero so one of them will be atleast one of them will be an eigenvector and so that number lambda, lambda knot is an eigenvalue ok that is the explanation for the real case ok (())(41:43) other properties I will prove in the next class.