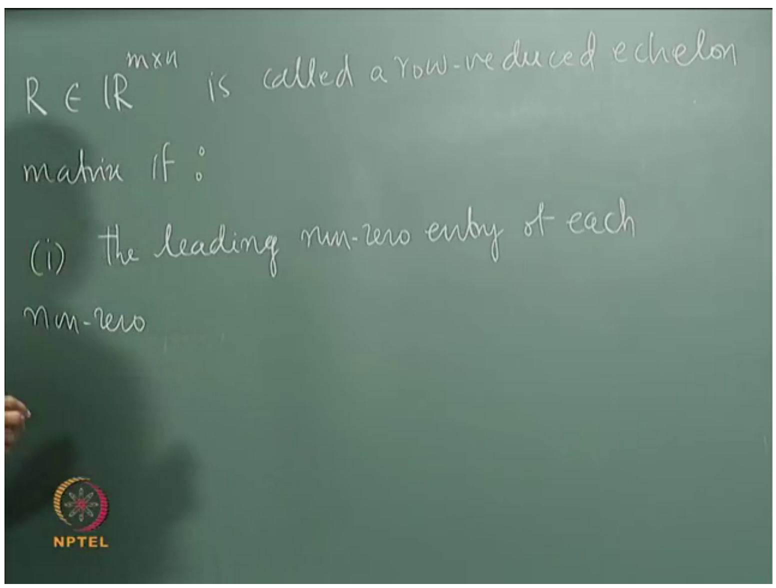
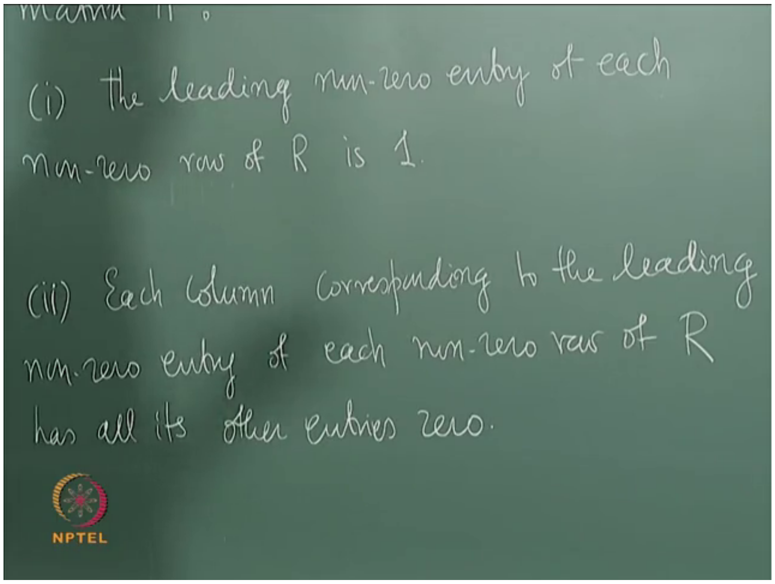
Linear Algebra
By Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 4
Row-reduced Echelon Matrices

Okay, in today's lecture we will discuss the proof of the fact that any rectangular matrix can be made row equivalent to a row reduced echelon matrix, okay this is one of the important results in linear equations. So we will prove this result and look at other consequences, let me first recall the definition of a row reduced echelon matrix.

(Refer Slide Time: 0:46)

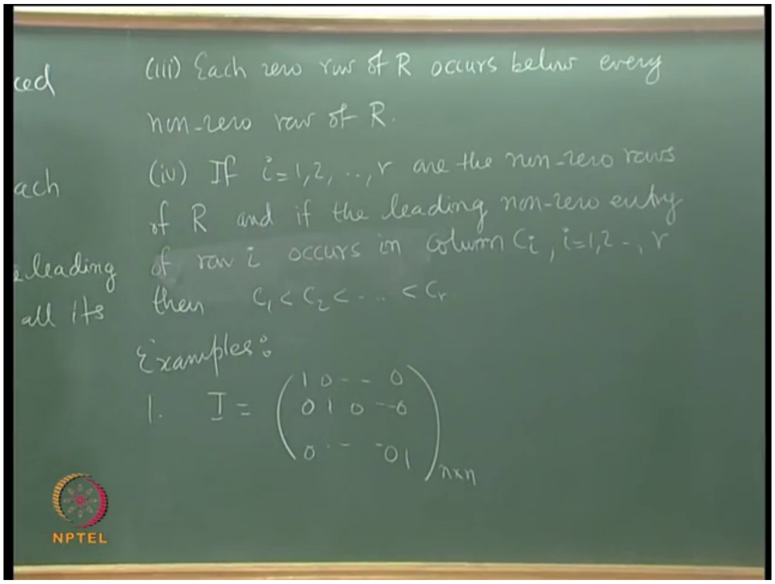


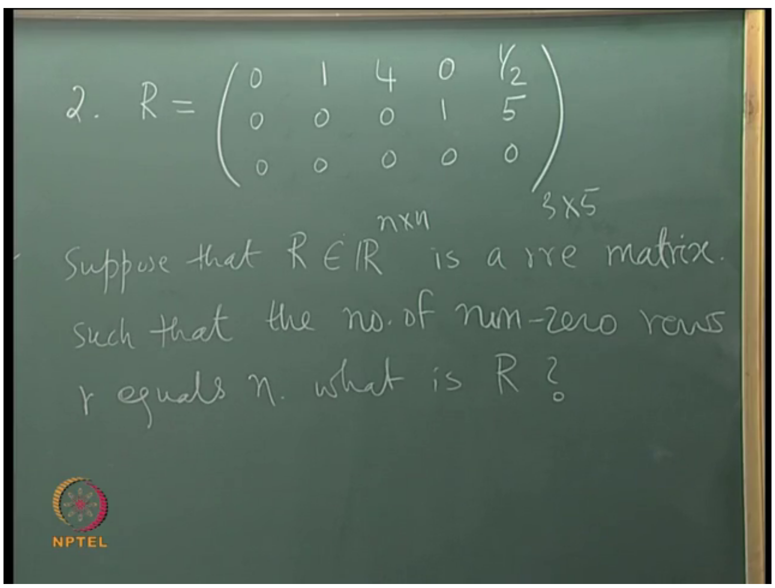


So let us say R a rectangular matrix with real entries m rows and n columns is called row reduced echelon matrix if it satisfies the following conditions, the first condition is that the leading non-zero entry of each non-zero row of R is 1 that is the first non-zero entry in each non-zero row of R as 1 this is the first condition, second condition look at each column corresponding to the leading non-zero entry of a non-zero row all the other entries along this column must be 0 this is the second condition.

So the statement is each column corresponding to the leading non-zero entry which is 1, okay so look at each column corresponding to the leading non-zero entry of each non-zero row of R has all its other entries 0, this is the second condition, okay. So look at the columns which have the leading non-zero entry of a non-zero row of R the entry all the other entries except this leading non-zero entry must be 0 this is the second condition we also have three more conditions so let me go to the other conditions.

(Refer Slide Time: 3:16)





Each 0 row of R occurs below every non-zero row of R and I said that this is like stacking the 0 rows of R at the bottom.

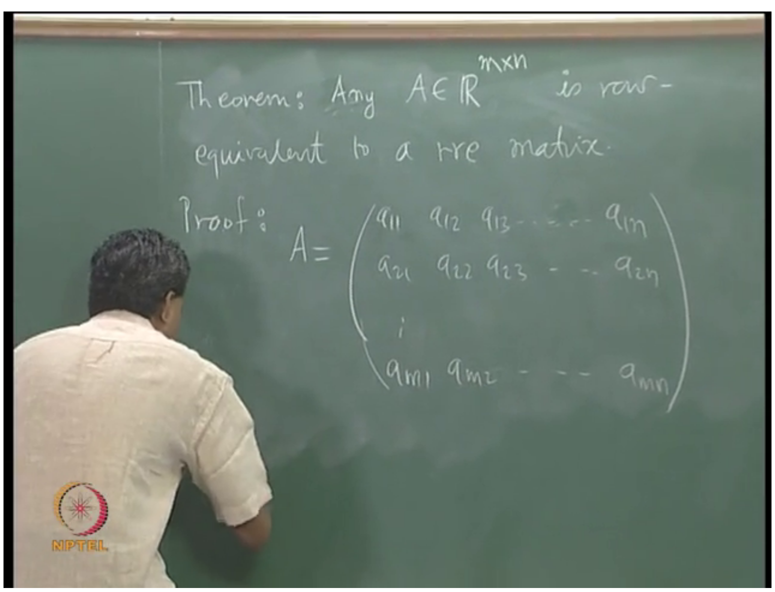
Condition 4, if i equals 1, 2, 3 etcetera r are the non-zero rows of R and if the leading non-zero entry occurs in column leading non-zero entry I think I need to make a slight change the leading non-zero entry of row i, okay that occurs in column Ci i equals 1, 2, 3 etcetera r then we require that C1 less than C2 etcetera less than Cr, okay so these are the conditions that a row reduced echelon matrix must satisfy, okay this we had seen yesterday, what I would like to do today is to discuss how a every matrix can be reduced to a row reduced echelon matrix see this is important how every matrix can be reduced to a row reduced echelon matrix that is what I am going to discuss today and also discuss some of the properties of row reduced echelon matrices, okay.

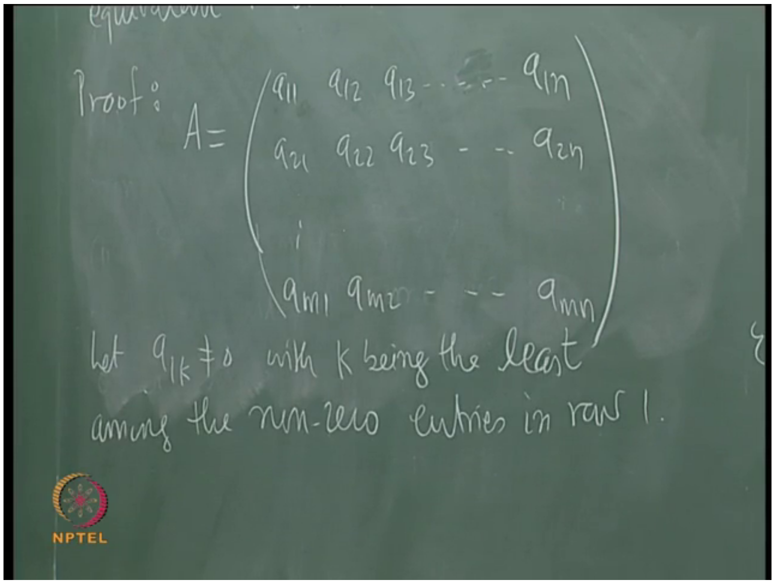
But before proceeding let us look at one or two examples, I have given a few yesterday let me give you a couple of examples today, identity matrix of any order, okay of any order this is a row reduced echelon matrix trivially let me give you one more example which we will also use a little later. The matrix I will call it R, R will here after stand for row reduced echelon matrix let us say I have something like this, okay 3 rows 5 columns this is another example of a row reduced echelon matrix, okay.

There is also another fact which we will need a little later you think over this question I will post this as a problem, suppose that R is a square row reduced echelon matrix with property that the number of non-zero rows, okay with the property that the number of non-zero rows r equals the order of the matrix I have a square row reduced echelon matrix with the property that the number of non-zero rows is equal to the order of the matrix, okay what can you say about R? Just think it over, okay we will come back and answer this question a little later, okay.

As I mentioned it is important to note that every matrix can be reduced to a row reduced echelon matrix, okay let me prove that now, okay.

(Refer Slide Time: 8:45)



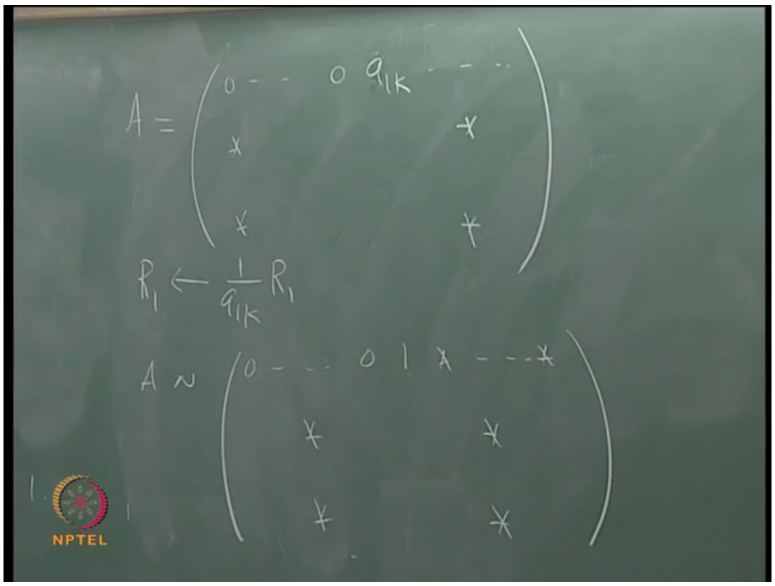


So what I will discuss next is the following theorem and its proof, okay let me just change this I will simply say every is row equivalent. Now that is a statement of a theorem so I would like to put it this way any matrix is row equivalent to a row reduced echelon matrix, okay by the way the reason why we need to reduce a matrix to a row reduced echelon form is that row reduced echelon form is much simpler matrix to handle than the original matrix A, okay you will see that a little later if you have not already seen it.

Remember that we are looking at the system Ax equal to b that system 1 we have reduced it to a system of the form Cx equal to d and we are looking at a particular structure for C, okay it is really this structure that we are seeking, we are seeking C to be a row reduced echelon matrix so we must know that every matrix can be reduced to a row reduced echelon matrix that is what we are going to prove now, okay. So proof let me write down the matrix A with its entries Aij as follows a11, a12, a13 etcetera a1n m rows n columns a21, a22, a23 etcetera am1, am2 etcetera amn, okay I want to do elementary row operations on this matrix and reduced it to a row reduced echelon matrix, okay.

We will handle this matrix row by row let us look at the first row if all the entries of the first row are 0 then I will go to the second row if all the entries of the second row are 0 I will go to the third row and so on, okay. So one possibility is that A is 0 that is trivially a row reduced echelon matrix, okay. Suppose A is not 0 and suppose the first row is not the 0 row, okay there is some non-zero entry, let us take the first one the first non-zero entry when you go from left to right the first non-zero entry when you go from left to right in the first row so I will call that as a1k let a1k be non-zero, and what is the property of k? k is the least k being the least among the non-zero entries in the first row among the non-zero entries I am looking at aik a1k k is the least a1k is not 0, okay which means to the left of a1k all entries are 0, okay so does a matrix a look like?

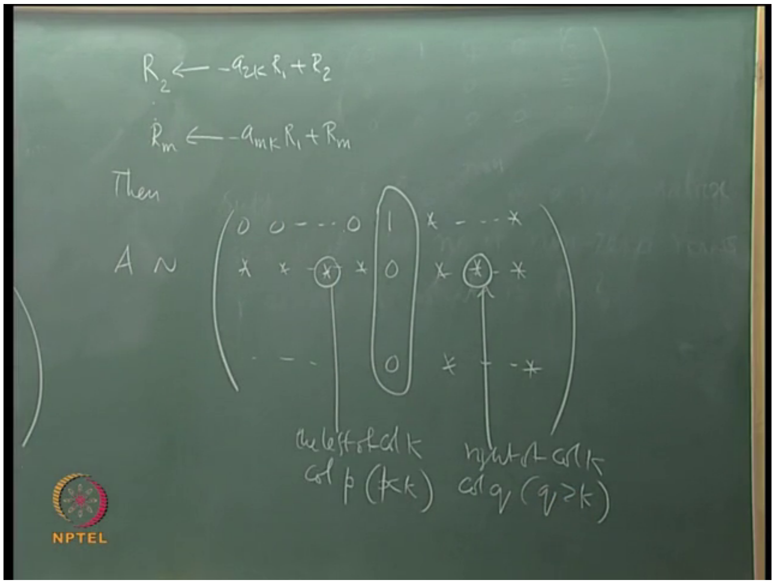
(Refer Slide Time: 12:50)



With this information the matrix A looks as follows, the first k minus 1 entries are 0 this entry is non-zero that is a1k the other entries are there as it is the other rows I will not disturb them now so they are left as they are I know that a1k is not 0 so I will do this first elementary row operations see what I want is that the leading non-zero entry of each row is 1 I know that this a1k is not 0 so I will do this first operation obviously look at 1 by a1k of the first row that is the first operation, upon this operation this is an elementary row operation, okay then A is row equivalent to this matrix all the entries to the left of a1k, okay they are all 0 the entry for a1k there has been replaced by 1 all other entries I will use as star to denote them the other entries are left as they are right now, okay that is the first step, next step.

Remember the condition that must be satisfied by a row reduced echelon matrix the second condition each column that has a leading non-zero entry has corresponding to some row has all other entries 0 which means I must make these entries 0 I must make the entries of the kth column of A below this 1 to be 0 but I know how to do it.

(Refer Slide Time: 14:30)



So that is that tells me what the next set of operations must be. Look at row 2 the new row 2 will be see this entry is a2k , okay etcetera this entry is amk entries in kth column. So I do these operations simultaneously, row 2 has been replaced by minus a to k row 1 plus row 2 etcetera row m has been replaced by minus amk row 1 plus row m, okay do this one after the other then the matrix A reduces to the following form A is row equivalent to this matrix 0, 0 etcetera 0 this entry a1k originally that was 1 leave the other entries of the first row as they are, now these entries I do not know what they are, okay this particular entry is 0 that much I know this entry is also they may be 0 they maybe non-zero but right now I am not concerned let me fill up the first the column corresponding to k that is the kth column of the original matrix a.

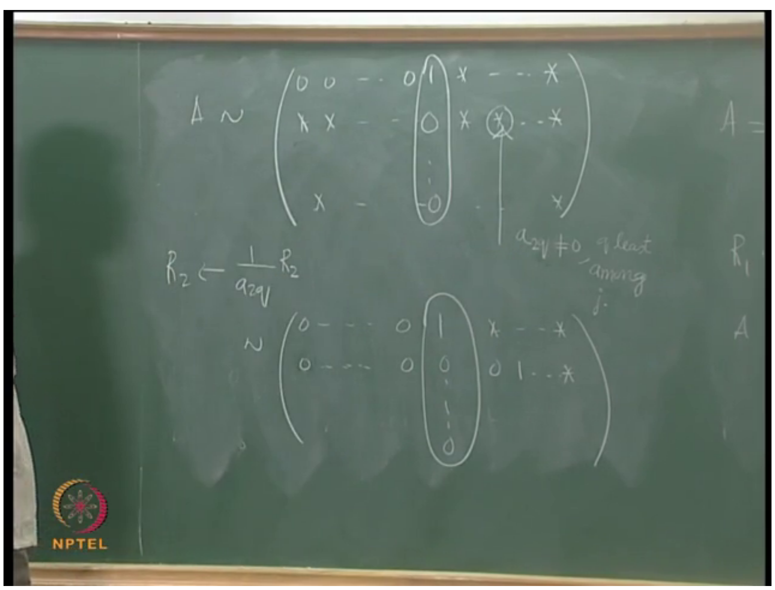
Now if you look at the entries in this kth column they are 1, 0, 0, 0, okay the first entry is 1 all the other entries are 0 and again these entries maybe 0 non-zero I am not concerned presently, okay so the first two conditions of a row reduced echelon matrix are satisfied for the first row at this stage, okay the first the leading non-zero entry of the first row is 1 each column containing the first column the column that contains a leading non-zero entry of the first row that is the kth column has all its other entries 0, okay. Now we are in two cases when I look at the second row, let me write down the matrix if all the entries of the second row are 0 I do nothing to the second row, okay presently if all the entries of the second row are 0 I do nothing to the second row I move to the third row if all the entries of the third row are 0 I do nothing I proceed like that.

If there is some entry of the second row which is non-zero, then there is one thing for sure this cannot occur in the kth column, if there is a non-zero entry in the second row that non-zero entry cannot occur on the kth column because kth column has been taken care of let me use this to denote the kth column kth column all the entries except the first row entry are 0. So the non-zero entry in the second row cannot be in the kth column, so it is either to the left of the kth column or to the right of the kth column I will consider these two cases separately, okay.

So let me introduce to arrows to denote the two possibilities, one happens somewhere here, okay this corresponds to the left of column k, okay I will call that p column p this happens in column p, p less than k, okay this is one possibility the other possibility is to the right that is the leading non-zero entry of the second row that is what I am looking at that either happens to the left of the kth column or the right of the kth column it cannot be the kth column that is clear now, so let us say this entry corresponds to the leading non-zero entry so I all call it to the right of column k I will call that q column q, q greater than k p or q cannot be equal to k that is clear.

So let us take these two cases separately and then see how this can be handled, let us take the case on the right, okay.

(Refer Slide Time: 18:49)

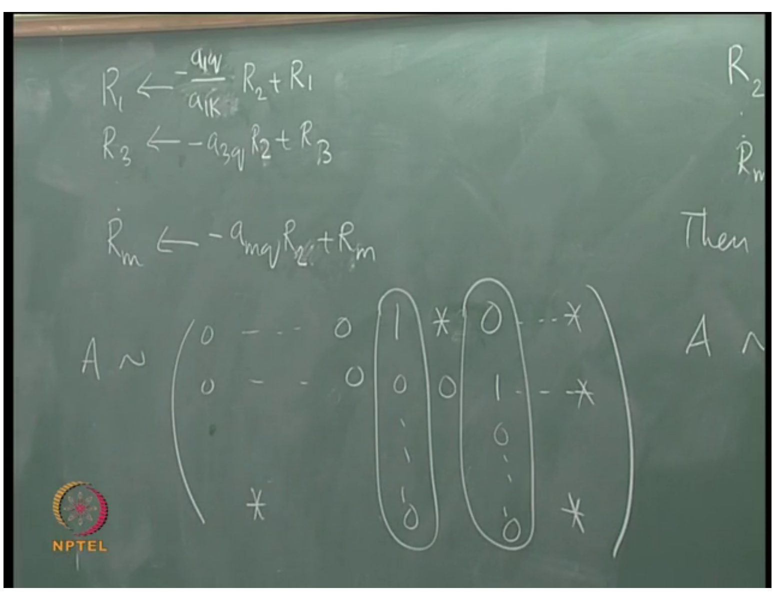


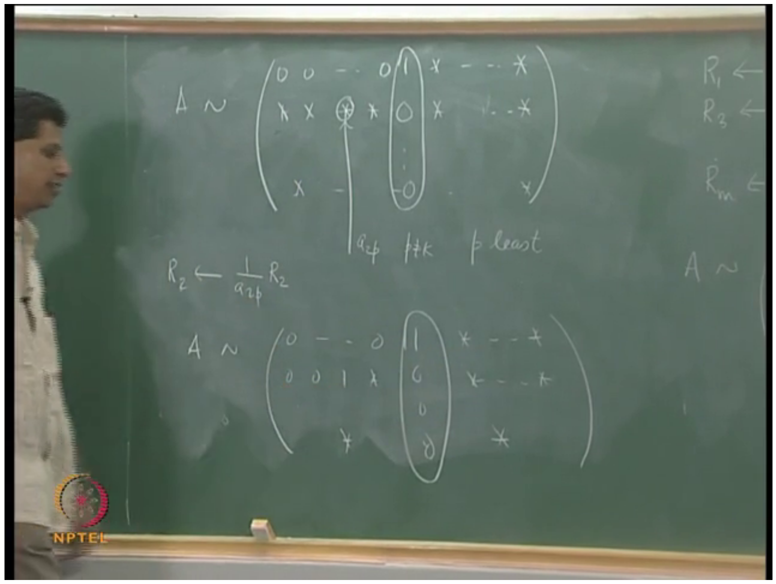
So let me write down this matrix once again, A is row equivalent to this matrix the other entries I will leave them as they are I know that these are 0 below 1, so this is the case we are this is non-zero, okay that is this corresponds to a2 I am calling that as q a2q is non-zero, okay leading non-zero entry q is the least among the second subscripts leading non-zero entry of row 2 q is the least among the second subscripts occurring in the second row, second row is we started with a21, a22 etcetera a2n they have not changed except a2k, okay.

So among these I take the one which is the first non-zero entry, so I am that is a2q I know that that is not 0, so what is the next operation? I will divide this second row by this number so the second row is replaced by 1 by a2q times row 2, okay so that this entry becomes 1 remember I am looking at the leading non-zero entry of the second row, okay. So this gives me the following matrix, 0, 0 etcetera 1 the entries along this column are 0 except the first one these remain the same the second row now changes I do not, okay the second row changes I know for certain that these entries are 0 this is also a 0 let me use the same picture here this entry is 1 all the entries other entries I am not concerned but I know for sure that these entries to the left of one are 0 because of the definition of a2q a2q is the first non-zero entry in the second row, so first non-zero entry second row which means all the other entries before that entry in the second row are 0, okay.

Then the rest of them are kept as they are presently then we do we need to do elementary row operations now using the second row, okay we need to do elementary row operations using the second row.

(Refer Slide Time: 21:52)





So what are these operations, okay that is clear we need to change the first row also because I must make this entry 0 this entry must also be made 0, okay only then the condition of the second condition corresponding to a row reduced echelon matrix will be satisfied corresponding to this leading non-zero entry of the second row, okay.

So I do these operations row 1 is see this entry is 1 and see this entry is as it is the original entry, so this is a1q, right? So I look at 1, I am sorry I look at minus a1q times row 2 plus row 1 I do a similar thing for all the other rows row 2 remains the same row 3 is minus this entry a3q row 3 plus row 1 etcetera row m minus amq row m plus row 1, yes you are right a1q divided by a1k yes, this is a1q divided by a1k, alright, yes we are doing the first operation is yes that is correct, first row has been replaced by the row divided by a1k the first non-zero entry of yes that is correct.

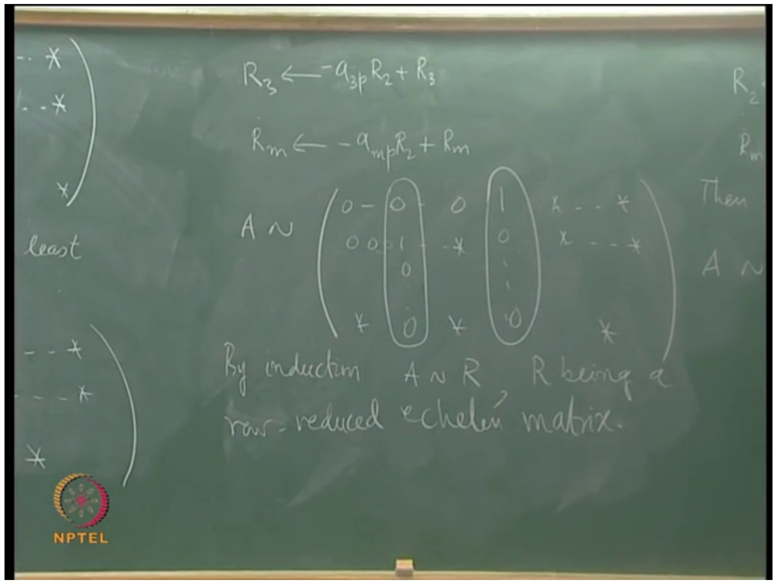
So this is a1q by a1k that is not 0 I want to make that entry 0 so I do these operations, okay I do these operations then A reduces to the following matrix all these entries are 0 this entry is 0 now, okay corresponding to the so let us say I put a star and then put a 0 leave the other entries I am not concerned about the other entries rather than all these entries are 0 1 occurs here this must be 0 other entries I am not concerned what I also know is that these entries now become 0 so first this column next column not concerned about the other rows presently, yes this is the pivot row is the second row, yes this must be R3 this must be Rm, yes the pivot row is R2 I am keeping R2 as the row with respect to which I do the other with respect to which I change the entries of the matrix, yes.

So this is what happens when the leading non-zero entry of the second row is to the right of column k that is it happens in column q, q is greater than k, okay. I need to look at the other case and then we will proceed by induction. What is the other case let me start all over from here, the second case corresponds to the leading non-zero entry being present here, okay this is a2p, okay that is a2p according to my notation p is not k and p is the least p is the least among the second subscripts of the second row entries of the second row, p is the least among the second subscripts of the entries of the second row.

So now it is to the left of k so I again do the same operation a similar operation row 2 has been replaced will be replaced now by 1 by a2p row 2, okay that will give me a 1 here so A is equivalent to, okay this is column k these entries I will not be concerned now this is the first non-zero entry so these two entries are 0 I have 1 here these are certain other entries these are not changed, is that okay? Is now I do now this is to the left of the kth column so I will not disturb the first row unlike what I did earlier, earlier it was to the right of the kth column and so I need to change this particular entry to make it 0, okay but now it is left of this column the 1kth the leading non-zero entry of the second row is to the left of column k and so these entries I know are already 0 so I do not touch row 1 now, okay I do the elementary row operations corresponding to row 3 etcetera row m using this second row to get the following matrix, okay.

So I am looking at row 2 has been row 1 has been fixed I do not change that now, row 2 has be fixed row 3 etcetera.

(Refer Slide Time: 28:02)



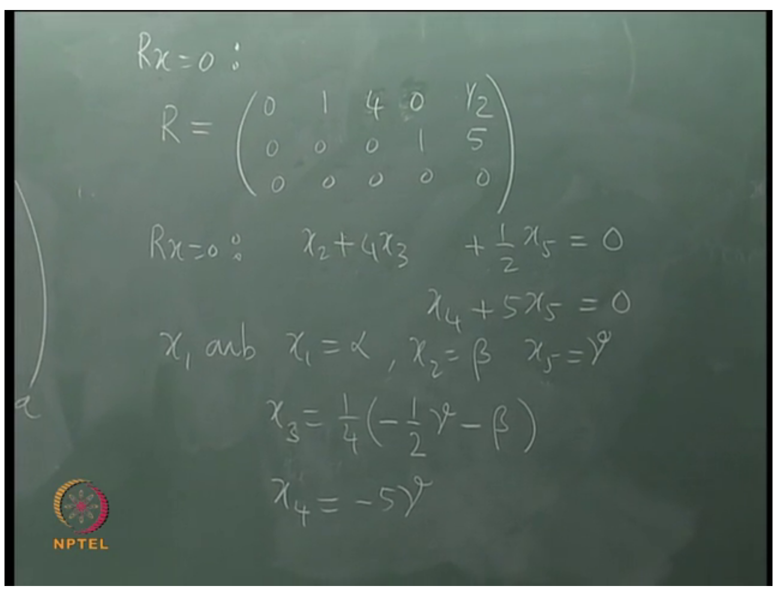
Row 3 for instance I must look at this entry that is minus a3q I am sorry a3p minus a3p times row 2 plus row 3 minus a3p times row 2 plus row 3 that is row 3 etcetera row m is minus amp row 2, row 2 has been used as so called pivot row amp row 2 plus row m to get the following matrix as a row educed rather the following matrix as a row equivalent matrix row equivalent to A first row entries they are 0, 1 here these entries are 0 first row entries are not disturbed I have a 1 occurring here 0, 0, 1 this entry 0 this is already 0 let us note that this is already 0 these entries have been made 0 now because of these operations these entries have been made 0.

So this is now corresponding this corresponding to the second row the leading non-zero entry appears in column p and I observe that the second condition is satisfies all the other entries are 0, okay. Now what is to be observed is the following when the leading non-zero entry occurs to the right of the column k, okay what we have observed is that the entries corresponding to the first row before the leading non-zero entry they are not changed any case we are, is that okay? If the leading non-zero entry of the second row occurs to the right of column k or to the left of column k the entries of the first row before 1 including 1 those entries are changed nor or the entries corresponding to the kth column under the entry 1 they are not changed in any case, okay.

So you right of k or to the left of k these entries do not change, so by induction one can show that these entries do not change when you do these elementary row operations, okay we proceed by induction go to the third row the third row if it is full of 0’s all 0’s then go to the fourth row etcetera if third row has a leading non-zero entry then do the same operation again and since there are n operation since there are m rows this procedure stops after you are down with the last row mth row, okay so this is the finite procedure in the last step you get a row reduced echelon matrix.

So I will simply say by induction A is row equivalent to a row reduced echelon matrix, yes I have not mentioned that but those, okay then the proof is not complete let me just include the following I will tell you orally it is still not row equivalent to a row reduced echelon matrix I need to look at these C1, C2 etcetera, right? And I also need to stack them in such a way that the zeros are at the bottom, okay but that can be done by row interchanges, okay so that does not much explanation at all that is, what is the third condition? Third condition is every 0 row must happen must occur below every non-zero row that can be done by row interchanges, okay what is the last condition? I have the first R non-zero rows and I have the leading non-zero entries occurring in column C1, C2 etcetera Cr these must be arranged in such a way that C1 less that C2 etcetera less than Cr, okay but this can again be done by row interchanges and so I get in the final step I get the row reduced echelon matrix R after doing these row interchanges if necessary, okay.

(Refer Slide Time: 33:42)

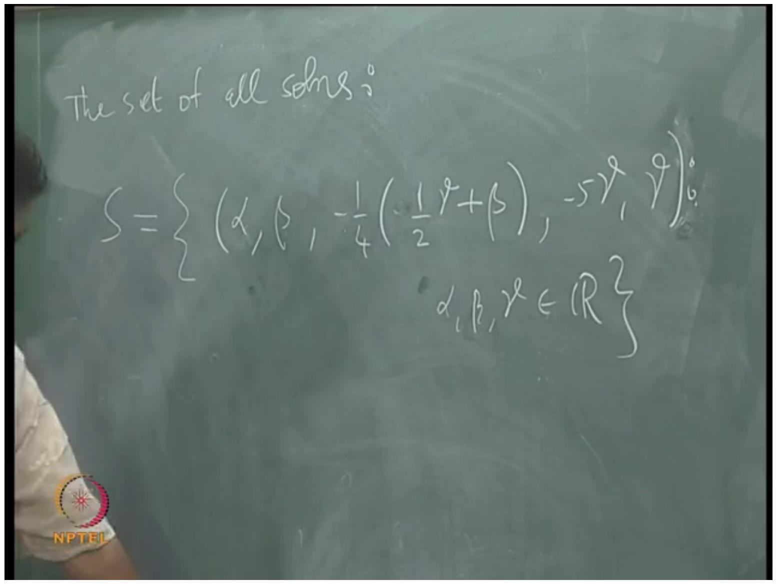


So let us consolidate by looking at a numerical example, I will take the row reduced echelon matrix that I had given at the beginning of the lecture I want to look at all the solutions of the system Rx equal to 0 where R is given by this matrix, please tell if entries are alright 0, 1, 0, 4 this entry is 2 go to this entry 4, 0, half, 0, 0, 0, 1, 5 and the last row is 0 this is a row reduced echelon matrix the leading non-zero entry of each row is 1 each column containing the leading non-zero entry of any row has all its other entries 0 non-zero rows are stacked above the 0 rows.

So this is a row reduced echelon matrix I want to look at the set of all the solutions of the system Rx equal to 0 homogeneous presently, Rx equal to 0 is the same as if you expand you get the following equation this is x2, x2 plus 4x3 plus 1 by 2x5 that is 0 first equation, second equation is this is x4, x4 plus 5 x5 is 0 the last row is 0 so it does not impose any condition on the unknowns X1, X2 etcetera X5, okay. So what I do now is what is clear that x1 is arbitrary x5 can also be chosen to be arbitrary all the other unknowns can be determined there are five variables essentially two equations I need to fix three.

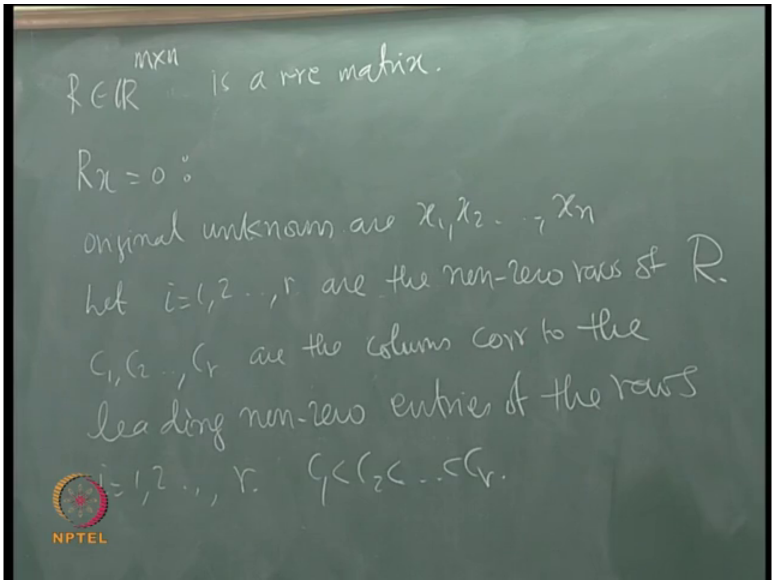
So x1 is arbitrary I will call it alpha x5 is arbitrary but before that I will also observe that x3 or x3 can be chosen to be arbitrary either x3 or x2 let us say I choose x2 I will call that beta x2 is also arbitrary finally x5 is arbitrary x5 can be chosen as any real number so I will call that gamma then with these choices for x1, x2, x5 I can determine x2 and x4 uniquely for this choice I can determine x2 and x4 uniquely, so x2 turns out to be what? Minus 1 by 2x5 minus 1 by 2 gamma minus 4x3, yes x2 is beta I want x3, x3 is 1 by 4 times, okay let me write this, okay I am looking at the first equation 1 by 4 times minus 1 by 2 gamma that is this term minus x2 minus beta that is x3 for me, okay please check the calculations and x4, x4 is minus 5 x5 minus 5 gamma, okay.

(Refer Slide Time: 37:26)



You substitute x1, x2, x3, x4, x5 into the two equation verify that they are satisfied, so I will write down the set of all solutions of the system Rx equal to 0 in this example I will call that S it is the set of all x1, x2, x3, x4, x5, x1 is alpha, x2 is beta, x3 is 1 by 4 so I will write it as it, okay I can take minus minus 1 by 4 minus 1, sorry half gamma plus beta that is a third component the fourth component is x4 minus 5 gamma the fifth component has been fixed to be gamma alpha beta gamma are arbitrary real numbers so this tells us immediately that there are infinitely many solutions I can rewrite it but I will leave it as it is, yeah I must close this this is the set of all solutions, for this homogeneous system, okay.

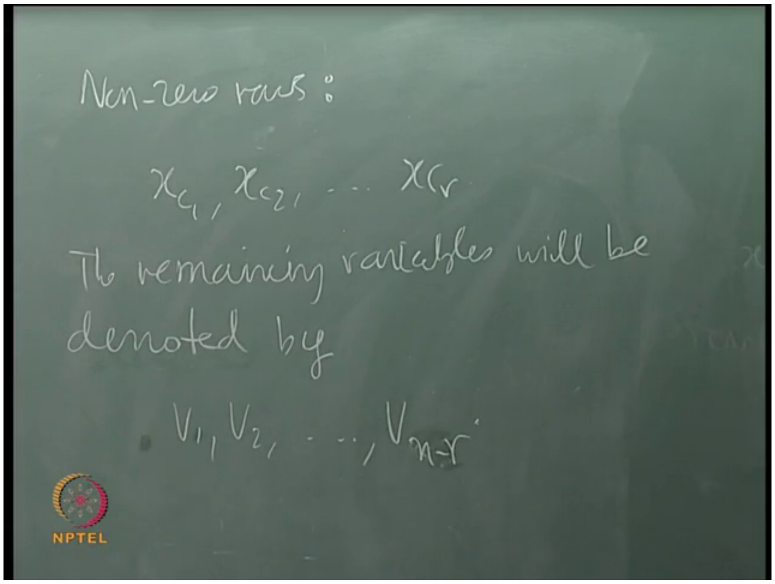
(Refer Slide Time: 39:00)

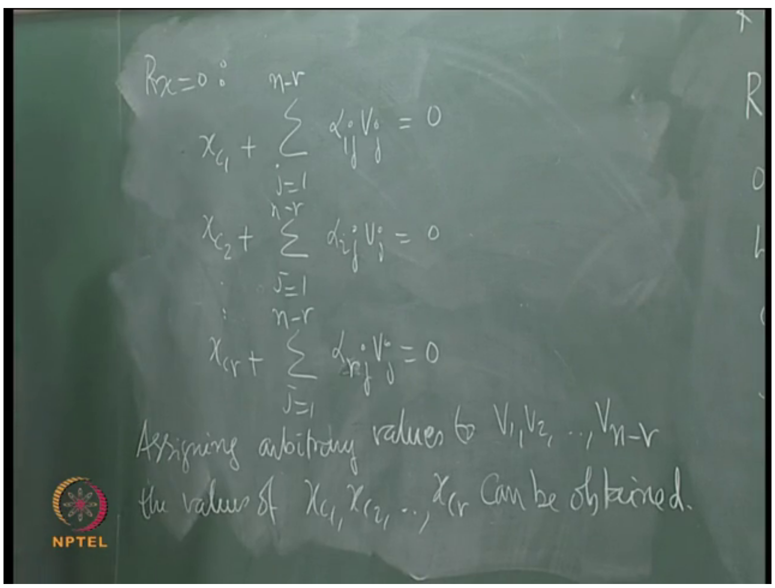


Let us now look at some of the other properties of row reduced echelon matrix, okay but before that using this numerical example as a motivating example let us also look at the general case, okay which is R is a row reduced echelon matrix I want to look at the system Rx equal to 0, okay to formalize what I have done for this numerical example R is a row reduced echelon matrix I want to look at the set of all solutions of Rx equal to 0, okay let us the original variables are the original unknowns are x1, x2 etcetera xn I need to re-label them, okay let us say that i equal 1, 2, 3 etcetera r are the non-zero rows of r, R is a row reduced echelon matrix the non-zero rows are at the top I am assuming that they are r in number, I also have these numbers C1, C2, C3 etcetera, okay etcetera Cr are the columns corresponding to the leading non-zero entries of the rows 1, 2, 3 etcetera r, okay I am just confirming that this is a notation we have adopted, we also know that C1 less than C2 etcetera less than Cr.

Let me now re-label the unknowns as those that corresponds to the non-zero rows and those that do not, re-label the unknowns as the rows that correspond to the non-zero rows.

(Refer Slide Time: 41:19)



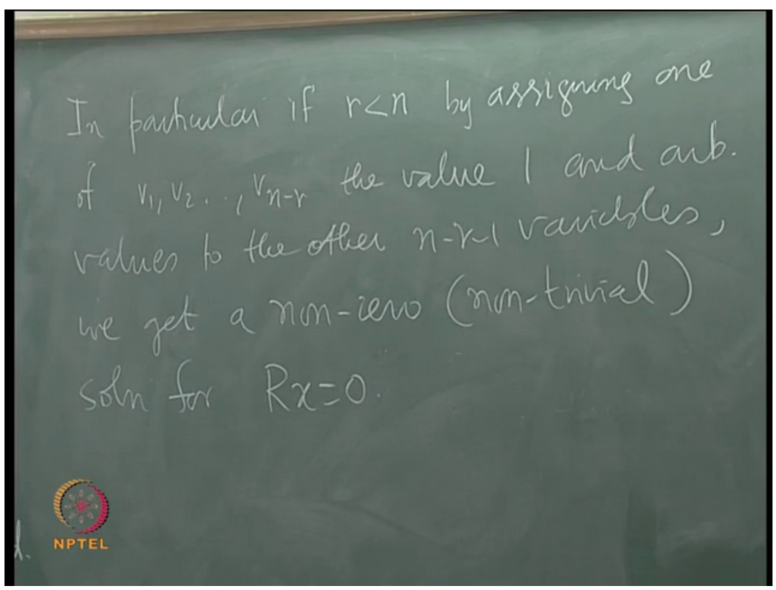


Those that correspond to the non-zero rows I will denote them naturally by xc1, xc2 etcetera xcr I am re-labeling the unknowns that we started with x1, etcetera xn xc1, xc2, etcetera xcr naturally correspond to the unknowns with respect to the non-zero first r non-zero rows of r capital R the remaining will be named the remaining variables or unknowns will be denoted by v1, v2 etcetera v n minus r the remaining variables will be denoted v1, v2, etcetera v n minus r.

So there are r corresponding to the non-zero rows and there are n minus r that correspond to the other rows, see I am looking at the number of variables then can you see immediately that Rx equal to 0 is precisely the following let me write that here Rx equal to 0 written in full gives me the following xc1 plus summation j equals 1 to n minus r let me say I have something like alpha 1 j vj equal to 0 that is the first row, second row xc2 summation j equals 1 to n minus r second row so I will call them alpha 2j vj equal to 0 etcetera xcr plus summation j equal to 1 to minus r alpha mj there are m equations this is rj, I am sorry alpha rj vj equal to 0 these are the equations corresponding to the first r non-zero rows of capital r the other equation do not appear they do not impose any condition on the unknowns because the correspond to 0 rows, okay.

So what this says is that you can fix these variables assigning arbitrary values to v1, v2, etcetera v n minus r the values of xc1, xc2, etcetera xcr can be obtained you assign arbitrary values to the so called free variables these are called the free variables v1, v2, etcetera v n minus r n minus r of them you can determine the other r variables xc1, xc2, etcetera xcr using these equations by assigning arbitrary values to these n minus r variables, okay that is what we have done in the previous problem.

(Refer Slide Time: 45:15)



In particular we have the following in particular if r is less than n, okay in particular of the number of non-zero rows of capital R is less than the number of unknowns, okay what this means is that there is at least one of these variables v1, etcetera vn minus r, okay n minus r is positive that means I have at least v1 at least one free variable, okay let me say I assign one of them the value 1 in particular of r is less than 1 by assigning 1 of v1, v2, etcetera v n minus r the value 1 I pick any of these n minus r variables I pick one of them give the value 1 to that variable and I can determine assign arbitrary values to the other variables and arbitrary values to the other n minus r minus 1 variables by doing this we get a non-zero it is called a non-trivial solution we get a non-zero solution for the system Rx equal to 0, okay.

This is an important observation if the number of non-zero rows of the row reduced echelon matrix capital R is strictly less than the number of unknowns appearing in the problem then the homogeneous system Rx equal to 0 has at least one non-zero solution, do you agree? Okay, I have given the proof here pick one of the free variables assign the value 1, you know that there is at least one free variable because r is less than n, okay.

Now this leads to a fundamental theorem that if you have a rectangular system of homogeneous equations with the number of equations being strictly less than the number of unknowns then it always has a non-trivial solution, okay that I will prove in the next class and also discuss other properties of row reduced echelon matrices.