Linear Algebra
Professor K.C Shivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Module 14-Self-Adjoint, Normal and Unitary Operators
Lecture 48
Unitary Operators

We are discussing the notion of isomorphism's on inner product spaces ok so an isomorphism on an inner product space let us recall must be a vector space isomorphism which also preserves the inner product ok and we had seen last time that a linear transformation preserves inner product if and only if it preserves the norm ok and we had also seen a characterization of isomorphism's on inner product spaces ok. Lets discuss the notion of unitary operators next, unitary operators.

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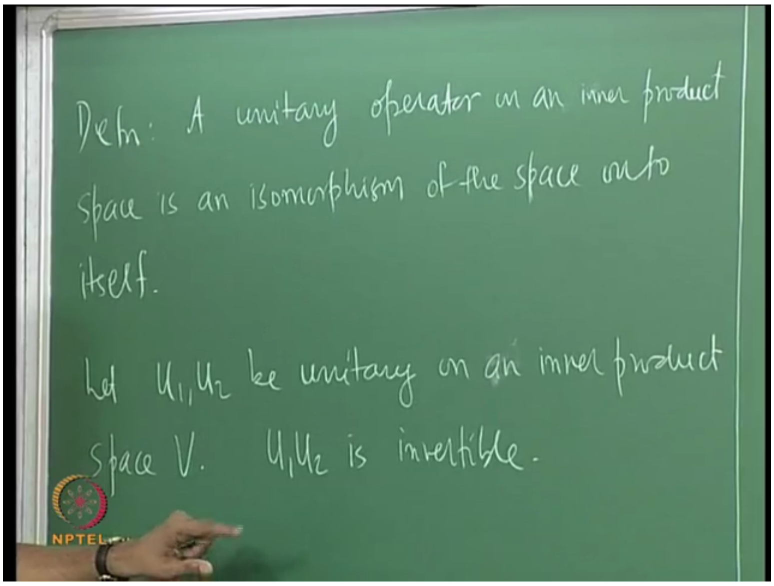

I remember having given the definition but I will recall this definition. A unitary operator on a vector space on an inner product space is an isomorphism of the space onto itself.

It is an isomorphism of this as an inner product space so it is a vector space isomorphism and preserves inner products. So that is the definition of a unitary operator. We will look at one or two results for unitary operators some examples and then for a linear transformation what happens to the matrix of the linear transformation when you change from one basis to another, this has been studied earlier we look at what happens when the basis are orthonormal basis ok. What are the properties of unitary operators? Suppose you have let U1, U2 be unitary operators on an inner product space V, let me say on an inner product space V.

Look at the ok then look at the product will show that the product is also unitary for one thing the product U1 U2 is invertible see the claim that I am trying to make here is that if U1 and U2 are unitary then I want to show that the product is also unitary. Now what is a unitary map? It must be an isomorphism so it must be a vector space isomorphism and must preserve inner products. For the product to be a vector space isomorphism it must be invertible but this is invertible because U1 and U2 are unitary so they are invertible the product is invertible infact there is a formula for the inverse also U1 U2 inverse is U2 inverse U1 inverse.

So the product is invertible I will next show that the product preserves inner products then it will mean that this product means composition, composition of unitary maps, also look at norm of I want to show that U1 U2 preserves inner product but we have seen that a mapping preserves inner products on a vector on an inner product space if and only if it preserves norms. So I will simply (so) show U1 U2 preserves norms.

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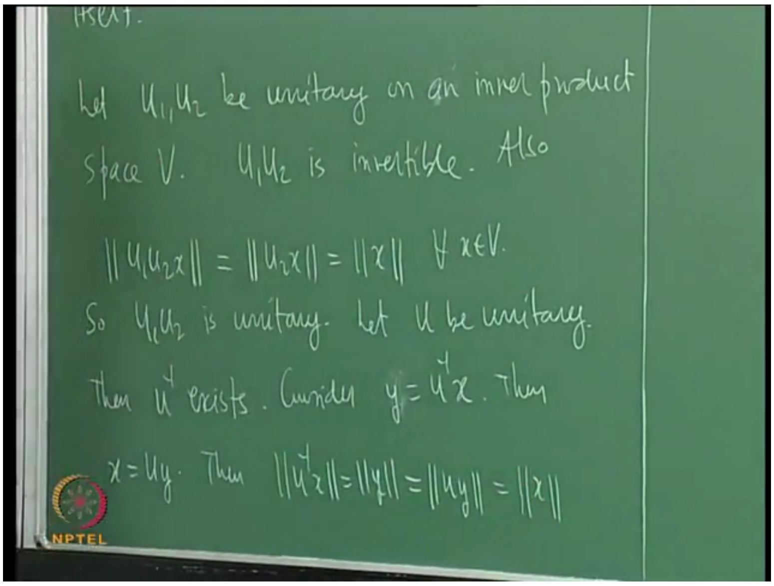

So look at U1 U2 x this is U1 of U2 x so since U1 is unitary it preserves inner it preserves a norm, it preserves inner product so it preserves norm. So this must be norm U2 x, because U1 is unitary, again U2 is unitary so this must be norm x and so you have shown norm U1 U2 x is norm x for all x.

And so U1 U2 preserves norm so it must preserve the inner product, this is the norm induced by the inner product ok. Ok so the product is so U1 U2 is unitary so the set of unitary operators on a vector space on an inner product space it is closed with respect to composition, what about inverse? Is the inverse also unitary? Let U be unitary then U inverse exists because it is an isomorphism then U inverse exists. We want to show that U inverse preserves norm so you consider norm of ok consider y to be U inverse x. Composition, these are linear transformations so whenever I write this it is a composition, when I write down the matrix of these transformations then it is a matrix product.

So here it is a composition operation, I want to show that U inverse preserves norm so it will follow that U inverse preserves inner products since U inverse-inverse exists it will follow that it will then follow that U is a unitary operator. So consider y equal to U inverse x, I want to show U inverse preserves norm, this means x equals U y I want to show norm U inverse x equals norm x ok. Then we have norm U inverse x equals norm y but norm y is norm U y because U is unitary but U y is x ok. So this is true for all x. So I have shown that norm U inverse x equal to norm x, U inverse is offcourse invertible U inverse-inverse is U.

So it is a vector space isomorphism that preserves inner product so it preserves norms so U inverse is unitary.

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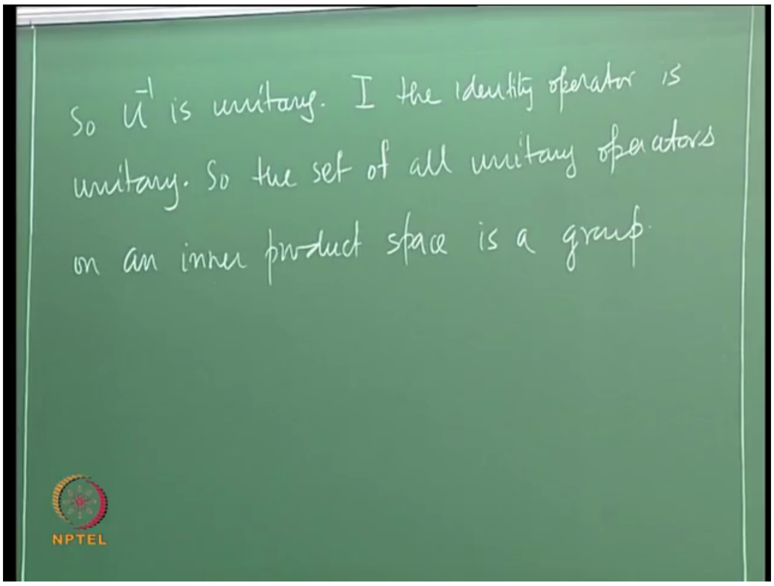

What about the identity operator, is that unitary? Obviously identity is unitary I the identity operator is obviously unitary, so what follows is that, the set of all unitary operators on inner product space forms a group. The set of all unitary operators on an inner product space is a group is a multiplicative group where the multiplication is defined as a composition of the operators, so is a multiplicative group offcourse its not necessarily (())(8:13)

Ok remember this result that we proved last time, if T is a linear transformation from V into W dimension of V is equal to dimension of W, V and W are offcourse inner product spaces then T preserves inner product is the same as saying that T is our inner product space isomorphism that is the same as saying that T takes an orthonormal basis to an orthonormal basis, which is the same as saying that T takes every orthonormal basis to an orthonormal basis onto ok.

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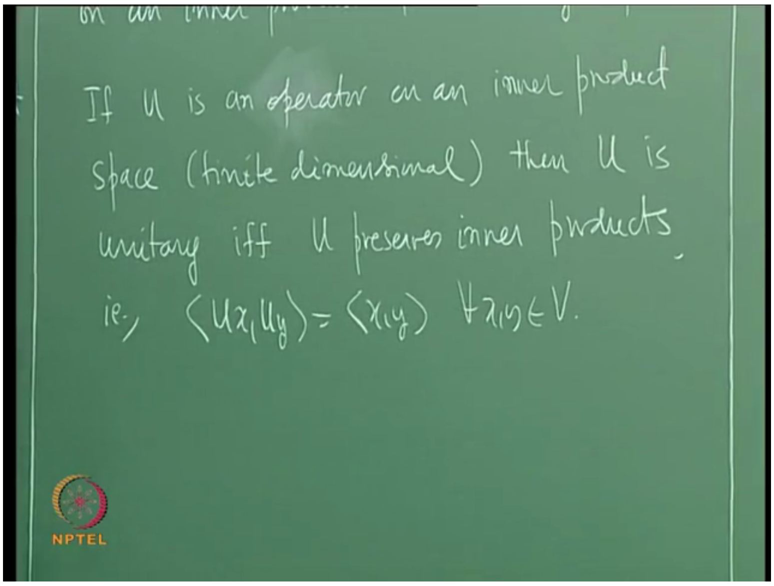

So what follows is that if U is a unitary operator ok just an operator if U is an operator on an inner product space finite dimensional.

If U is an operator on a finite dimensional inner product space then U is unitary if and only if U preserves inner products. So just let me recall what it means when we say that an operator preserves inner products. Inner product U x U y is equal to inner product x y this must be true for all x y ok this follows easily. U is an operator on a finite dimensional vector space now dimensions coincide then U is unitary if and only if U preserves inner products this follows from that theorem ok. Let’s also recollect the notion of the adjoint of an operator ok, the adjoint of an operator we proved in the case of finite dimensional space ok.

There is characterization of unitary operators in terms of the adjoint operation, so that is what I want to discuss next. So this next result holds in general in an infinite dimensional space ok. So the result is dimension free, so what I am trying to say is that there is a connection between the unitary operators on a general vector space and the adjoint operation, so that is the following.

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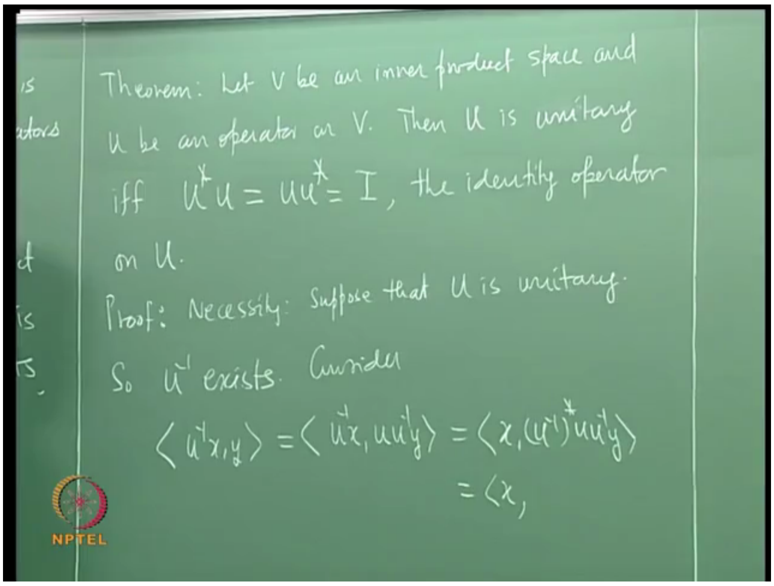

So I will prove this is as a theorem. Let V be an inner product space and U be an operator on V linear operator on V then U is unitary if and only if U star U equals U-U star equals I the identity operator on U.

V is an inner product space not necessarily finite dimensional. In order to see remember we have shown that the adjoint operator of a linear operator exists on a finite dimensional space ok. So we are already talking about the and remember that in general the adjoint operator over an infinite dimensional space does not exists ok but we are already talking about adjoint operator on a possibly infinite dimensional inner product space, that is possible if the operator is unitary, that is what this result says ok proof, suppose U is unitary there are two implications (right).

So let's prove the first part, necessity, suppose that U is unitary, then what is a definition of a unitary operator? It is an isomorphism of the inner product space onto itself. So in the first place it is an invertible linear transformation, so U inverse exists. We must show that U preserves inner products then it follows that U is then it follows I am sorry given U is unitary we want to show that this identity holds. Now this two equations hold, now this two equations can you figure out that it is the same as saying that U inverse equal to U star ok. So we will just show that U inverse equal to U star, U inverse exists the claim is that U inverse equal to U star ok.

So you consider I want to show U inverse is U star ok. Consider U inverse x, y I can write this as U inverse x, lets see, U-U inverse y no I want to show U star U equal to identity , what is given is that U is unitary so this is yeah, U star is adjoint, star is always the adjoint operator. Do you first agree with this? I will just applied the fact that U inverse exists so y is written as U-U inverse y so but ok I need to show U inverse star but ok U inverse star is U star inverse lets see this is x, U inverse star U-U inverse y this is x, U inverse star is U star inverse ok I think it is not easy I mean I could have I can write U star inverse but first of all I want to show that star exists right so ok lets see I need to give a different proof.

I want to show U inverse is U star and write it as ok.

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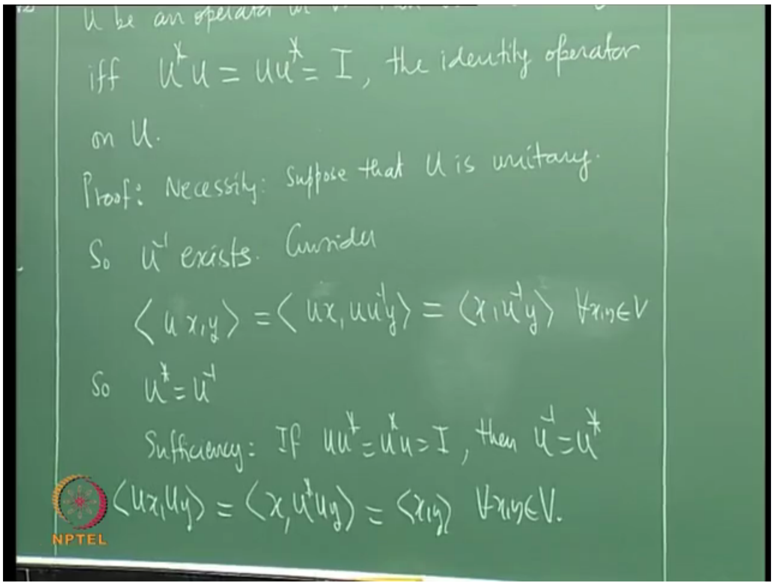

I should have started with inner product U x, y yes that is the same as U x U inverse y ok, that is the same as U x U inverse y U is unitary so this U kind of can be removed this is x, U inverse y, yes, is x, U inverse y so what this means is that by the definition U star is that unique operator which satisfies the equation U x, y equals x, U star y but what we have shown is that U x, y is x, U inverse y. So it means that U star must be U inverse. So if U is unitary then U star is equal to U inverse so this two equations hold trivial in that case, that is the first part.

Sufficiency, if U satisfies these two equations then I want to show that U is unitary for one thing U is invertible if U-U star equals U star U equals identity then it follows that U inverse equals U star. So U is invertible I will simply show U preserves inner products ok that will prove that U is unitary, so it is invertible I need to show that it preserves inner products. So consider U x, U y this is x, so what we know is that U star is U inverse so I will first write x, U star U y, U star us U inverse so this is just x, y. So U preserves inner products, U is invertible and U preserves inner products so U is unitary

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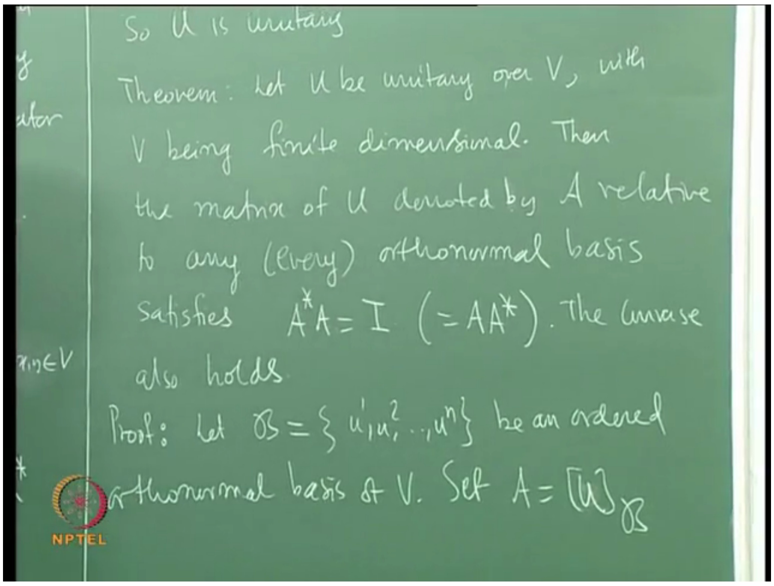

Ok so its intimately connected to the operation of taking the adjoint U star. U star exists and U star is equal to U inverse for a unitary operator. The original definition is just an isomorphism of a (vec) inner product space onto itself. Isomorphism over vector spaces plus the fact that it preserves inner products ok. So this is dimension free, ok, before I look at examples I want to look at the matrix case that is we have it is really a little theorem over finite dimensional spaces what happens to the matrix of unitary operator with respect to an orthonormal basis. Let U be unitary over V with V being finite dimensional then the matrix of U denoted by A relative to any, any means every orthonormal basis then the matrix of U denoted by A.

So A is a matrix a few relative to an orthonormal basis satisfies A star A equals identity. What will also follow is that is equal to A- A star. See this is like a square matrix acting on a finite dimensional space so if it has a left inverse then it must have a right inverse and those two must coincide ok. So there is last part comes from that result. The converse is also true, the converse also holds. What is the converse? You take a matrix that satisfies the equation A star A equal to I and look at a unitary operator which equals look at the unitary operator which satisfies a property that there is an orthonormal basis such that the matrix of this U with respect to that (ormal) orthonormal basis is A then that U must be unitary ok that is a converse.

Take a matrix A that satisfies this equation and then look at the unitary operator it can be this can be easily defined all that you have to do is look at the operator U that satisfies U x equals A times x. This U has a property that the matrix of U relative to some basis will be equal to A then you can show that the unit you can show that the operator U is unitary ok, so that is the converse. So proof, by the way an operator that satisfies this equation is called I am sorry, a matrix that satisfies this equation is called a unitary matrix. Matrix coming from a unitary operator relative to an orthonormal basis is called a unitary matrix.

If the space is real it is called an orthogonal matrix. The space is a real space then it is called an orthogonal matrix ok. I want to show that this is satisfy, converse is also easy ok. First part, I must show A star A equals identity. Lets take a basis, script a B, B be an ordered orthonormal basis relative to any ordered orthonormal basis our basis will always be ordered. Let B equal let me use U1 U2 etc U n let B be an ordered orthonormal basis of the vector space V. Call the matrix of U, U is the unitary operator that is given to me, call the matrix of U relative to this basis as A. We must show that this A satisfies this equation.

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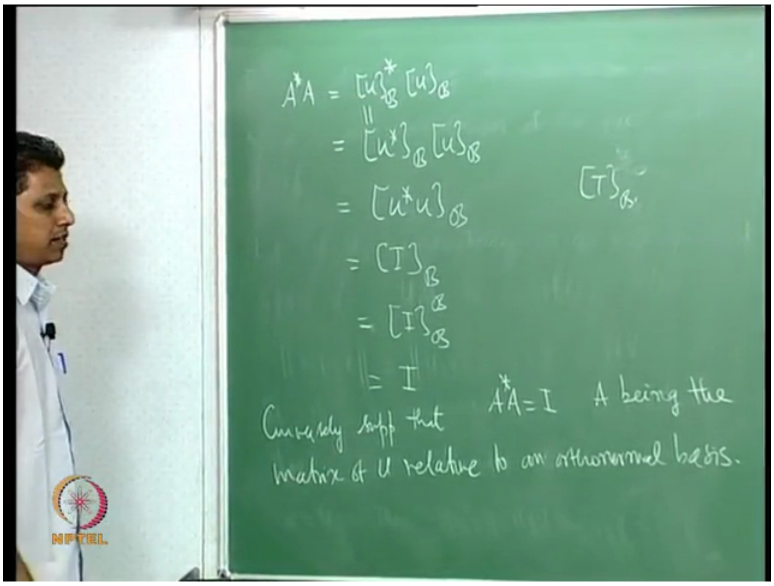

Ok look at A star A, this is matrix product, A star is the matrix of U relative to B star into the matrix of U relative to B, this is matrix multiplication. But this is we have seen last time that this star can be taken inside it is U star relative to B into U relative to B. The matrix of U relative to B. This is the product of two matrices this will correspond to the composition this will correspond to the matrix of the composition of U star U which we have done even for finite dimensional spaces not there is no inner product involved here. Ok but then U star U is U is unitary so U star U is identity. We have proved it last time.

So this is identity relative to B, identity relative to B simply means indentivity with respect to B-B right that is in our notation it is a same basis with respect to which we are doing this. Which is the identity matrix ok. Remember the notation that we have employed, in general we talk about the matrix of T relative to two basis B1 B2. When B1 is equal to when T is a linear operator on a vector space then it makes sense to use only one basis, in that case B1 equals B2 in that case will simply say T B, so that is the notation I am using here.

Identity B means identity, you write every basis element in terms of the basis element and write down the matrix that is the identity matrix. So A star A equal to I that proves the first part. The matrix of U relative to any orthonormal basis will satisfy the equation A star A equal to identity. Converse, conversely suppose that A star A equals I and U is a matrix of A relative to some basis, A is a matrix of (())(26:50) relative to some basis ok A being the matrix of U relative to an orthonormal basis that is so U x equals A x corresponding to that basis. In which one, first part? U star U equal to identity using orthonormal basis I don't think I have used, that is only one property that I have used that this product will be this one otherwise I have not used orthonormal basis.

Ok there is one place where we have used, in writing this down, this is not true in general please observe this is not true in general because see I have made this remark I remember the time when we discussed I think I called it A and B, if A is the matrix of U relative to an orthonormal basis B and B is a matrix of U star relative to the same orthonormal basis then A is equal to B star. If you use that is this is a very simple relationship between orthonormal between the matrices of U and U star corresponding to the same orthonormal basis. If you don't take an orthonormal basis then this relationship does not hold anymore, so that is where is being used.

For this you don't need unitary, this is true for any operator see this relationship is true for any operator provided B is an orthonormal basis that is correct, U star, U is a unitary is being used here when I wrote down U star U equal to identity see this identity transformation inside is identity transformation that is why because U is unitary yes, so it is being used here, that is you can take a simple example of a 2 by 2 or a 3 by 3 vectors space and the transformation over that and then determine the (ortho) determine the matrix of U relative to an ordinary, basis compute U star, write down the matrix of U star relative to this ordinary basis not necessarily orthonormal then they are not related like this ok yeah so that is why this is being used.

Ok the converse, converse is this matrix A is the matrix relative to it is a matrix of U relative to some orthonormal basis it satisfies this equation I must verify that U is unitary ok.

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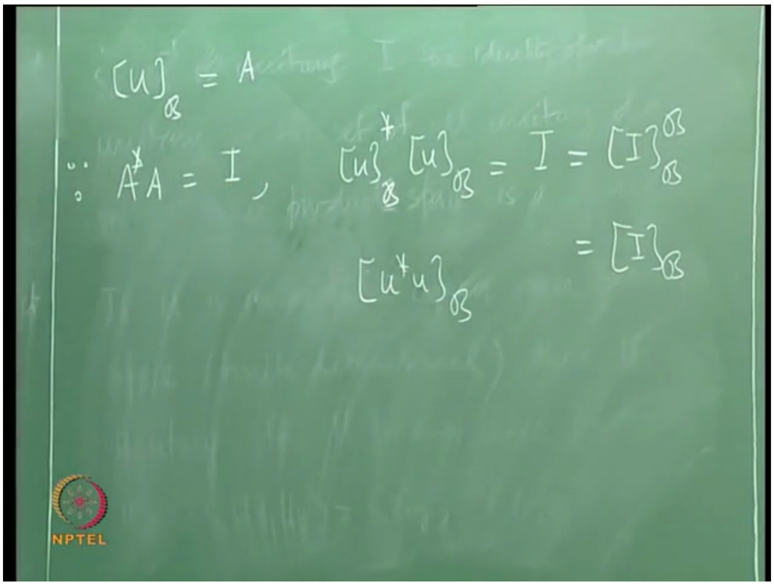

So U matrix of U relative to that orthonormal basis U relative to an orthonormal basis I will call it B, relative to the orthonormal basis which I will call script B. So the matrix of U relative to B is A by definition. Now look at A star A, this is equal to identity. Since A star A equal to identity we have A star is the matrix of U relative to B star U B equals identity matrix. Left (hand) but identity matrix is as I have written there it is identity operator relative to the same basis ok which the short notation is identity operator relative to just these spaces.

You remember this that if you take the identity operator and then take two basis then the matrix will not be identity ok. The matrix of the identity linear transformation with respect to two different basis will not be identity but what's important is that, the matrix will be invertible if two basis are the same then the matrix of the identity transformation relative to this basis is the identity matrix ok. So this right hand side is identity matrix. On the left hand side I have again B is an orthonormal basis so the star can be taken inside it is U star relative to B that is equal to identity relative to B.

See both sides I have matrices but what is important is this identity operator inside this is operator product U star U, operator composition U star U. So can you see from this that there is a basis, what is the meaning of this statement?

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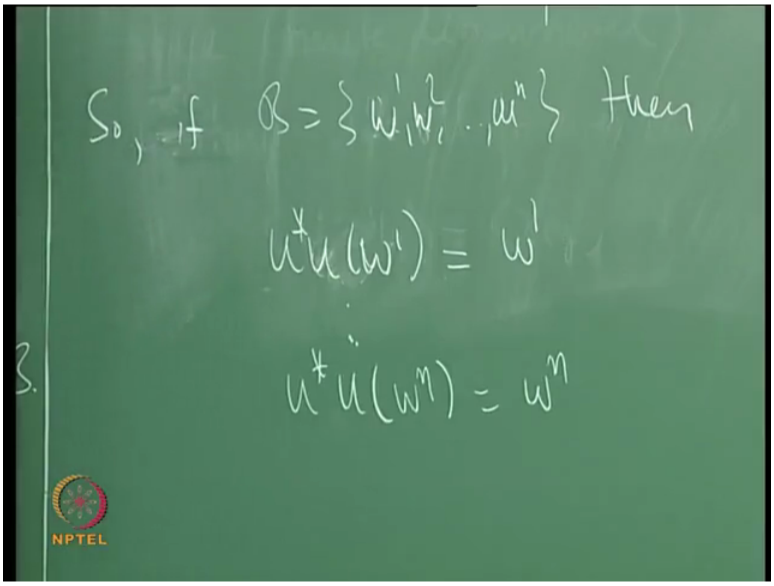

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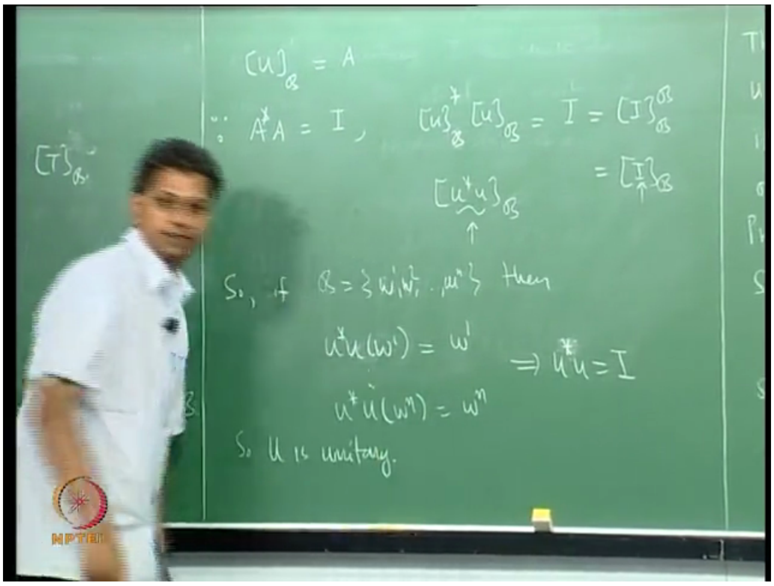

The meaning of this statement is that, if just write it elaborately, if B is lets call U1 U2 etc U n then this equation tells me that U star U of ok U1 is not a very good notation. Let's say W1 W2 etc W n then U star U W1 must be equal to W1 because of this etc U star U of W n must be equal to W n. This happens if this happens then this must hold, otherwise you won't get this right if this doesn't happen this is not true. So from this it, this is a basis, W n etc W n is a basis so it means from this U star U is identity. U star U is a identity operator the same thing you can do for U-U star to conclude that U star equals identity. So what we have proved is U is unitary ok.

Student: (())(33:42)

Professor: Composition.

Student: (())(34:01)

Professor: But that is not the definition, U star of U of x is not U star of x into U of x, infact there is no meaning to that. See U star x into U x means what? U star x is a vector U x is another vector so into means what? The multiplication means what, there is no meaning to that. It is always a composition operation. This is the composition operation of all base ok and see there is no complex number involved it is just the adjoint operator and it is the composition of the adjoint with the operator U, so there is no complex number involved here.

These are equations involving operators, U star U is an operator defined by U star U of X equals U star operating on U of x ok. So this is what we have then the matrix of a unitary operator relative to an orthonormal basis satisfies the equation A star A equal to A star equal to identity the converse is also true ok.

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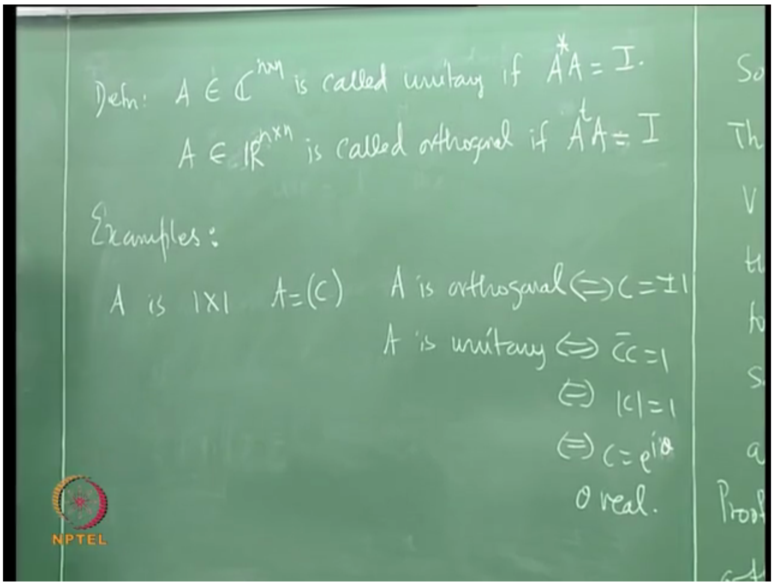

Lets now look at some examples ok before that I will just put down this definition A in C cross n is called unitary if A star A equals identity so this is just for matrices unitary matrix.

A element of R n cross n is called orthogonal if A transpose A equals identity that is if the space is real space and if you take a unitary operator and write down the matrix of the unitary operator relative to any basis then lets call it A then A star is actually A transpose, real space

A star is A transpose so this will hold, such matrices are called orthogonal matrices. So when I say orthogonal matrix is the real space in general unitary means it is a complex space. So lets look at some examples, examples of unitary matrices. The one dimensional case A is 1 cross 1 so there is just one vector C, A is just this then the real case A is orthogonal if and only if C equal to plus minus 1.

A is unitary if and only if, what is the adjoint in that case? What is the adjoint of? See what is C star when c is a complex number? Just C bar no, so if and only if C bar C is 1 which is true if and only if mod C is 1 that is if and only if C is e power i theta, theta real. C is e power i theta where theta is real number. So one dimensional case is easy, two dimensional case lets look at the orthogonal case unitary I will leave it as an exercise.

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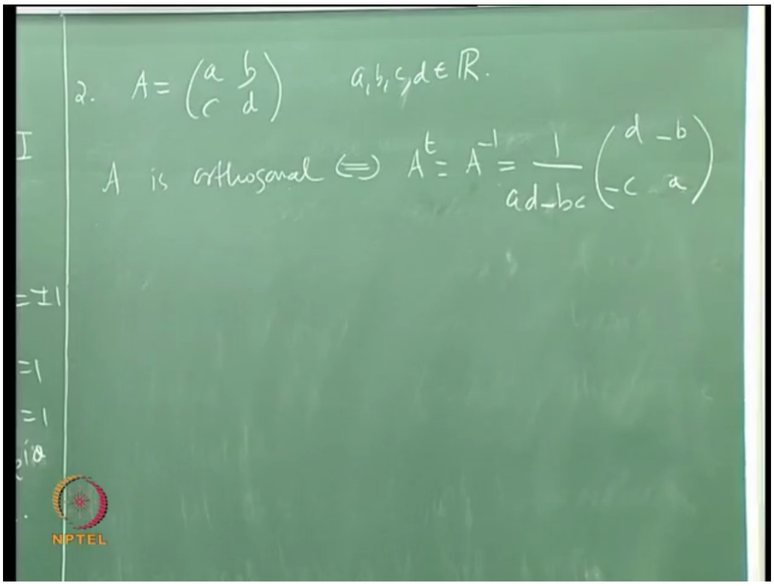

Example 2, take the case a, b, c, d I am looking at the real case. I want to determine conditions on a, b, c, d. What is the form of the matrix A if it is orthogonal? That is the real case, A is orthogonal if and only if A transpose equals A inverse similar to U star equals U inverse for the unitary case.

What is A inverse? A inverse you must, see I am trying to determine conditions under which A is orthogonal, so A inverse exists 1 by a d minus b c into d a minus b minus c that is A inverse right.

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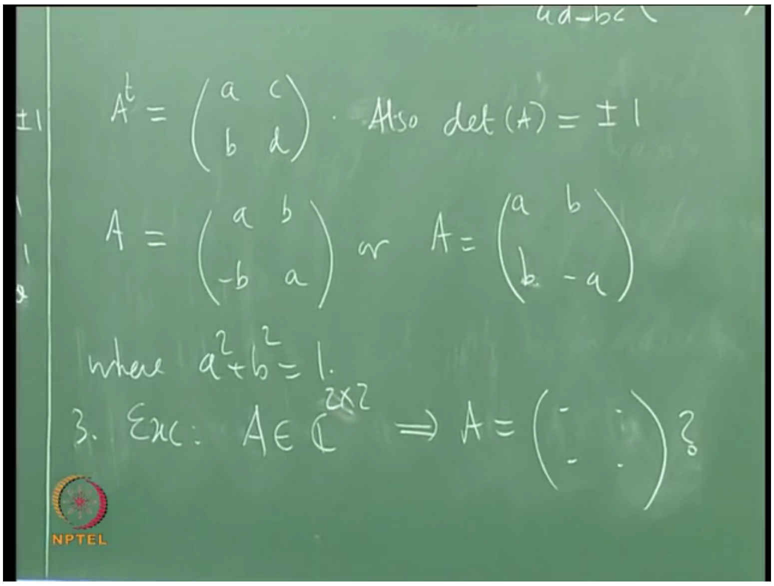

But what is A transpose on the other hand? A transpose is a, c, b, d so these two matrices must coincide. So A must be of the form so can you tell me yeah by the way what is this a d minus b c? The determinant of an orthonormal matrix, so let me say also determinant of A if it is orthogonal plus or minus 1 ok. So this a d minus b c is plus or minus 1 so that leads to the following two cases, if it is plus 1, minus 1 those are the two cases.

I will write down those two expressions, you must compare this with this, this is equal to that with a d minus b c equal to 1 this is equal to that with a d minus b c equal to minus 1. So A has either this form, just check this b minus b or can just fill up the other two entries. Is that ok? In the first case, d is equal to a, in the case when a d minus b c is 1, d is equal to a, b is equal to minus c. So when I write b here I get a minus b here. In the case when this is minus 1, a is d is minus a and so I get this and these two must have the same sign so b and minus b so this is minus a square minus b square that is the determinant, that is minus 1, whereas this a d minus b c in the second case is also minus 1.

Ok so please check these calculations. What is the corresponding result for complex case? That will be an exercise A element of C 3 cross sorry 2 cross 2 implies A is what? A similar analysis can be carried out.

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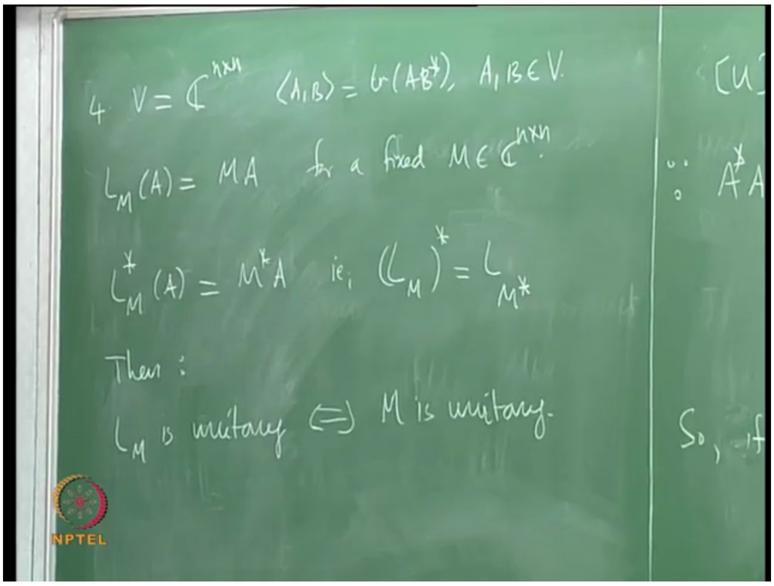

Lets also look at one more example, look at the vector space C n cross n with the inner product of A B is trace of A B star ok we have encountered this before, we have also encountered the operator left multiplication L M of A is equal to M times A for a fixed M. Left multiplication by the matrix M then we have computed the adjoint of this, so what is that? Left multiplication by M star ok that is we have computed that L M star of A is M star of A, M star into A that is L M star is L M star.

For this operator we have L M is unitary if and only if can you make a guess? M star is, M is unitary, L M is unitary if and only if M is unitary ok. Let me prove one part I will leave the other part for you. Let me take the case when M is unitary prove L M is unitary ok, so I will just prove one part.

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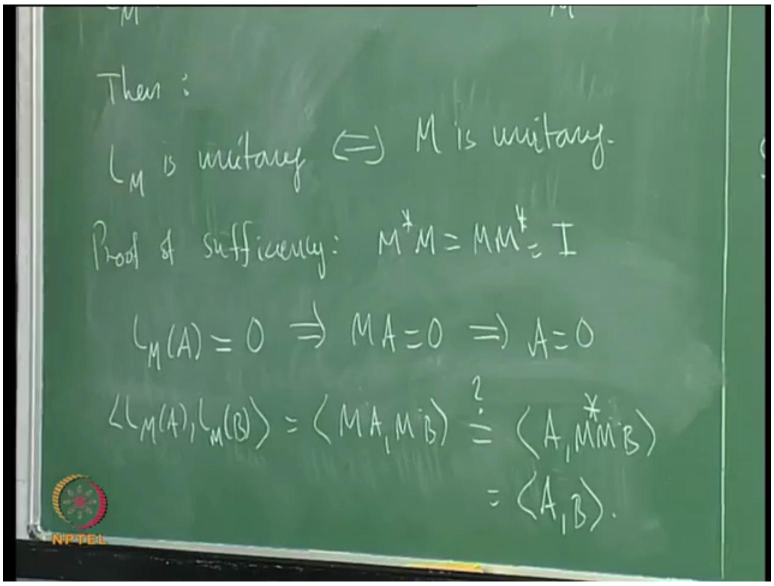

Proof of sufficiency, M star M equals M-M star equals identity I want to show L M is unitary. First L M is invertible, consider L M of A to be the zero matrix this means M A is a zero matrix. But M is unitary in particular M is invertible so this means A is 0. So the null, this is linear map offcourse so a null space of L M is single term zero so L M is invertible because it is on the same space finite dimensional.

So L M is invertible, I must show that L M preserves inner product then it could follow L M is unitary. So consider L M A, L M B inner product, L M by definition is M star A, L M B is M star B, this is I will without appealing to this I will straight away ok lets appeal to that. So we need to I can actually write this as but that needed verification I can write this is equal to A, M-M star B, M-M star is identity to it is just A, B ok. That means I will leave this as an exercise I will still proceed. This is equal to this needs a little justification but I am leaving that as an exercise again this is A, M-M star B. But M-M star is identity so this is A, B and so L M preserves inner products.

Sorry, it is M A, M B, M star M yes M A, M B and then you can push this M star that needs proof but that can be done ok so this is another example so there are two parts that you need to verify, one is this equation the other one is the necessity part if L M is unitary show that M is unitary it is almost time.

Ok maybe I will stop, what I want to discuss next is how are the this question I address in the beginning also. Given a matrix of a linear transformation corresponding to two basis we know that they are related by the formula that is let us say I have B as the matrix of A relative to a basis B2, A is a matrix of the transformation T relative to B1 then B is equal to M A sorry, that is correct M A, M inverse right we have proved this before.

What happens in the case of orthonormal basis? There is something more that we can say for the invertible matrix M that will turn out to be unitary, in the ordinary basis case M is invertible in the case of orthonormal basis M will turn out to be unitary matrix ok will prove this result and then consider the notion of self adjoint operators, normal operators next time ok I will stop.