Linear Algebra
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Module 13-Adjoint of a Linear Operator
Lecture 47
Properties of Adjoint Operation

We are discussing notion of the adjoint ok, we have seen that the adjoint exists for any operator over a finite dimensional inner product space. Let's look at some properties of this adjoint remember the correlation between similarities between the adjoint and the operation of complex conjugation ok I told you that they are similar the following result will tell you why this is so.

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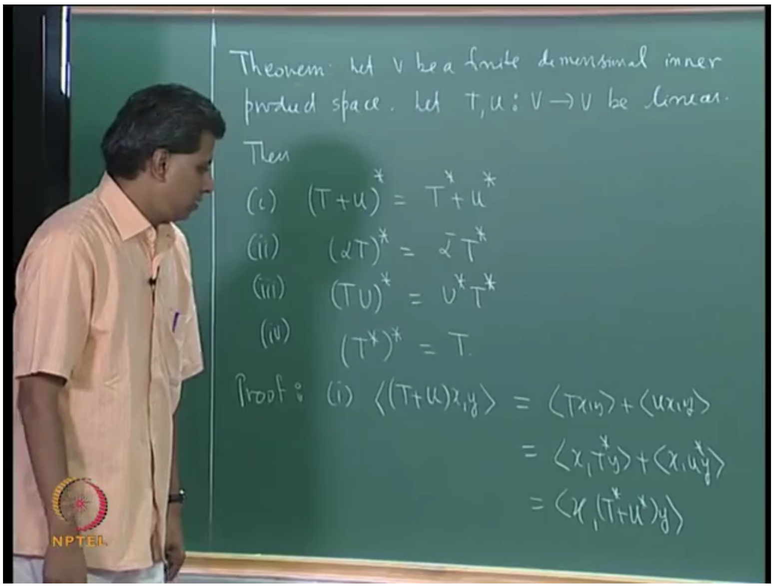

So I have the following theorem which lists the properties of the adjoint operator, let V be a finite dimensional inner product space let T, U be linear operators then the following properties hold, first one that is if you look at T plus U star this is T star plus U star.

Property 2, look at alpha times T star this is alpha bar T star, property 3 look at T U star that is U star T star and property 4 is that this operation of taking the adjoint is an operation of period two, you do it two times you get the same operator T ok proof is to just verify the basic equation for the first one you look at T plus U x, y T and U are linear so and this addition that is also a linear operation that is I can split this as T x, y plus U x, y that is x, T star y plus x, U star y this is possible because T and U have adjoints, it is a finite dimensional inner product space. This can be written as x, T star y plus U star y that is T star plus U star of y.

Now the definition of adjoint is if you have L x y equals x, M y then M is equal to L star, L x y equals x, M y then M is equal to L star, that is T star plus U star is T plus T whole star that is the first one ok. Second one similar so second one I will leave the thing is for the second it is a conjugate it will come with the conjugate, let me prove the third one so second is straight forward.

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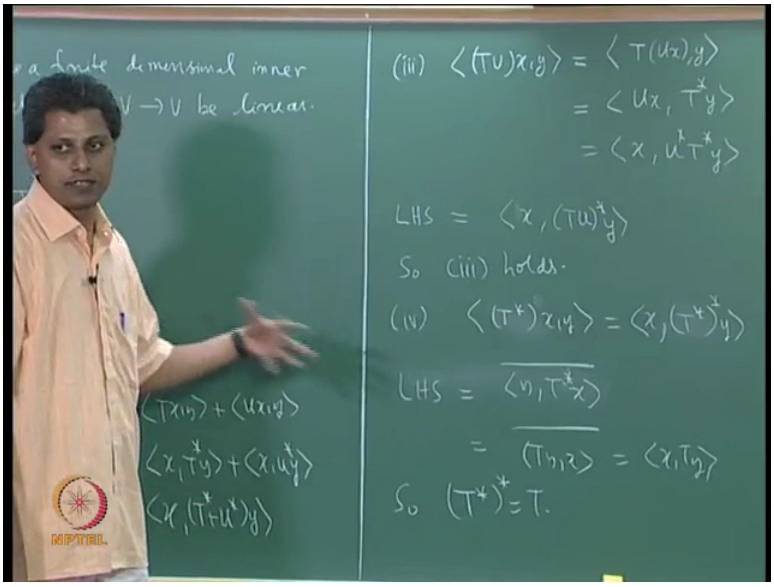

Third is also straight forward but I will prove it in any case. You consider what is the order in which I take, T U x, y, T U x, y, is this is the composition ok so that is T of U x, y then this is U x, T star y that is the definition of T star this goes here it goes with the conjugate apply it once again x U star T star y when I write U star T star it is the composition of this linear maps.

So what is the meaning of this? x but this left hand side is ok just to emphasize the left hand side is x, T U star y and so three holds we have shown that T U star is U star T star, this is called the reverse order law for the adjoint operation similar to the reverse order law for the usual transpose, transpose of linear transformations that we studied earlier similar to the operation of taking transpose for matrices similar to the operation of taking conjugate transpose for complex matrices, so that is property 3.

Property 4, you, look at T x , y ok I want to show T double star right so let me start with T double star this time, T double star x equals y this is by definition when it goes here one star is removed, one star actually appears so the problem becomes complicated so what I will do is I will start with T star x, y this is x, T y which is what I want to prove so that is x, T star-star y but the left hand side is x, T y right hand side is x, something y.

Student: (())(07:08)

Professor: Ok we don't know but ok, this much we know left hand side is y, T star x conjugate ok which is the conjugate of T y, x which is the same as x, T y that is the left hand side ok yeah so that needed explanation, so x, T y is x, T double star y so (form this) this is true for all x and y it follows that T star-star is T ok so just to list some properties now you will see that the reason why this adjoint operation is compare to the operation of complex conjugation.

If you have Z1 Z2 two complex numbers then Z1 plus Z2 bar is Z1 bar plus Z2 bar, Z1 ok look at this constant times Z is that constant times bar into Z bar and look at this, this is Z1 Z2 bar their complex multiplication is commutative so you don't have to be you don't have to impose this route ok, this is just T star U star there if you want and when you do it twice Z bar-bar is Z ok.

What is also to be compared? For an operator is the following, for an operator you can think of a real part and an imaginary part using the adjoint operation. So let me illustrate that and then move on to unitary operators.

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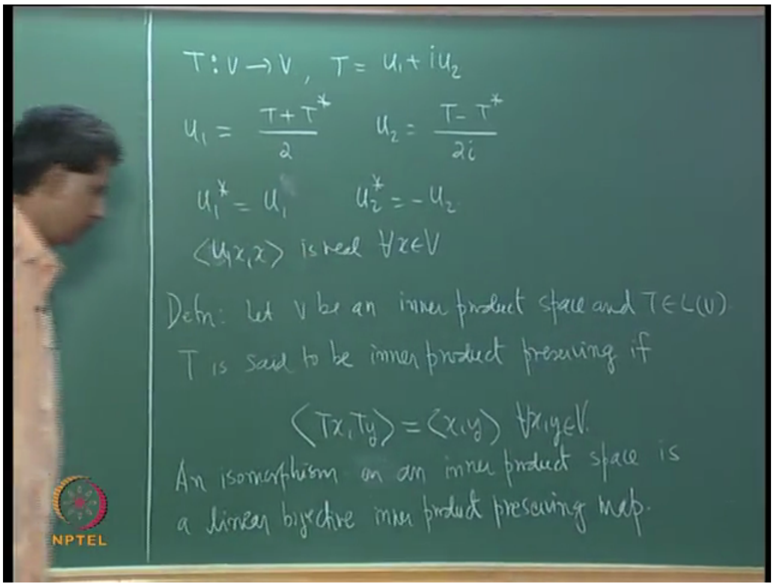

So if T is an operator on a finite dimensional inner product space V then I can write T as U1 plus I U2 ok where U1 is T plus T star by 2 and U2 is T minus T star by 2 I ok you can verify this. Now U1 and U2 are unique because T star is unique and you can also observe that U1 has the property that U1 is its own adjoint, it is a self adjoint operator. What about U2? U2 is that self adjoint? It is, see this is called Hermitian.

So what is the thing with U2? (())(10:18) so U2 star is minus U2 you can just check this calculations, so there is a kind of a U1 star is U1 is what I want to say is U2 star is minus U2. U1 is self adjoint so this is just to tell you that any operator on a possibly complex finite dimensional inner product space can be written as a real part and imaginary part now U1 is self adjoint so that is a kind of real part we talk about, why is it the real part? For instance a complex a number is real if and only if Z bar is equal to Z, so what is real about this with the thing that is in the mind when one says that this is a kind of real part, is the following.

Any self adjoint operator satisfies the following in particular this U1 satisfies the following, U1 x, x this is real for all x element of V so that is what we have in our mind when we say that this is a kind of a real part U1 x, x is real for all x in V whenever U1 star is U1 infact the converse is also true. First U1 star equal to U1 implies U1 x, x see this is a kind of a quadratic form U1 x, x is a quadratic form so if U1 is self adjoint then this is real that’s straight forward, the converse will use polarization identity. So I am going to leave that as an exercise I will discuss polarization identity a little later.

But the converse is also true, T is self adjoint if and only if T x, x is real ok. Why are self adjoint operators important? This is a self adjoint operator the real part U1 is a self adjoint operator, self adjoint operators are important for one single reason which is that (they) there exists an (ortho) if T is a self adjoint operator on a finite dimensional inner product space V then there exists an orthonormal basis of V each of whose vector is an (())(12:40) vector for T ok. So this is saying this is the same as saying that T can be diagnosed by means of an (orth) by means of a unitary matrix, I will explain this terms later but this is just a kind of a (()) (12:56).

Any self adjoint operator on a finite dimensional inner product space V has a property that there is a basis there is an orthonormal basis of V each of whose vectors is an I gain vector for T ok this is another very important property that a self adjoint operator satisfies. So this is just to kind of relate to the notion of the adjoint with the conjugate operation for a complex number. Lets look at this further look at subclasses of operators which satisfy certain other properties with regard to the adjoint ok. One of them is a unitary operator but before discussing unitary operator I want to discuss a notion of isomorphism between inner product spaces ok.

Using the operation of star I want to discuss notion of a unitary operator but before that isomorphism between inner product spaces. Let V be an inner product space not necessarily finite dimensional and t be a linear operator on V then T said to preserve the inner product said to be inner product preserving. If T satisfies the following, inner product of T x with T y is the same as inner product x with y this is true for all x y in T set to preserve the inner product if this equation holds ok this leads to the definition of an isomorphism.

Remember isomorphism between vector spaces is a 1-1 it is a linear injective surjective map ok linear bijection. For inner product spaces the extra requirement is it must be inner product preserving ok. An isomorphism, this isomorphism means same structure morphism structure (iso) is the same. In addition to this being a bijective linear map it has got to do with vectors (spaces) inner product spaces V, V and W for instance, so it must preserve the inner product also, that relation should also come. An isomorphism I will say now between an isomorphism on an inner product space is linear bijective inner (product) preserving map.

It is a linear bijective inner product preserving map, linear bijective map which preserves inner product. Ok before I go to characterizing isomorphism between inner product spaces let me tell you one property that an inner product preserving map must satisfy. I will give examples a little later but there is one property that this inner product preserving linear transformation must satisfy, in relation to the norm there is a relationship with the inner product this should lead to an relationship between norms so that is the following.

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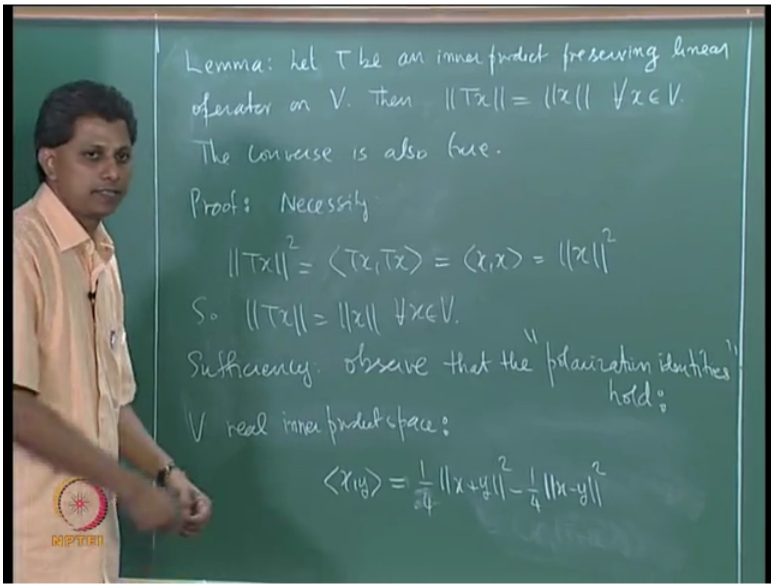

I will take this as a lemma, let T be an inner product preserving linear map linear operator on V, V is an inner product space not necessarily finite dimensional any inner product this is (definition) also you will see for a general inner product space then norm of T x equals norm of x for all x in V. We say in this case that it is an isometric norm of T x equals norm x for all x in V. The converse is also true, the converse also holds. So if T is an isometric linear then we can show that it will preserve the inner product. Remember that this norm is a norm induced by that inner product ok whenever we talk about a norm in inner product space in this course this is the norm induced by that inner product ok.

One way the proof is easy if it preserves the inner product then it preserves the norm is easy. So necessity, look at norm T x square this is inner product T x with T x that is what I told you that this is a norm induced by this inner product but T is inner product preserving so this is inner product x, x that is norm x square. You take the positive square root you get this are non-negative numbers you get T is norm preserving ok. So norm T x is norm x. For the converse you need a little work which is actually to first proof the polarization identities. So for the converse sufficiency we first observe but we can quickly see the proof also.

Observe that the polarization identities hold, so what is this one for real, one for the complex case. V is a real inner product space in this case the polarization identity is inner product x , y equals 1 by 2 sorry, 1 by 4 norm x plus y square minus (1 by 4 norm x minus y square) when it is real. For a complex inner product space it is a little more complicated.

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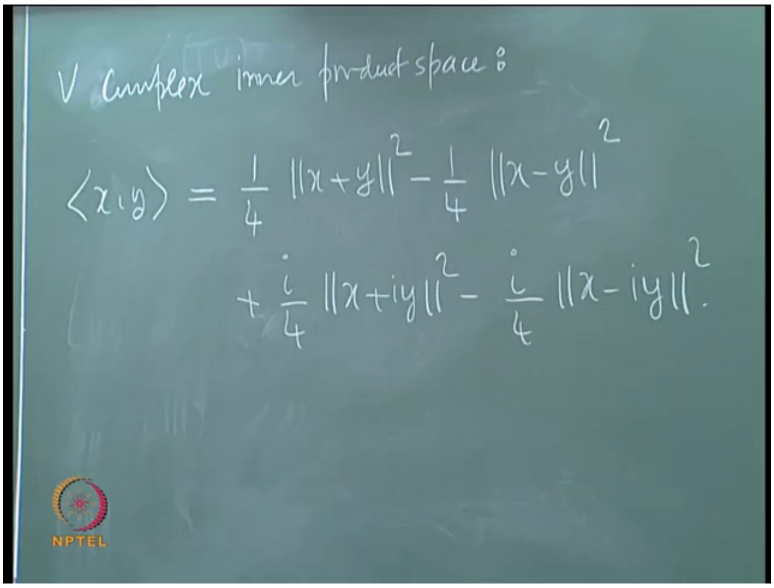

V is a complex inner product space then the polarization identity takes this form, you can guess the next two terms i by 4 norm x plus i y the whole square minus i y 4 norm x minus i y the whole square, this hold for all x y ok out of which let me prove see what you can do is in order to prove this look at the right hand side expand by the inner product (and) you will get the left hand side.

For example norm x plus y square is inner product x plus y, x plus y that is norm x square plus two times real part of x, y plus norm y square when you subtract the two times real part will go sorry it goes with the minus sign for the second term so that is the only term which will remain 2 plus 2, 4 that 4 will get cancelled with this so it is simple verification proof. So I will not get into the proof of this that is really straight forward. Using use ok either lets I will prove it for the case when V is real that if it preserves the norm then it must preserve the inner product ok complex case is similar.

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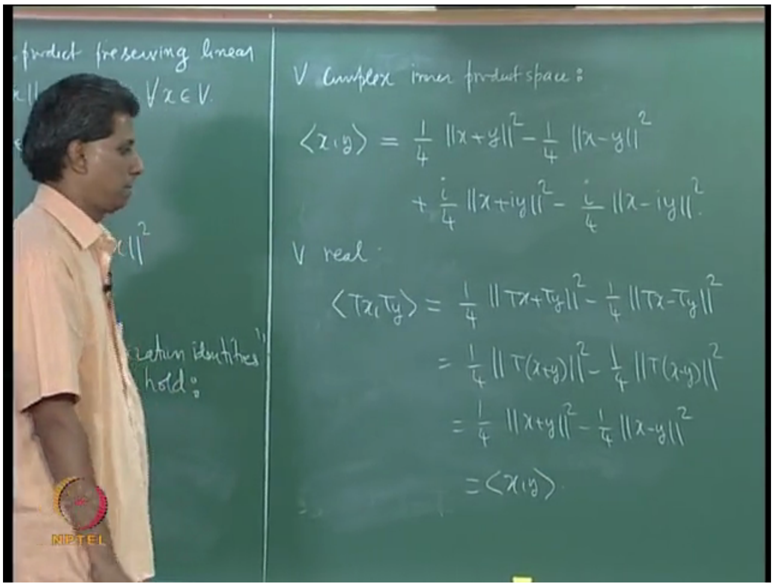

So lets take the case when V is real I want to show that inner product T x T y is x, y. So let me start with inner product T x T y, this by definition is 1 by 4 norm of T x plus T y the whole square minus 1 by 4 norm of T x minus T y the whole square. I am assuming I am proving the sufficiency part I am assuming that T preserves the norm I will show that T preserves the inner product. This is 1 by 4 norm of T of x plus y. T is linear minus 1 by 4 norm of T x minus y, use the fact that T preserves the norm so this first term is 1 by 4 norm of x plus y square that is norm of T x equals norm of x, so norm of T x square is norm of x square minus 1 by 4 norm of x minus y square.

The real polarization identity tells you that this is inner product x, y. So T preserves the inner product. Ok so if it is inner product preserving then it must be norm preserving and conversely. I have proved it for the real case, complex case is similar. So lets now look at necessary sufficient conditions for a linear transformation to be an isomorphism between inner product spaces. By the way I don't have to write two inner product spaces V and W are said to be isomorphic inner product space that is isomorphic if there is an isomorphism between the inner product space.

Isomorphism is not just a vector space isomorphism it must also preserve the inner products ok, then this relation is an equivalence relation, V is isomorphic to itself by means of the identity map, V isomorphic to W means W is isomorphic to V by means of the inverse transformation, inverse exists because T is bijective.

Inverse is linear because T is linear inverse preserves inner product because T preserves inner product that takes a little observation but it can be done and again if V is isomorphic to W, W isomorphic to Z then V is isomorphic to Z through the composition map ok so that's quickly it is an equivalence relation, isomorphism between vector spaces is an equivalence relation. But lets look at the following result which gives isomorphism in my means of other conditions.

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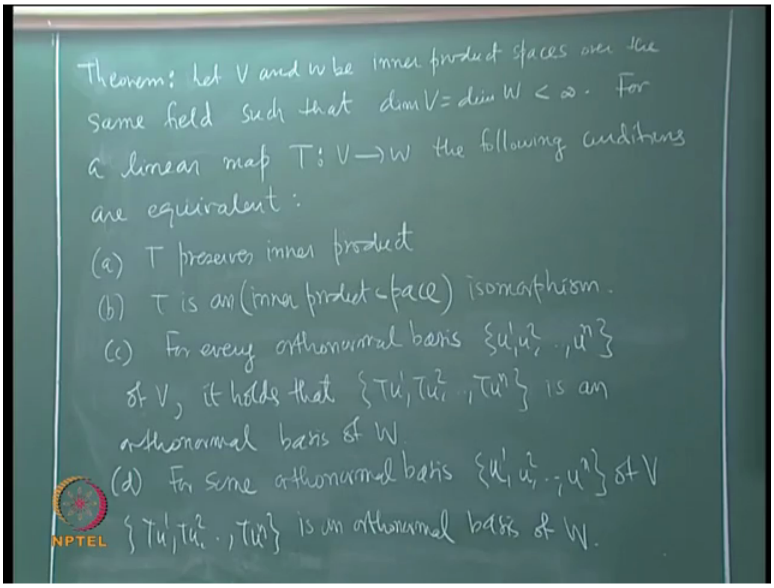

So I have the following result, you must compare this theorem with its corresponding result that we proved for vector spaces finite dimensional vector spaces where we characterized isomorphism. See here we have finite dimensional inner product spaces. Let V and W be (inner product) finite dimensional inner product spaces over the same field such that dimension of V is equal to dimension of W and I am assuming that V is a finite dimensional inner product space.

So V and W are finite dimensional inner product spaces with a same dimension. For a linear map T from V into W the following conditions are equal, the following conditions are equivalent ok so same dimension. The following conditions are equivalent, first condition is ok we call it a, T preserves inner product just this much T preserves inner product dimension V is dimension W, condition 2, T is an isomorphism to emphasize it is an inner product space isomorphism. Second is apparently stronger than the first one but under in this framework this are the same this are equivalent conditions.

So T is an isomorphism where I am just emphasizing that it must be an inner product space isomorphism. Condition c, for every orthonormal basis of V, let me list the elements for every orthonormal basis ok just write down the elements U1, U2 etc U n for every orthonormal basis this of V it holds that T U1 etc T U n is an orthonormal basis of W, now this you must compare this I told you with the vector space isomorphism it will be not be an orthonormal basis it will be just a basis ok, you include orthonormality because there is an inner product here, so it holds that T U1 T U2 etc T U n see I am given that the dimensions are the same it holds that this is an orthonormal basis of W.

For every orthonormal basis we may call this a image this is an orthonormal basis. Apparently weaker is condition d, which says for some orthonormal basis V1 ok I will use the same U1 U2 etc U n of V, T U1, T U2 etc T U n is an orthonormal basis of W. Condition d looks weaker than condition c, but in this framework they are the same the conditions are equivalent. Ok so lets prove this again as before will prove a implies b, b implies c, c implies d, d implies a.

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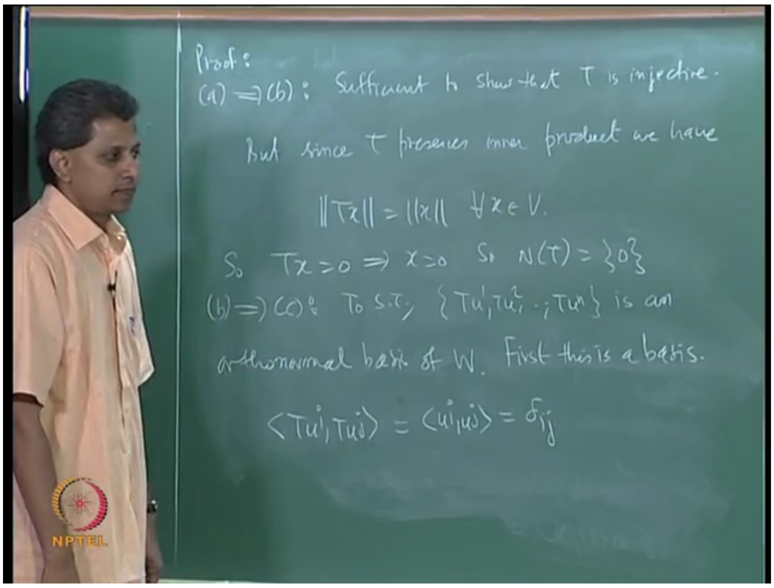

A implies b, T preserves inner product so you can give the proof so tell me. T preserves inner product I must show that it is a nice inner product space isomorphism so I need to show that it is bijective that is all. How does that follow? (())(31:37) are the same is to be used, but how do you prove it is bijective? That is the only thing left, you need to show T is bijective ok, null space is zero so how do you show that null space is zero? Preserves inner product ok,

Student: (())(32:01)

Professor: You can use that, norm T x equal to zero implies (x) norm x equal to zero just use that. So that is the first part, only to show T is bijective ok, sufficient to show you must show that T is bijective but sufficient to show that T is injective because the dimensions are the same by Rank Nullity dimension theorem it follows that it is injective if and only if it is surjective. T is injective but then we know that but since T preserves inner product we have the T preserves norm so norm of T x equals norm x for all x in V so T x equal to zero implies norm x equal to zero it implies x equal to zero.

So null space of T is single term zero and so T is injective so T is bijective, so b holds it is infact invertible that is what b says, b implies c, for every orthonormal basis we must show that this image is also an orthonormal basis of W, what is given is that, T is an inner product space isomorphism. I must show that ok let me just write down, given that orthonormal basis U1, etc U n I must show that T U1, T U2 etc T U n is an orthonormal of W, but why is it a basis for W?

Student: (())(34:28)

Why don’t you just give me one word, no, one word, T is an isomorphism, over, T is an (isomor) V implies c, T is an isomorphism so this must be a basis, we want to show it is extra it is orthonormal basis. See T is an isomorphism so this is U1, U2 etc U n is a basis implies T U1, T U etc T U n is a basis to begin with so first of all this is basis, first this is a basis. I need to show it is orthonormal but just use the fact that it preserves inner product. So just look at T U y, T U j I must show that this is equal to delta i j then it follows that it is an orthonormal basis.

But this is by definition U i, U j because it preserves inner product U i, U j I started with this as an orthonormal basis so this is delta i j the chronicle delta, so this proves that this set of vectors T U1 etc T U n forms an orthonormal basis of W, that is b implies c, c implies d, there is no need for the proof. If it holds for every orthonormal basis it must also hold for some orthonormal basis. So only thing is d implies a.

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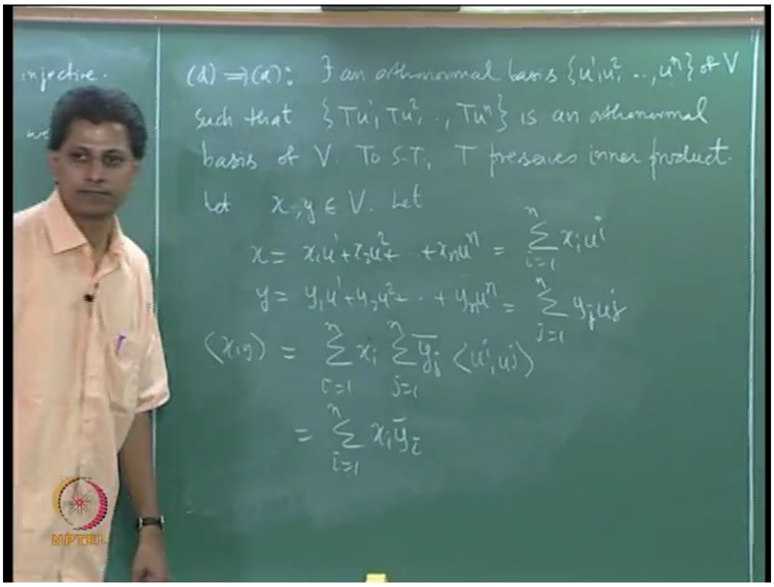

So c implies d, there is nothing trivial d implies a, for some orthonormal basis U1 etc this is an orthonormal basis if this is satisfied then I must show that T preserves inner product ok.

Ok let me just write down the condition d, there exists an orthonormal basis of V such that T U1 T U2 etc T Un is an orthonormal basis I know this I must show that T preserves the inner product. Muthukumar you wanted to say something? C is stronger than d is that ok? So c implies d is straight forward ok. So I am just proving d implies a ok, how do I go about proving this? I must show that inner product T x, T y equals inner product x, y for all x y ok. So we have to take (())(38:02) to that lets take x as say x is a linear combination of see I have I will take x y in V, I must show that inner product x y equals T x inner product T x, T y they are in V they can be written in terms of this and I also know what the coefficient must be I will call the coefficients x1 etc x n, y1 etc y n ok because they correspond to x and y.

So I have the following representation, x equals x1 U1 plus x2 U2 etc plus x n U n, I know what this coefficients must be, y will be y1 U1 plus y2 U2 etc see we need to use the linearity of T so you write x and y in terms of this basis vectors,, what is the inner product of x with y? Let's calculate that let me use a summation notation here this is summation, what do I use? I? I equals 1 to n xi ui this is summation j equals 1 to n yi ui sorry y j U j I need to calculate this so can I just write down now we have seen this several times i equals 1 to n xi j equals 1 to n y j inner product U i U j y j bar ok.

But you see what is going on we started with an orthonormal basis the outer sum I mean the inner summation is for j and for the inner summation i is fixed so when j takes a value i it is one all other terms are zero j takes the value i this becomes y i bar so this is just summation i equals 1 to n xi yi bar. Something that is similar to C n for instance something (that's) this is the precisely inner product in C n with respect to the standard orthonormal basis ok. Lets calculate Tx Ty similarly.

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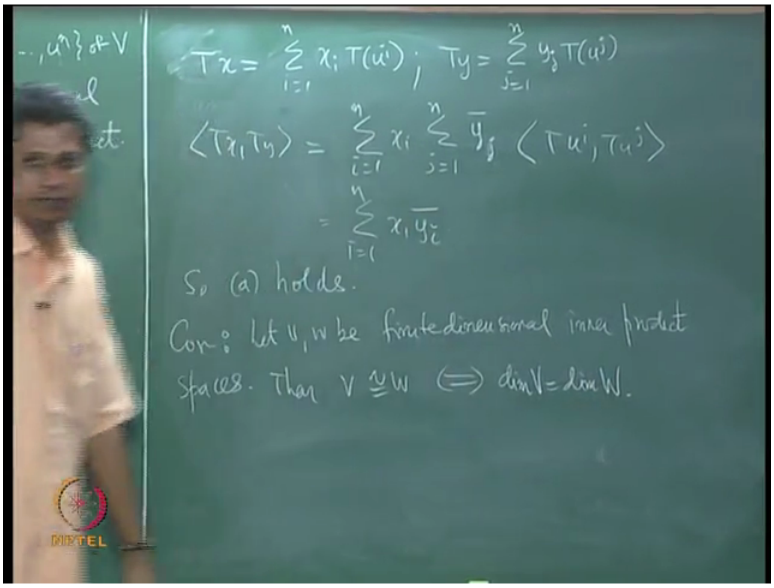

Consider Tx ok let me write down Tx, Tx will be summation i equals 1 to n xi T of Ui and Ty is summation j equals 1 to n y j T of U j I take the inner product compute show that it is the same as this.

So inner product Tx Ty is again I will reduce the calculations i equals 1 to n xi j equals 1 to n y j bar inner product T Ui T Uj again the same thing T Ui T Uj that is a part of a orthonormal basis when j takes a value i it is 1 all other terms are 0 because that's an orthonormal basis. So this is again i equal to 1 to n xi yi bar so he holds. So T preserves inner product if you know that T is a linear transformation which takes some orthonormal basis to some orthonormal basis of V to an orthonormal basis of W.

Dimension W equals dimension of V ok an easy corollary I am going to leave it as an exercise for you. Let V, W be finite dimensional inner product spaces then V is isomorphic to W where it is isomorphism of inner product spaces between inner product spaces then V is isomorphic to W if and only if dimension V equals dimension W. I will not write down the proof I want you to tell me how it goes. There are two parts, suppose v is isomorphic to W as inner product spaces then dimension V equals dimension W, how does it follow?

See there exists an inner product space isomorphism from V into W but that must be in the first place a vector space isomorphism but we know vector space is when they are isomorphic and they are finite dimensional the dimension must coincide one way, converse. Suppose dimension V equals dimension W, finite dimension spaces I must show that V is isomorphic to W, that is all I must exhibit an isomorphism so take an orthonormal basis of V take an orthonormal basis of W, orthonormal basis of V we call it V1 etc V n for W, W1 etc W n define a map T, which takes Vi to Wi then you can show that it is linear bijective preserves inner product ok, this is an easy corollary of the previous result.

Before I stop lets look at two examples one over finite dimensional spaces another one over infinite dimensional spaces. In a product preserving maps (())(44:19) product preserving linear maps ok.

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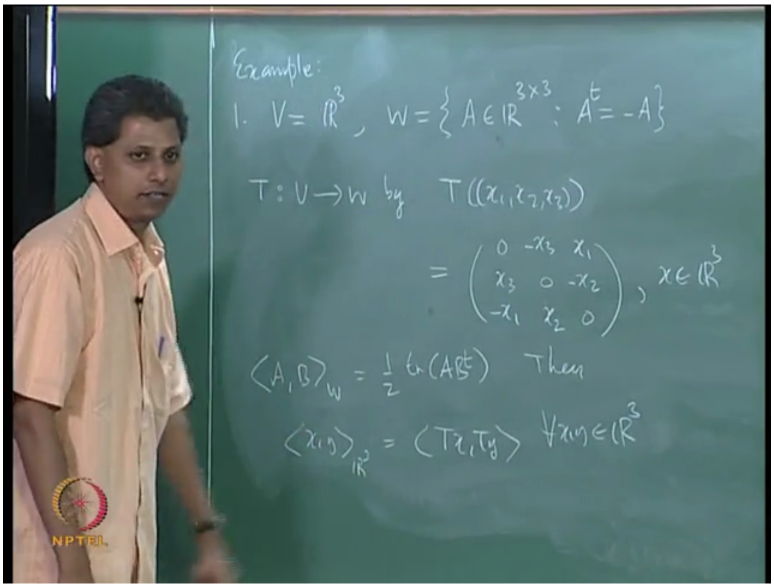

I told you two examples the first one is I will take V to be R3 with a usual inner product W to be the subspace of the space of all 3 by 3 real matrices that R Skew is symmetric space of all real Skew symmetric matrices of order 3. I want you to tell me the dimension of W. What do you expect the dimension of W to be first? I want to give example of inner product space isomorphism.

So what do you expect the dimension of W to be? 3 ok, can you show it is 3, how? A transpose equals A see a matrix of order 3 square matrix of order 3 has 9 independent elements, 9 independent ways of filling the elements the entries of A. A transpose equal to minus A make sure the diagonals are zero so three conditions are gone. I need six independent apparently six independent conditions, can I reduce it to three? Skew symmetry transpose is minus A so just fix either the upper part or the lower part the other part is determined. So dimension W is three ok, I define a map T from V to W by T of the vector x.

Take a vector x, how do I fill it up as a matrix in such a way that I get a Skew symmetric matrix? The definition is it must be Skew symmetric the diagonal entries must be zero. So 0, 0, 0 this is minus x3 this is minus x2 this is x1, this is negative x3 minus x1, x2 ok then it is an easy exercise to see that T is linear dimension of V is the same as dimension of W, so in order to show that it is invertible bijective it is enough if I show null space is single term zero, can you see that? Null space of T must be single term zero so T is injective dimensional same T is bijective I only need to show that T is an inner product space, till now I have only vector spaces, I will (())(47:13) R3 with the usual inner product (())(47:15) W with the following inner product.

For two matrices in W, the inner product is defined as 1 by 2 times trace of A B transpose this is not new, it is just trace of B star A if you want the extra thing is 1 by 2 you will realize why it is what that constant 1 by 2 appears there. Given two matrices A B in W ok I will leave it as an exercise for you to verify that inner product x y over R3 the usual inner product Tx, T so please verify this is just elementary calculation

You calculate A B transpose look at the trace it will be two times the usual inner product that is the reason why this 1 by 2 is taken to cancel out the two ok finite dimensional case this is an inner product space isomorphism. Just an example in the infinite dimensional case, its inner product preserving but not an (isomor).

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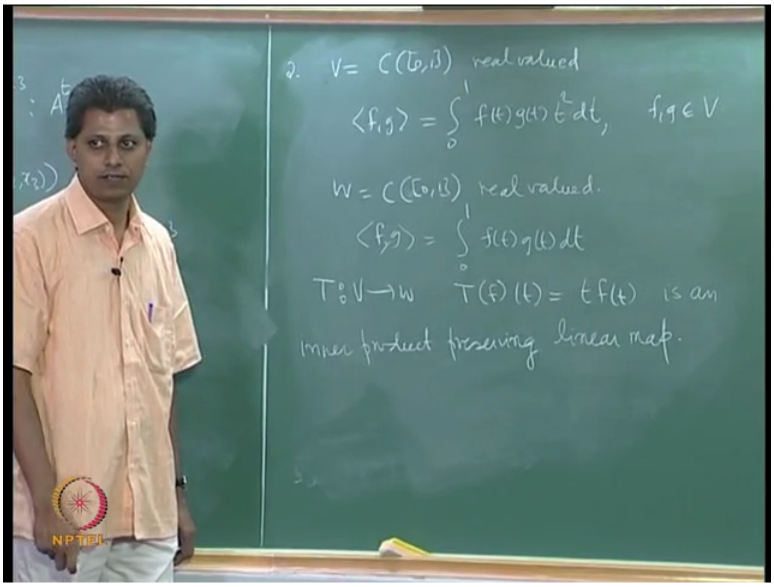

Example 2 I will take V to be C 0, 1 real valued continuous function over the interval 0, 1 with this inner product it will be 0 to 1 f of t, g of t together with a weight t square matrix ok so I will include a weight t square any positive weight can be included so this is an inner product on V. W is a same space C 0, 1 real valued continuous function over 0, 1 with a usual inner product, same space with a usual inner product. Define the mapping T from V to W by T of f continuous function f at T. So T f at T is T times f of t then you can show that this t is linear its inner product preserve. This t will come two times first argument, second argument so this t square comes out ok.

So you can check that this t is inner product preserving, inner product preserving linear map its an inner product preserving linear map is T invertible? T f equal to 0, does it imply f equal to 0? Offcourse T f equal to 0, offcourse implies f equal to 0, T is injective T is not bijective because take the constant function 1 in W the question is does there exists a function small f such that t time f of t equal to 1? Does there exists a continuous function f such that f of t equals 1 by t?

So t is injective but not bijective please check this. So t is an inner product preserving linear map injective but not bijective ok. See I am emphasizing this example because see remember we have shown that if t is if dimension V equals dimension W and if t is inner product preserving then it is invertible ok, that doesn't happen here because we are over infinite dimension spaces that is the purpose of this example, the result that we proved earlier holds for finite dimensional spaces, dimension V equals dimension W finite it is the same space W V. W equals V same space inner products are different ok.

So this is injective inner product preserving but not bijective but not on to but not surjective ok, let me stop see I want a continuous function f of t such that t f of t equal to 0 for all t, the only way that can happen is f is zero.