Linear Algebra
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Module 13-Adjoint of a Linear Operator
Lecture 46
The Adjoint Operator

Ok so we are looking at linear functionals leading to the notion of adjoint of an operator ok, so let me recall this result that we proved last time.

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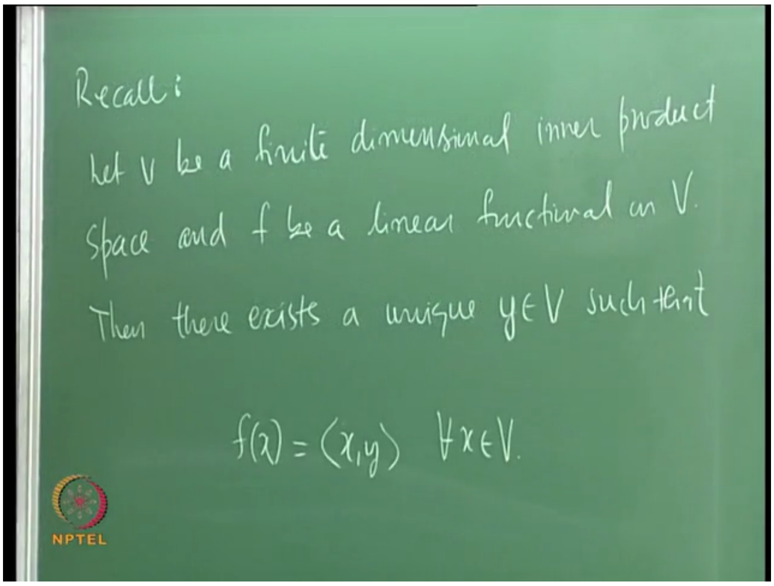

Let V be a finite dimensional inner product space and f be a linear functional on V then there exists a unique vector y in V such that f of x is nothing but the inner product of x with this fixed vector y for every x in V. This y will offcourse depend on f because we have seen the formula for y ok.

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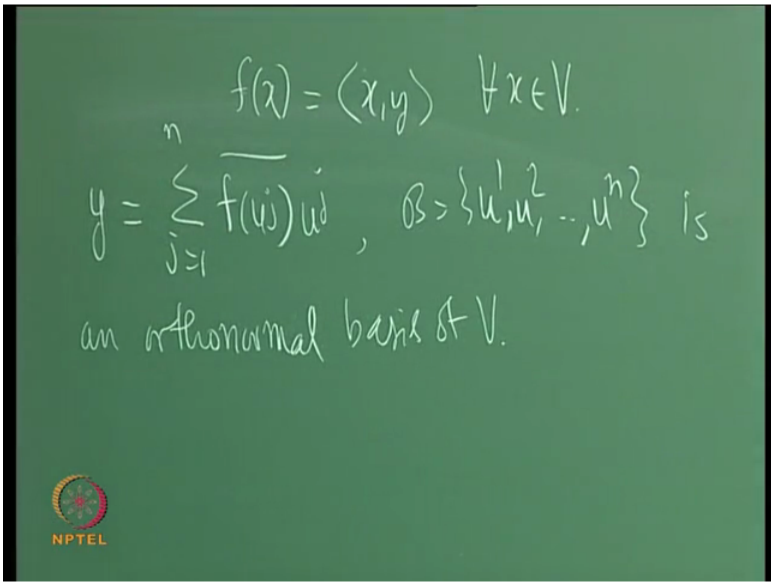

The formula for y is summation j equals 1 to n f of U j bar U j so it depends on f and this U j comes from a basis is an orthonormal basis of V ok using this will now look at the notion of the adjoint of an operator so lets first prove this result.

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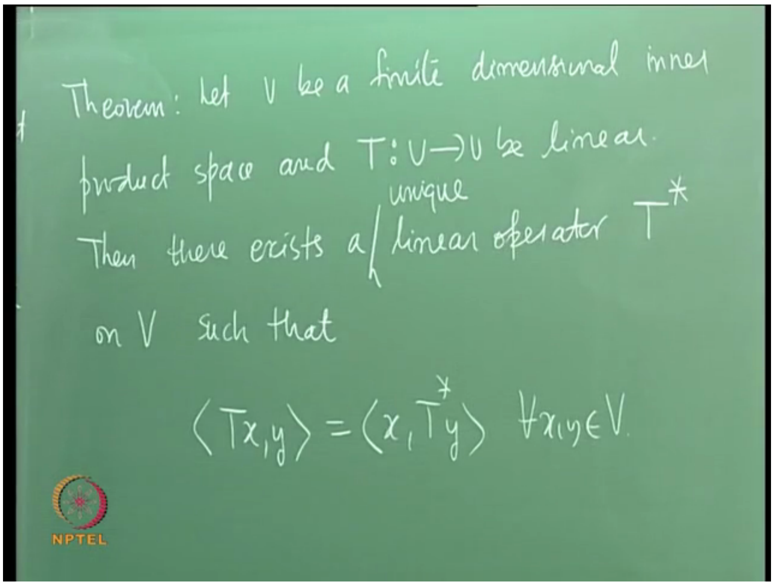

Let V be a finite dimensional inner product space and T be a linear operator T is a linear transformation on a finite dimensional inner product space then there exists a linear operator T star on V, so will call this operator T star there exists a linear operator T star on V such that the following holds remember we are calling it T star so it does depend on T and it in particular satisfies the following, it is linear offcourse it satisfies the following property such that if you look at the inner product of T x with y this is the inner product of x with T star y this must be true for all x y in V.

So what this result claims is that there is on a finite dimensional inner product space V there is this operator this is unique so let me mention that also, there exists a unique linear operator which will call T star this T star will call it as the adjoint of the operator T that will do a little later first to prove this, to prove this we will appeal to the previous theorem so in order to apply the previous theorem we need to look at our functional.

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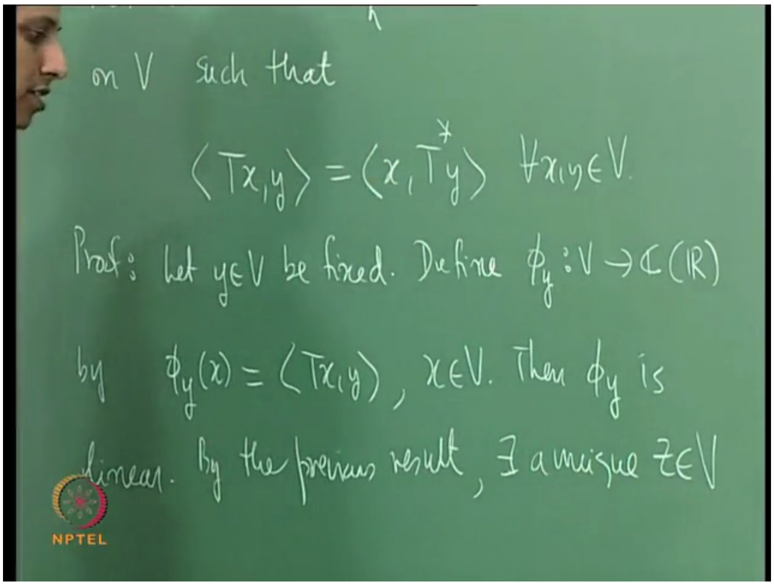

So the proof is as follows, let y belong to V be fixed I will tell you what T star y is ok then I need to see that it is unique T star y is unique and then T star satisfies this and finally T star is linear ok.

Define the following operator, define phi y, it depends on y, phi y from V to (())(4:54) within brackets R define this operator by phi y of x equals I will now invoke the operator T use the operator T in the definition of this functional phi y, y is fixed this x varies so this is an inner product ok phi y of x is a number real or a complex number. Now T appears here so you can show that phi is linear phi y is linear then phi y is linear phi y of alpha x plus beta Z you can show that it is equal to alpha phi y of x plus beta phi y of Z so phi y is linear it is a linear functional (())(5:52) previous theorem.

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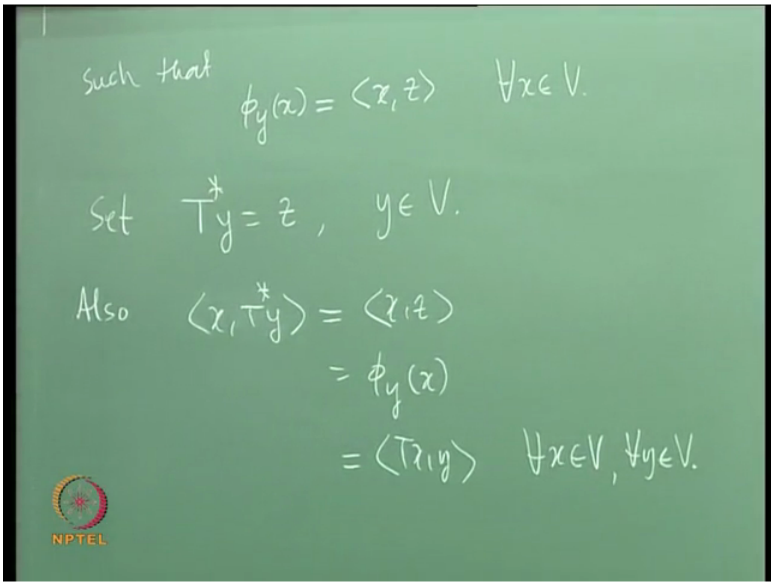

By the previous theorem there exists a unique I will call it Z by the previous result there exists a unique Z in V such that the action of phi y on any vectors x is the inner product of the vector x with Z. What is important is to realize again that it is unique this uniqueness will imply the uniqueness of this operator T star infact ok. So we started with Y we started with Y in V we have got a unique vector z that satisfies this equation for all x so will associate this Z to Y under T star so the definition is set T star y equals Z. Now you can do this for every fixed y that is given a y I will define this linear functional and then I know how to get this unique Z then for that while I will associate this Z so this is, since this right hand side is unique for this y this is well defined in the first place.

We need to verify it is linear but before that lets verify that this T star satisfy this equation that straight forward. Also if you look at T x, y I will start with x to T star y, x with T star y is by definition T star y is Z so that is x with Z but x with Z is here that's phi y of x and phi y of x by definition is T x y, this is true first for all x in V and then I vary y then this is true for all y in V ok. First I fix y and then this is true for all x then I vary y appealing to that Z which comes for this particular y. So this equation holds x , T star y is equal to T x, y for all x and y. We need to show T is T star is linear so that is also not difficult so.

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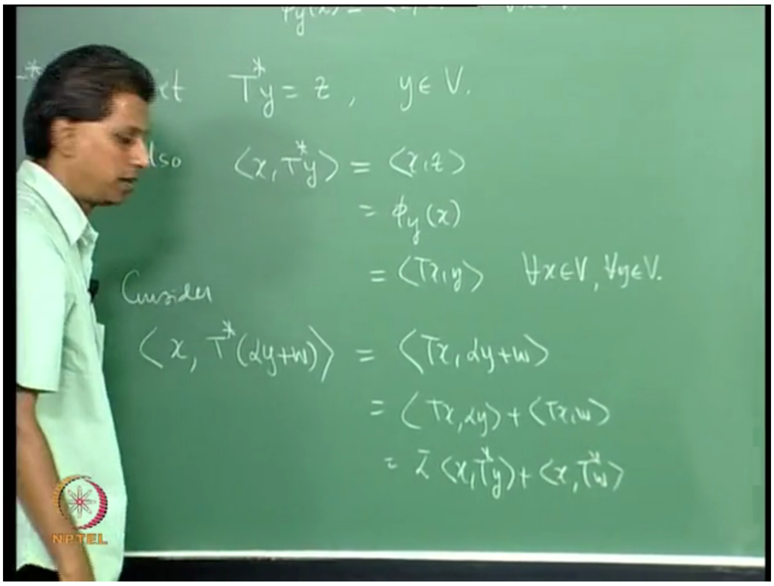

We need to show T star is linear so lets consider I have used a variables x y and z ok let me use x will remain a variable I want to show T star is linear lets say T star alpha y plus W now I know that T star satisfies this equation I will make use of that. So I must so lets look at this I will show that this is equal to inner product x with alpha T star y plus T star W, this is true for all x so T star is linear.

So consider this, by definition T star satisfies this condition that x with T star y must be T x with y, so this is equal to inner product T x with alpha y plus W that can be written as T x, alpha y plus T x, W now here I will take alpha bar outside I will do two steps I will take alpha bar outside and then T x, y is what I will get that I will write as x, T star y the next term is x, T star W.

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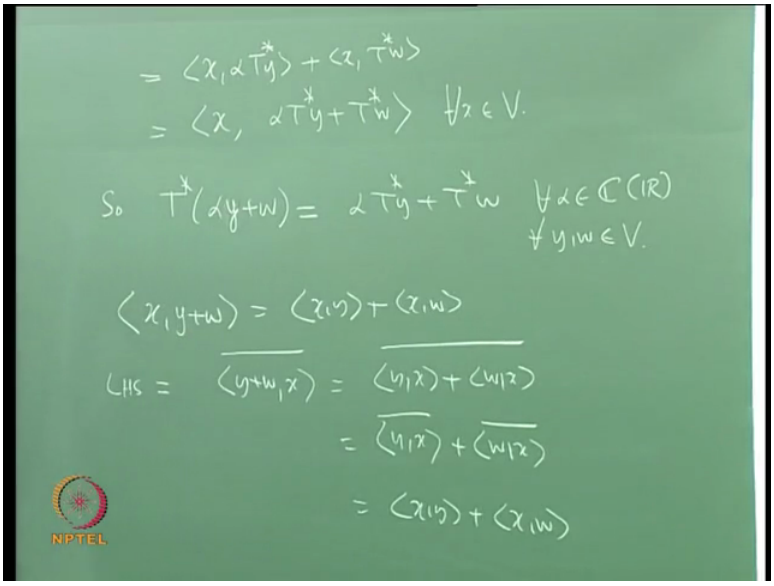

Now I will take this alpha bar inside so this gives me x, alpha T star y plus x, T star W this x can be taken common x alpha T star y plus T star W this is true for all x ok.

Hence I started with x with T star of something I show that it is x of alpha T star of y plus T star W this is true for all x so it means that T star of alpha x plus W sorry alpha y plus W I have shown that this is equal to alpha T star y plus T star W C or R again we have made use of the fact that if inner product of x with Z equal to zero for all x then Z must be zero.

Student: (())(11:28)

X Z plus y Z, which term? What, where is your doubt? This one? It is an inner product no? So inner product of x, y plus W is inner product of x, y plus inner product of x, W. (you can use that). See you can use that whenever there is no scalar ok, your question is what happens, why is this your question is why is this true? Ok start with the left hand side left hand side is y plus W x conjugate now it is you know it is linear with respect to the first one, y x plus W x conjugate then y x conjugate plus W x conjugate that is x y plus x W. If I have note mentioned it please take this as the explanation.

See the inner product is linear with respect to the first variable and with respect to the second variable it is additive it is conjugate linear with respect to the second variable but it is additive with respect to the second variable that is something we are we have been using probably without mentioning ok so for we have shown that this T star is offcourse well defined and its linear, why is it unique? Unique because Z is unique that is if there is ok I leave that part to you if there is some other operator S that satisfies the equation inner product E x with y is equal to x, s y if there is some other operator S such that inner product E x y equals x, s y for all x and y then S is equal to T star ok.

You can show that S is equal to T star, so T star is unique.

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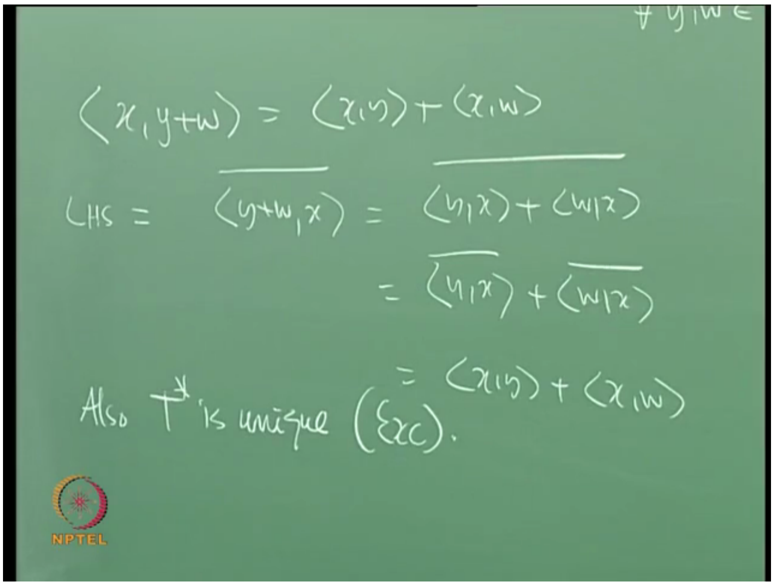

So I will just mention that you fill up the last line also T star is unique I have told you how to prove it but in any case that is not difficult so T star is unique. So lets go to the definition of the adjoint using this result.

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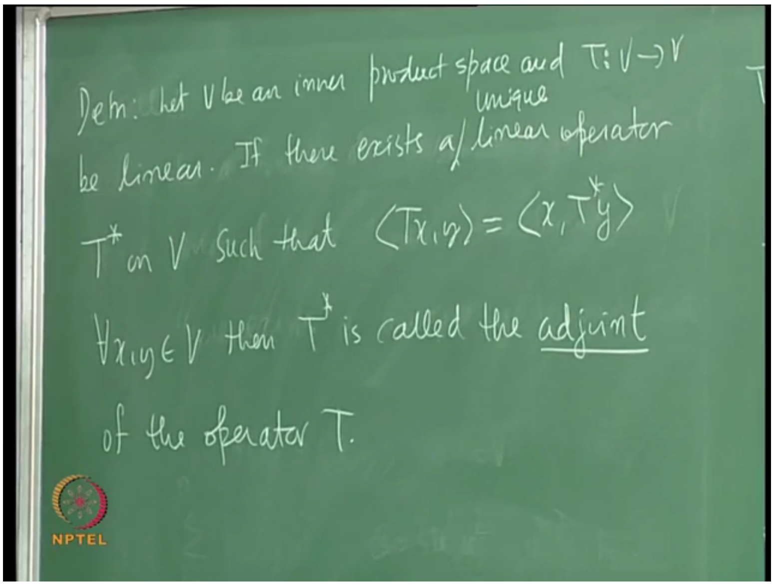

Let V be an inner product space and T be linear. If there exists an operator a linear operator T star on V such that inner product T x with y equals x, T star y, if this is satisfied for all x, y in V then T star is called if there exists a again unique linear operator then T star is called the adjoint of the operator T, the adjoint operator.

So what we have proved? Now is that if V is finite dimensional then any linear operator has the adjoint ok. So remember that this T star is defined by mean of this equation so it depends on inner product T star offcourse depends on T it will also depend on the inner product because it must satisfy this equation ok. We look at some examples of adjoint operators how to compute the adjoint. Infinite case the adjoint may not exists. You can look at I will just give values answer you can look at the differentiation operator, differentiation operator on the space of continuous function.

You can show that this does not have the adjoint if time permits I will discuss this also if its infinite dimensional adjoint may not exists for a linear operator ok. You will discuss this in your functional analysis course, infinite dimensions so we discuss in the functional analysis course ok. Look at some examples but before that lets settle one or two easy questions that would arise naturally for instance given a inner product space given a finite dimensional inner product space along with an ordered orthonormal basis I write down the matrix of T relative to that ordered orthonormal basis. What will be the ordered what will be the matrix representation of T star?

What do we expect the answer to be? It can't be the same unless T is self adjoint ok but before that lets address this question ok recall that when we looked at the notion of why an orthonormal basis is better than an ordinary basis. In the representation of vectors that is if you look at the representation over vector with respect to a basis, x equal to let us say the basis is U1 U2 etc U n just an ordinary basis then x is alpha 1 U1 etc alpha n U n where alpha 1 etc alpha n one needs to compute by solving a system of equations whereas if U1 U2 etc U n where an orthonormal basis you don't have to compute this numbers they are just inner products of x with U1 x with U2 etc.

So computation of the coefficients that becomes easier similarly remember the matrix representation of a linear transformation that we have discussed. So let me discuss recall this and then go to the inner product space. How is the matrix of a linear operator relative the basis is written?

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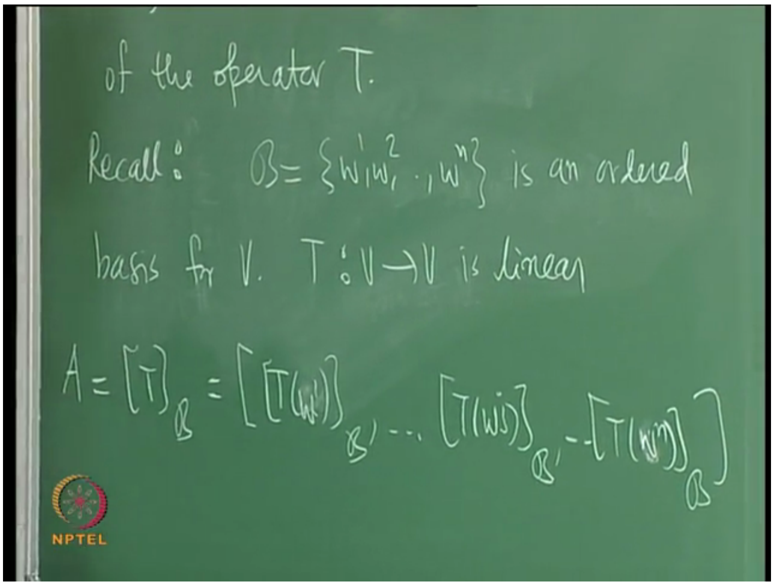

So this is also something we can quickly recall, I will use U1 U2 etc for the orthonormal basis lets take B to be say W1 W2 etc W n this is an ordinary basis this is an ordered basis for a vector space V ok there is no inner product space, no.

For vector space V T is also linear that is given T is linear then how do we write down the matrix of T relative to this basis B? We look at a more general notion of how we write down the matrix of T relative to two basis if the vector space see we discuss V to W if W equals V then it is convenient to deal with just one basis that is this, so what is this? This is the matrix of T relative to B I will call that A that is given by the image of first vector I am calling it W1 write down T W1 this is the vector in V that can be written in terms of B collect the coefficients that is the first column that is this etc T W j this is the jth column T W n this is the nth column, this is how you write down the matrix of A relative to this basis ok.

This means, what is the formula then? Connecting the entries of A and the right hand side images T of W j for instance, do you remember that formula? What is the formula? All you have to do is just look at the left hand side take the jth column the jth column of A, what are those entries, jth column means the second entry varies is that sorry the first entry varies. Is it ok? Tell me if this is clear?

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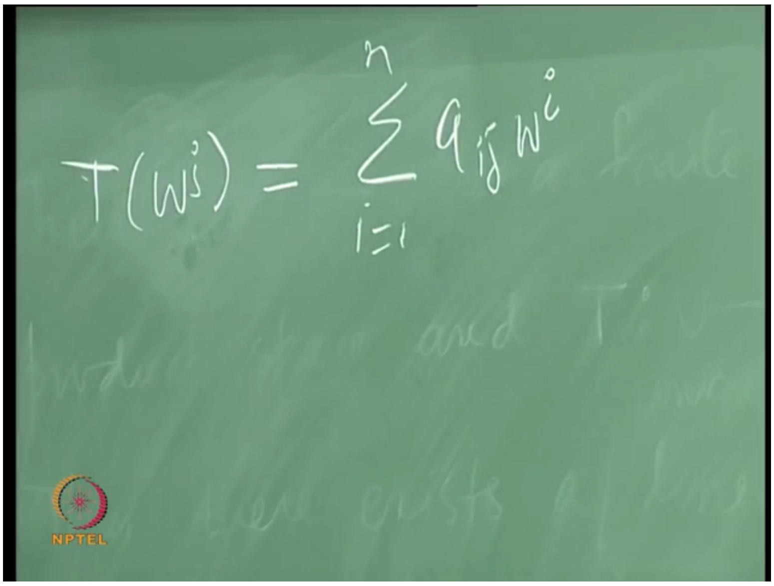

The formula for A i j is what I want, this is I want the formula for T U j, T W j I want the formula for T W j so what is that?

In terms of the matrix A in terms of the entries of A, do you remember this? J summation i equals 1 to n a i j Wi is that ok? That is because if you write T W j is the jth column does this give the jth column, the first entry is what vary so this gives the jth column ok this is the formula for the matrix of a linear transformation relative to a basis and how do you compute this a I j again a I j's are (compu) it is the jth column so you go to the jth column that is you look at W j image of T under W j, image of W j under T relative to this basis you write a representation that is amounting to solving a system of equations ok.

So in general if you have an ordinary ordered basis this coefficients are obtained by solving equations but if it is an orthonormal basis then you would expect the computation to be easier, so let us first prove that.

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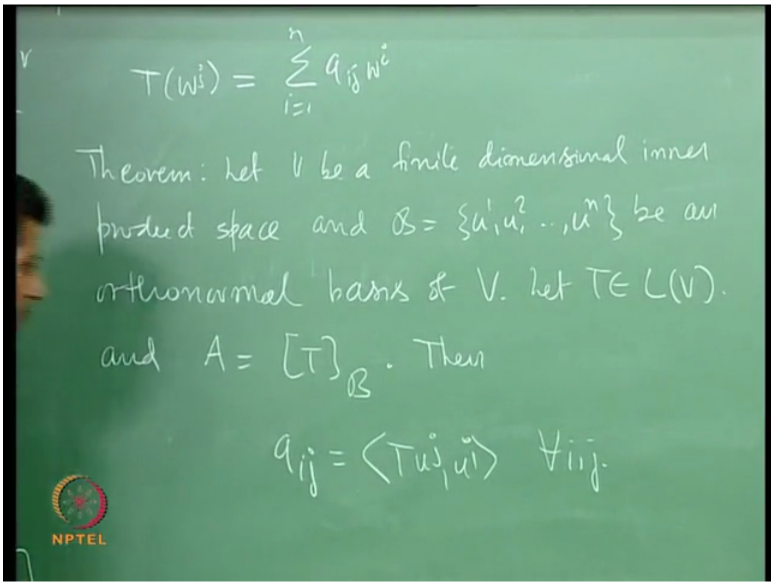

Let V be a finite dimensional inner product space I expected altleast one of you to answer this, let V be a finite dimensional inner product space and B equals U1 U2 etc U n be an orthonormal basis of V let T be a linear operator, you remember this notation L of V is a set of all linear transformations from V into V.

Let T be a linear operator and A be the matrix of T relative to this orthonormal basis B then can you make a guess? That is not correct but let me ask you the question, what is the formula for a i j? What is the formula for a i j? Yes it is an inner product yes inner product of what with what? No, now the basis is orthonormal basis U1 etc U n ordered orthonormal basis it is an ordered orthonormal basis I have not written but this an ordered orthonormal basis I want the formula for a i j, what is a formula for if I write x as a linear combination of an orthonormal (basis) in terms of an orthonormal basis? Ok so what is the inner product here?

There is no x, it is which the other way around, T u j with U I, this is the formula ok. Which means what you simply take the take one inner product look at the image of U j under T take the inner product of that with U i then that gives the entry i j, this is the formula for a i j as before the entries the coefficients or the so called coordinates of a vector x relative to an orthonormal basis are obtained by simply taking the inner product of that vector with each of those orthonormal basis vectors, you get this ok this is analogue to that result.

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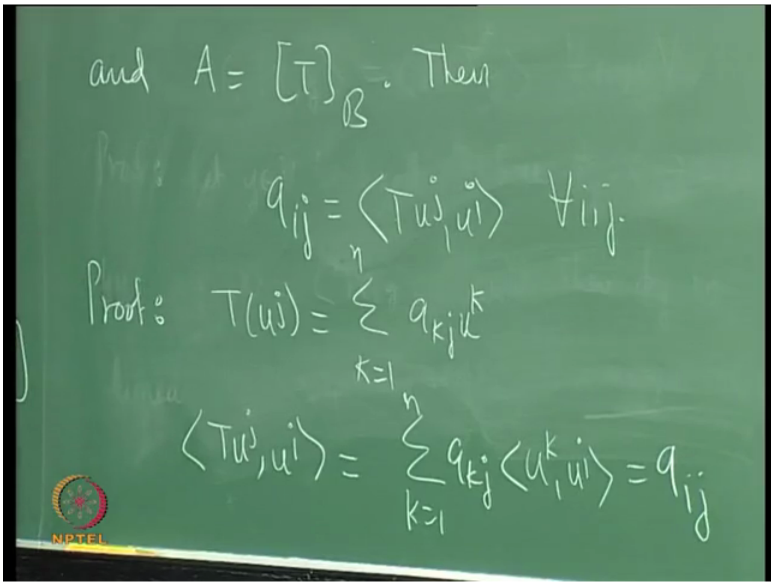

Now to prove this just use this formula, proof I want to look at the U j T U j so I will use this formula instead of W j W i it will be U j U i so T U j is summation ok I am using i here so I will use a k here k equal to I am just rewriting this formula I am rewriting this formula relative to the spaces. Summation k equal to 1 to n a k j Uk ok that is this formula this is T U j all I have to do is look at inner product T U j with U i lets not do all this just k equals 1 to n a k j that comes out inner product this is the first term so this is Uk Ui k is a running index k is the summation index when see this i is fixed, when I write this formula this i is fixed for me.

So when k is running index when k takes a value i this is 1 all other entries are 0 because an orthonormal basis when k takes a value i it is a i j all other terms are zero, which is what we wanted to prove T U j U i is a i j so this is a formula connecting this is a formula for computing the entries of the matrix A it is obtained by this n square inner product ok remember this formula connecting ok again this problem go back to the vector space case not the inner product space, vector space case we define the so called transpose of a linear transformation, the transpose was shown to be unique.

So the question is if A is the matrix of a linear operator T relative to a basis B, what is a matrix of T transpose as a linear transformation? We have seen that it is a transpose of the matrix of T ok and entirely similar results holds for the adjoint operator that’s is very easy, so you can think of this star as a kind of generalization of the transpose operation.

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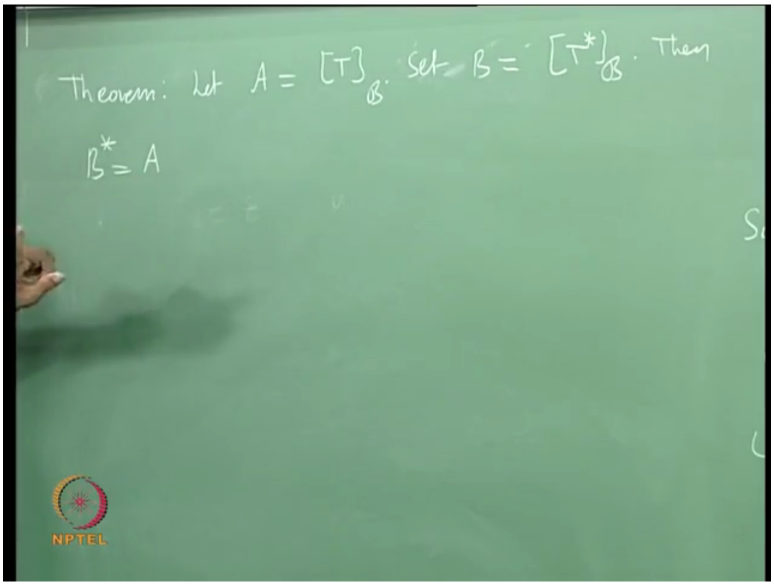

So I will state that as lemma or a theorem, I will be cryptic I will follow what the notation that we have used before I will simply say that my matrix A is the matrix of the transformation T relative to B then the matrix of T star ok.

Let I will call B as the matrix of T star relative to the orthonormal basis B then the conclusion is B star equals A. Remember the star of a linear transformation on an inner product space as been define just today ok but what I am writing down here is the complex conjugate of matrix see this A and B are matrices that we know the complex conjugate of matrix we know. You interchange rows and columns after taking the conjugate transpose after taking the conjugate right. This matrix A has complex entries take the complex conjugates of that of the entries of A then take the transpose you get A star so this is how B star is constructed

So the matrices of T and T star relative to this particular basis relative to any basis that you start with provided you stick to that basis for T star they are related by this formula.

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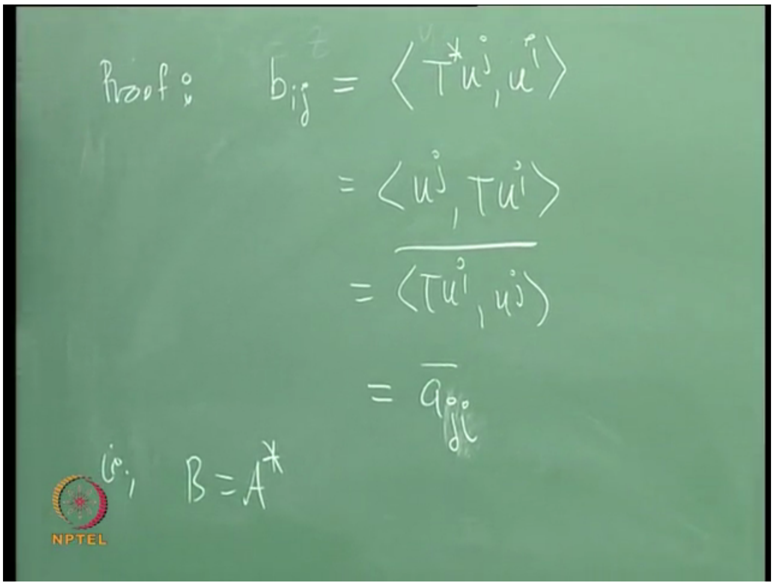

So proof again you have to simply use the formula for the entries of B star and A ok. So ok lets start with lets use the previous one. Let me write down b i j, what is the formula for b i j? b is defined through this b i j by definition must be T star U j U i is that ok? T star U j U i but this is U j T U i because that is how the adjoint is defined but this is T U i, U j the conjugate of that.

But T U i, U j go back to the previous result is a i j, a i j bar, a j i bar yes this first and then this a j i bar that is what we wanted to prove that is B equals A star a complex conjugation we know you do it once more you get the same matrix.

Student: (())(32:08)

See this formula is correct right, b i j next step, how is the adjoint defined? We are not using that, see T star is T is correct but we are using that, I have not proved it so I am not using it. The same thing can be used like what the argument that I gave for the second argument that can be given here. No but where I have used that I don't understand see T star U j U i this T star ok see you can take the conjugate and then go back to this, you want that explanation? Ok lets you have a objection then, what you want is an explanation for this, right?

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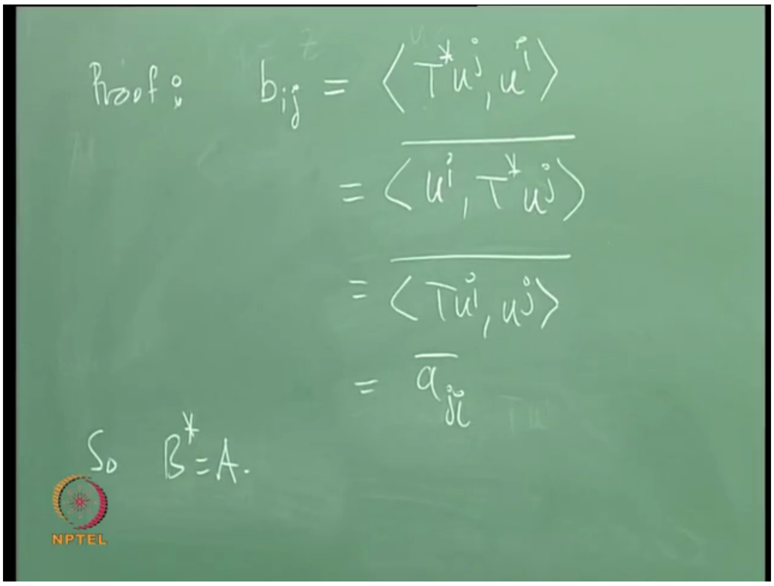

So I can write this as U i T star U j conjugate ok, then it is a conjugate of T U i, U j but that is the same as U j, T U i sorry the proof of ends here right, T U i U j, will be a j i going along with the bar, so agree? Is that ok? b i j, I j that is T star and then it is Ui with T star U j conjugate keep the conjugate then I use the adjoint operation T Uu U j but T i Ui U j is a j i that goes along with the bar so probably a more direct proof if you want and so as I mention before B star equals A I have shown that B is equal to A star but the star operation is of order is of period 2, you do it once more you get the same matrix, any problem again?

This proves B equal to A star but you do the star once again because it is a matrix now then I get this so I still don't prove that for a linear transformation T, T star star is T I am not using them, I am using it for a matrix I am using what it tells me for matrices. For matrices you can verify that by just writing down the entries right ok look at some examples both

Student: (())(35:15)

Professor: You can't expect this to have, it is only with respect to the same basis that the relationship quotes infact if you look at just an ordinary basis which is not an orthonormal basis then this relationship does not hold if it is not an orthonormal basis just an ordinary basis then this relationship is not true, it is a more complicated formula. How A and B are related if it is just an ordinary basis will not be the same as this that relationship is different, it is more complicated. But what is more meaningful and what is more applicable is writing it relative to the same basis ok.

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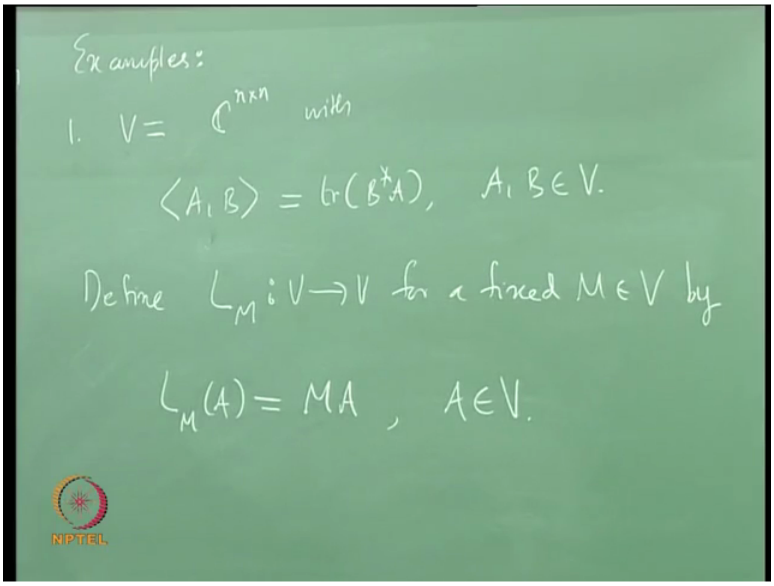

I wanted to discuss examples let me take to one finite dimension the other one infinite dimension, examples of transposes, how do you compute the transpose of certain operators. Let’s take V as C n cross n with the trace inner product with let us say if I have A B in V then this I trace of B star A with respect to the trace inner product this was given as one of the examples of inner product space. Define the operator L M from V to V for a fixed M in V by this formula L M of A is M times A this is a left multiplication that is why this L is given.

Left multiplication by M, A is the variable M is fixed A is a variable left multiplication by M, what is the adjoint of this operator? Lets calculate by the way this is linear that is an easy exercise L M is linear that is left to you. I want to calculate the adjoint, to calculate the adjoint I must appeal to the formula the first principle definition I want to look at L M A, B I want to write it as say I want to compute T star so I must write T x, y as x , T star y, so I must write this as A, something of B that something will give the T star L M star.

So this is what this is by L M A I will apply the definition its MA, B that is trace of B star the second one.

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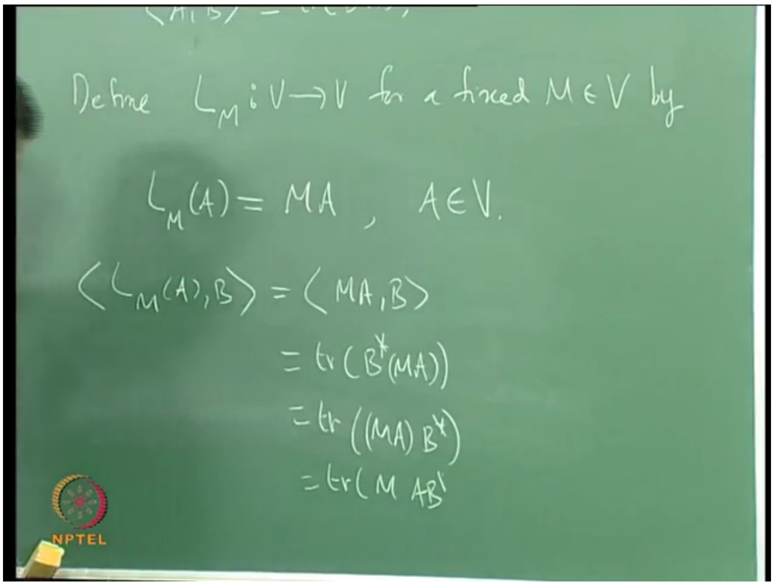

So this is trace of B star MA, multiplication is matrix multiplication (())(38:42) I still insist that I write this bracket I appeal to this fact the trace of the product that is trace of let us say C times D is trace of D times C, I will keep applying that trace of till I am satisfied, trace of MA, B star that is the same as trace of M times AB star right, B star MA is trace MA B star I will write this as trace MA B star so I wanted to take A outside ok. No I will not write like this, I want to keep A to the left.

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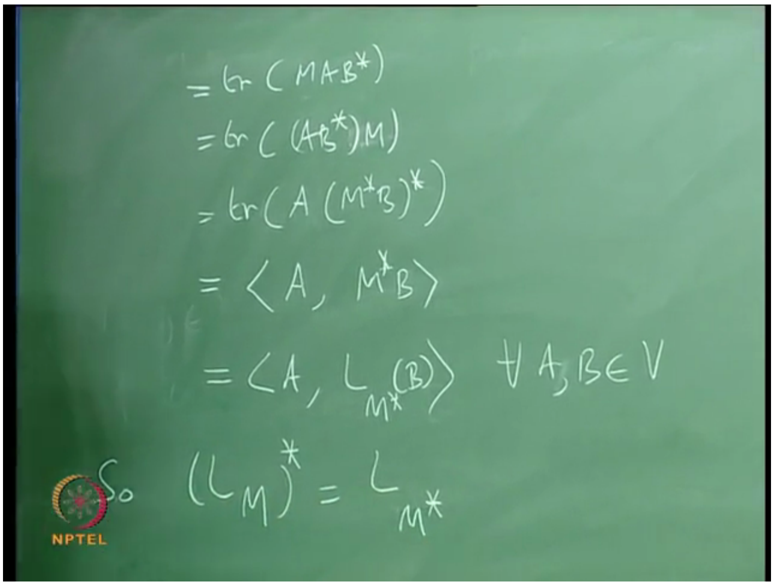

I will write this as again the same thing AB star M which is trace of A M star B whole star this is A into B star M, B star M is this whole star again I am dealing with this reverse order law for matrices not for transformations which I have not yet proved, so I get this but this is the same as now I have taken A outside and star of something so remember that the formula for the inner product then is given by AM star B ok, do you agree? This can be written as A, M star B is L M star of B left multiplication of M star with B, L M star of B and so what does it mean?

If you look at L M star that is L M star this is true for all A B, so the conjugate of L M is L of the conjugate with respect to M star M is what I started with. Left multiplication the conjugate is left multiplication with the conjugate of that matrix ok, this is for the finite dimensional case. Lets look at an infinite dimensional example.

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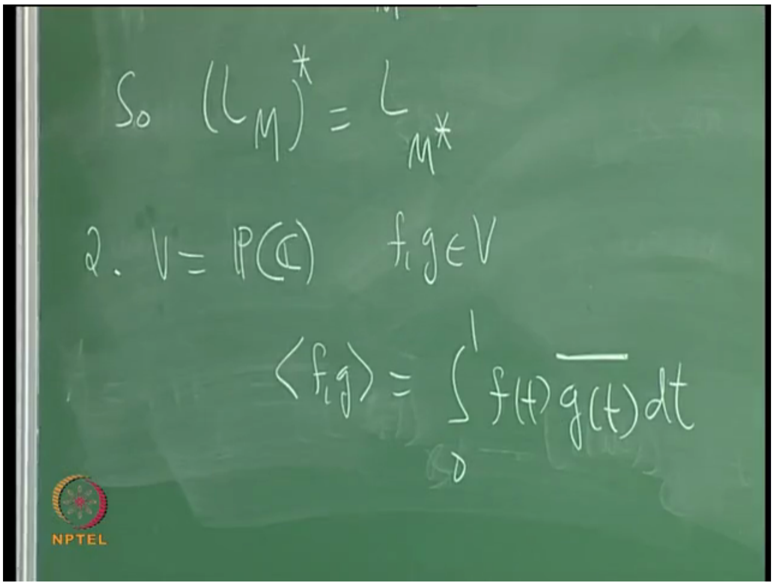

Let's consider the space of all polynomials with complex entries complex coefficients space of all polynomials with complex coefficients the inner product is the usual inner product integral 0 to 1, f of t g bar f g bar.

Now what is g t bar? G is a complex polynomial the variable is real coefficients are complex so g t bar, if g equal to let us say a1 a knot plus a1 t etc a n t to the n where a knot a1 etc are complex numbers g t bar will be a knot bar plus a1 bar t plus a2 bar t square etc plus a n bar t to the n ok. So I take the complex conjugate of the coefficients that is my definition, so with that definition I give this inner product this is an inner product so V with this you can show is an inner product space I will look at the following operator.

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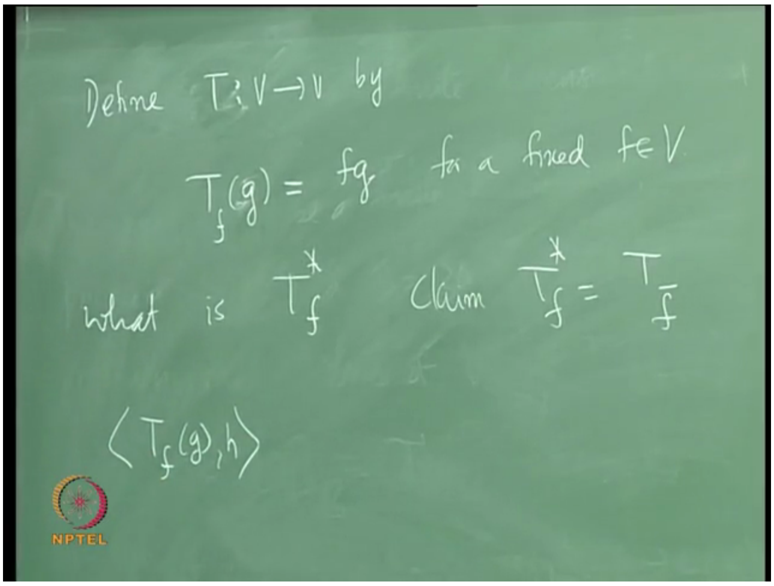

Again something similar to the previous one left multiplication by a fixed polynomial define let us say L or T, T from V to V by T of g is f with g for a fixed (())(43:41) so I can actually call T f of g, take a fixed f and g fixed f and V take a fixed polynomial simply multiply given any g simply multiply on the left on the right doesn't matter polynomial (())(43:58) is commutative unlike the previous one. What is T f? This is T f, what is T star f? That is a question? What do you expect the answer to be? What is f star for a function?

For a matrix M, M star is known, what is f star for a function? It should be f bar, will can show that T f, T star is T f bar ok lets do that quickly again first principle, use the first principle definition, I want the inner product of T f g, h I want to calculate this I must write it as g, something and then that something will give me T star.

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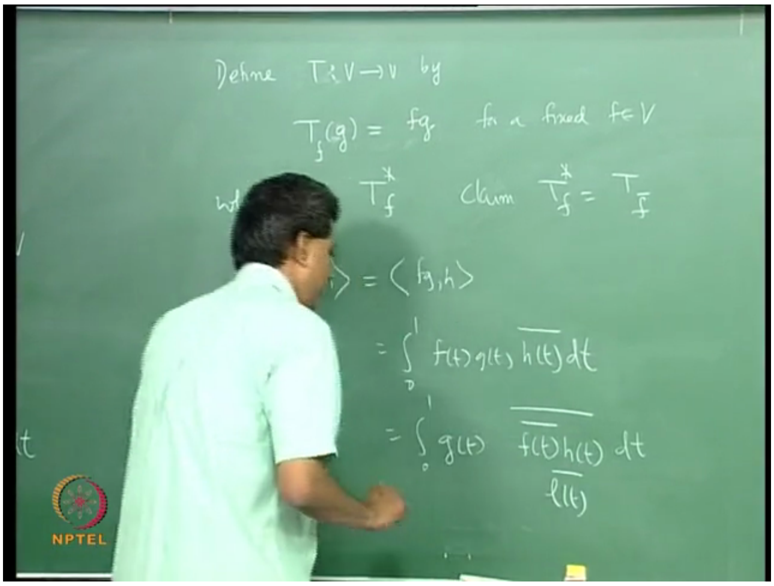

So this is by definition, T f g, h by definition T f g is f g, h for f g , h the definition is integral 0 to 1 f of t g of t, h t bar I must kind of take out g but it is clear how we should do that, 0 to 1, see this is matrix I am sorry, this is polynomial multiplication so it is commutative I can take g t outside and then combine the other two and out them under a bar. I want f to be without bar so f t bar h must go with this. So do I get what I want? Double bar gives me f single bar gives me h bar, I can call this L of t bar if you want and then write this.

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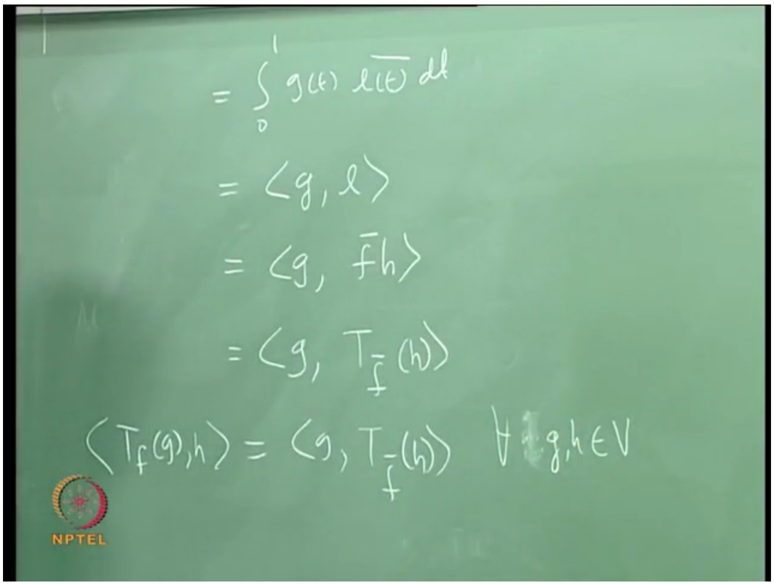

This is integral 0 to 1 g of t L of t bar dt but that is by definition inner product of g with L which is g with L f bar h which is I observe g t f bar of h.

So what I have proved is that T f g with h is g, T f bar h for all f g for all f g h for all g h I am sorry, f is fixed g h are the variable polynomials.

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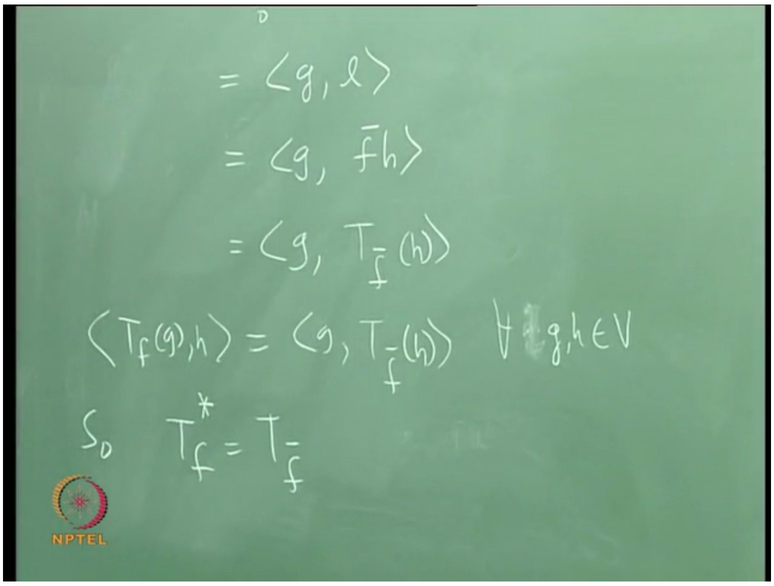

So this means T f star is T f bar ok so this example tells you that star behaves like conjugation, adjoint behaves like complex conjugate that is what I mean, the adjoint operation star behaves like complex conjugate taking the bar, it also behaves like complex conjugate see when I say complex conjugate it is Z is a complex number Z bar is what I am calling complex conjugate maybe just conjugate.

The previous example also has the conjugate L M star is M star M star already has conjugate ok, so star the adjoint operation kind of behaves like the conjugation that you do for a complex number ok. Will explore this a little further and see how some of the properties of the adjoint are similar to the properties of the complex conjugate ok, so let me stop here.