Linear Algebra
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Module 11-Inner Product Spaces
Lecture 42
Bessel's Inequality, Parseval's Identity, Best Approximation

Ok so let us continue, see I want to discuss applications of the Gram-Schmidt Process. It follows immediately from the Gram-Schmidt Procedure that the following result holds.

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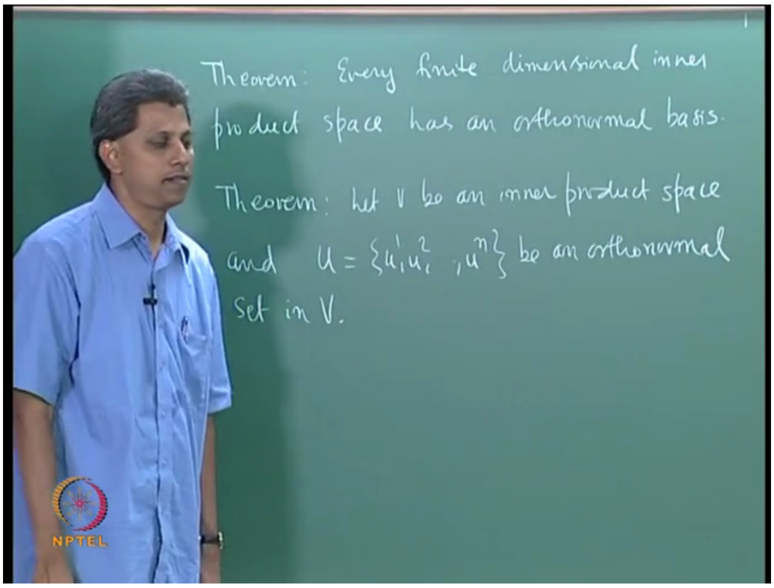

So I will simply state this as a theorem and skip the proof. Every finite dimensional inner product space has an orthonormal basis I not write down the proof, the proof is it is a finite dimensional inner product space so it is a finite dimensional vector space so it has a basis consisting of finitely many elements since it's a basis is linearly independent apply Gram-Schmidt Process to that to get a orthonormal set.

The fact that these spans are the same proves that this is an orthonormal basis also ok. So you can write down the proof on you own. I am more interested in applying this GS procedure to what is called as a best approximation problem ok. Let us develop some machinery before stating this problem. So I have the following result which will be useful later. Let V be an inner product space and let us call U as the set of all vectors U1 U2 etc U n this is not necessarily a basis but an orthonormal set of V. Let V be an inner product space and U be an orthonormal set orthonormal subset of V.

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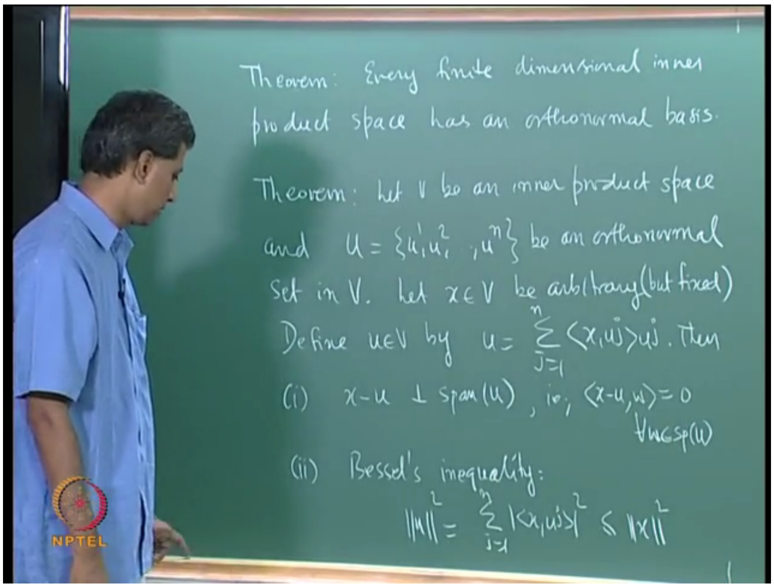

Let us take an arbitrary element x in V arbitrary but fixed element in V and define a vector U as a linear combination of this Uj's in this particular manner define a vector U in this manner for a fixed x. Then this vector U satisfies the following properties. Remember x is fixed, the first property is that x minus U this vector is perpendicular to span of U the vector x minus U is perpendicular to span of U what is the meaning? The meaning is that the inner product of x minus U with W equal to 0 for all W in span of U, it is orthogonal to every vector in the span of U that is the first property.

Property 2, is called Bessel's Inequality which is the following inequality, the first one is an equation, what is norm U square? Norm U square is summation j equal 1 to n modulus of the inner product of x with Uj the whole square and this does not exceed the norm of x. That inequality is Bessel's Inequality. So there is a formula for norm of U square if you know the coefficients. If you know the numbers inner product x Uj and this number does not exceed the norm of x square that is the second property. So norm U square less and nor equal to norm x square is Bessel's Inequality.

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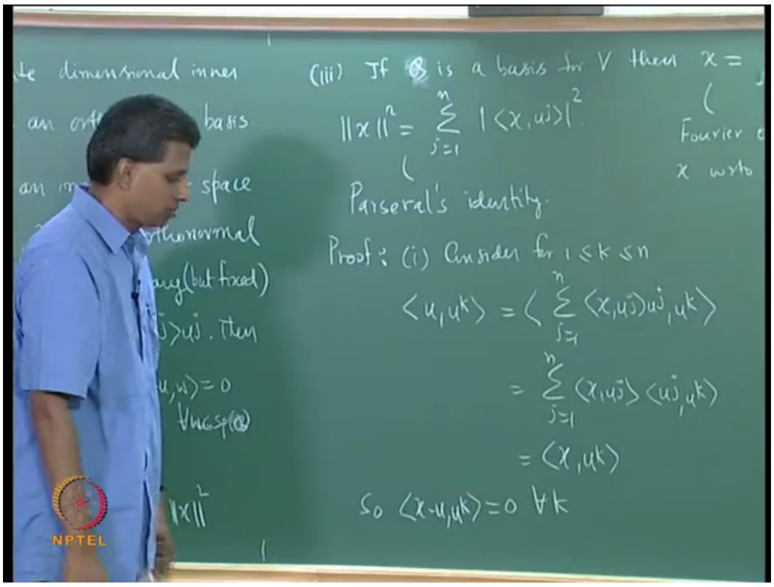

Property 3, if U is a basis for V, U is already an orthonormal set so it is an orthonormal basis, if you have that condition also then x equals summation j equals 1 to n x Uj U j and norm x square in that case is summation j equal to 1 to n modulus x Uj the whole square ok. That is a U is a basis for V then x coincide with this little U that we started with. Once you realize that the other two are straight forward. This is called the Fourier expansion of x with respect to the basis that we started with ok.

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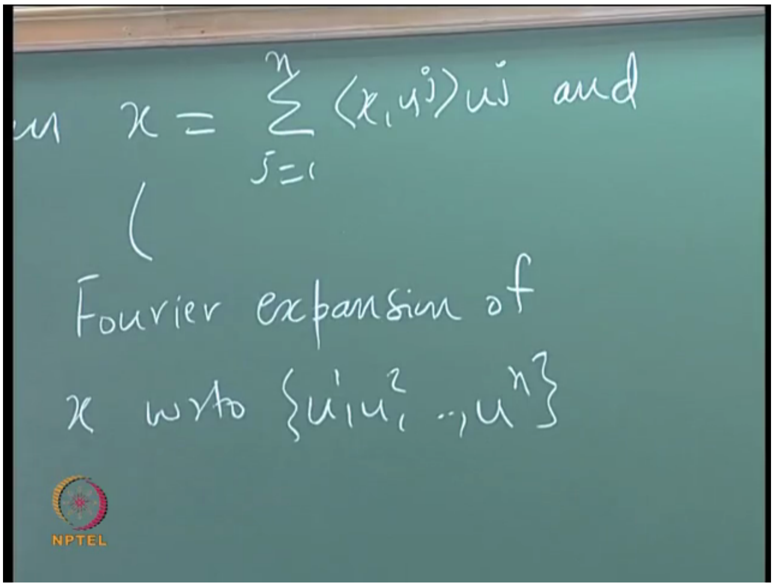

So this one is called Fourier expansion of x with respect to the basis that we started U1 U2 etc U n and this is sometimes called Parseval's Identity ok.

So let us look at a proof, first I must show that x minus U is orthogonal to span of U, U is this orthonormal set so its enough if I show that x minus U is orthogonal each of this vectors, span of U anything in this span of U is a linear combination of this and so this is enough. So let us consider U with Uk for a fixed k

Student: (())(08:01)

So I will change that everywhere

Student: (())(08:40)

Ok I replace U by B, so I am just renaming the set U by B script B ok, proof. So for each Ui coming from that set script B I will look at the inner product of U with Uk I have the definition for U, summation j equals 1 to n inner product x Uj Uj I take the inner product of that with Uk but this is the first term goes out with goes out as it is. The first scalar then I look at the inner product of Uj with Uk, j is running index k is fixed and k runs between 1 and n so when j is equal to k that is the only term which will remain all the other terms is zero because it is an orthonormal set, script B is an orthonormal set.

So this is delta j k so this takes a value 1 when j is equal to k and in that case it is just x with Uk and j equals k, this number is 1 all other terms are zero and so what we have proved is that, inner product x minus U with Uk zero for all k and so it follows that x minus is perpendicular to span of B ok, that proves the first part ok.

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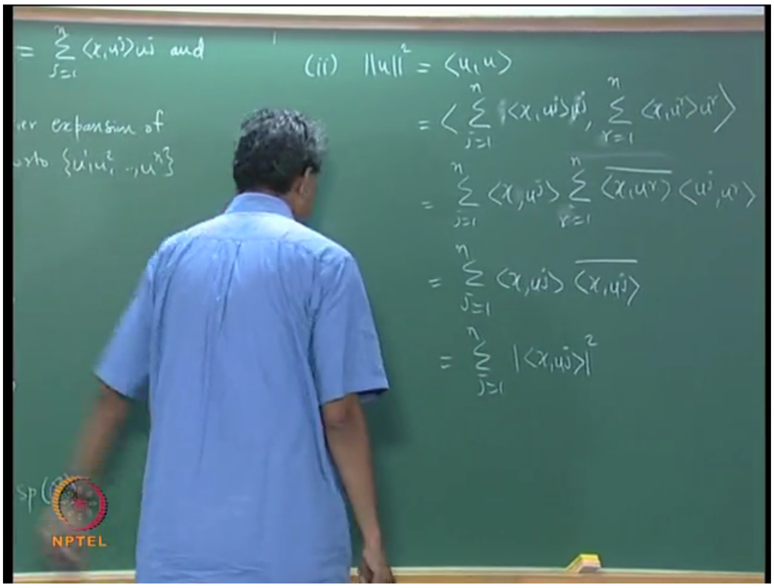

Second Bessel's Inequality, just use this relationship but doing the norm square and the inner product and expand, this is summation j equals 1 to n mod x Uj Uj second term I will use let us say r, r equals 1 to n x with Ur Ur, this is summation j equals 1 to n x with Uj summation j equals 1 to sorry r equals 1 to n x with Ur the whole thing goes with the conjugate.

I can apply the conjugate to each other terms, so and then the rest is Uj Ur ok. You can do it either way first look at this sum r equals 1 to n where j is fixed and then the summation is over j. So when j is fixed r takes a value j this will be 1 all other terms is zero, so when r is equal to j this will be x, Uj with a conjugate. Summation j equals 1 to n x, Uj into conjugate x, Uj bar. See for the second sum r is running index and r takes a value j, j is fixed. Now this is simply complex number into its conjugate so modulus square ok so that is the formula for norm U square so if you know the coefficients then you can compute its norm.

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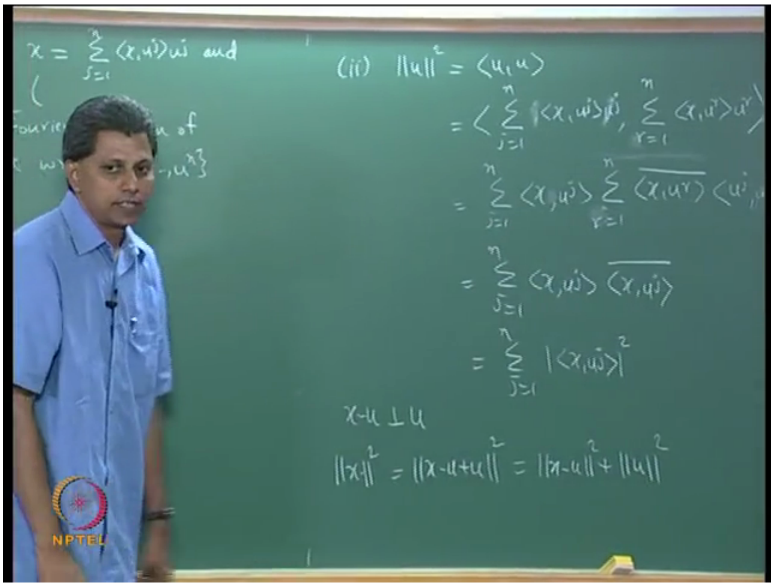

Ok we have a second inequality that we need to prove, norm U square is less and or equal to norm x square but remember that x minus U is perpendicular to U that is from the first part use Pythagoras theorem. Norm x square is equal to norm x minus U plus U square by Pythagoras theorem this will be norm x minus U square plus norm U square ok. If you look at norm x minus U square it is a norm negative number so this is I have added something non-negative so that much be greater than or equal to norm U square.

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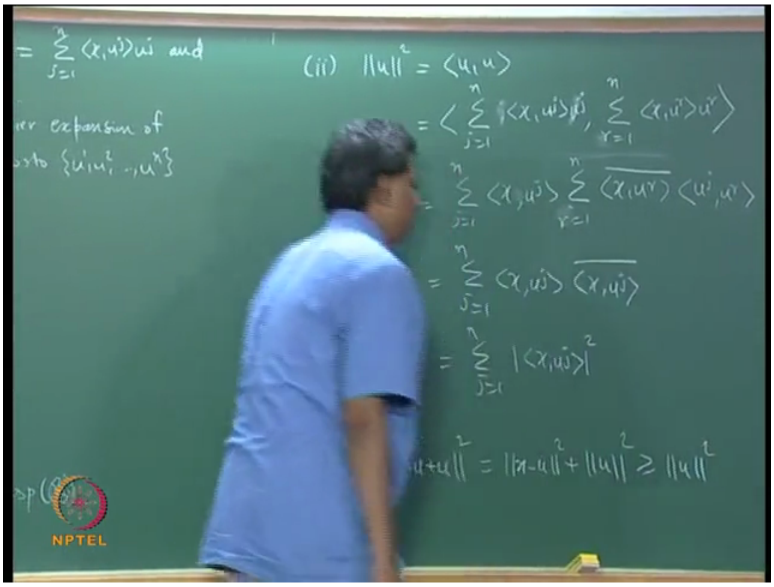

So norm x square is greater than or equal to norm U square so norm x is greater than or equal to norm U that is Bessel's Inequality ok.

Use the fact that x minus U is perpendicular to U, U belongs to span of B by definition U belongs to span of B that is what that is why I could use the first part x minus U perpendicular to U ok third is immediate once we prove that X is equal to U ok.

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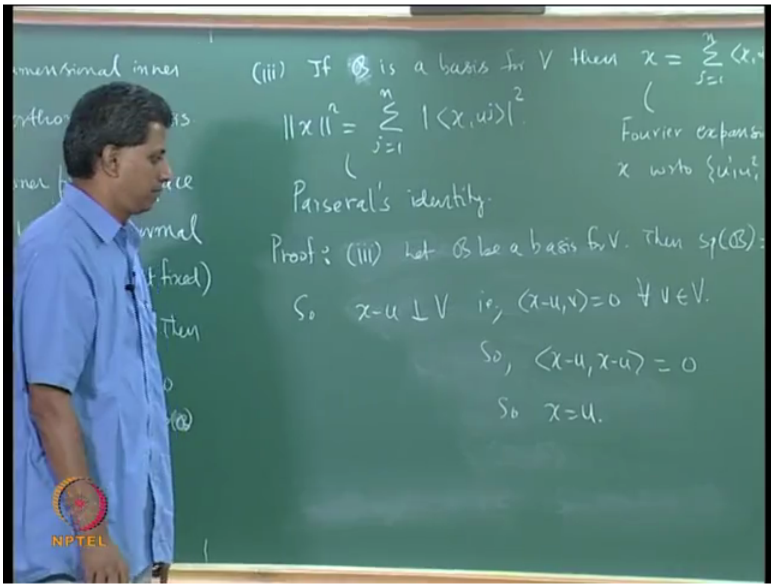

So let me I will prove that here itself, I will prove that x is equal to U then it follows from what we proved into that these two hold. If I prove x equal U then we know the formula for U that I have written down, this is a formula for U, if x is equal to U that first thing holds if x is equal to U from norm U square equal to summation modulus square this follows ok, that is easy.

Look at if let this be a basis for V then span of B is V till now it is not a basis in three we are assuming it's a basis also so span of B is V, now what does the first part say? First part says x minus U is perpendicular to span of B so x minus U perpendicular to V which means x minus U inner product of that with V this is zero for all V in B, in particular for V equal to x minus U we get the following, x minus U x minus U must be zero but x minus x minus U is norm x minus U square ok you can even use the thing from the inner product. So I will say this is zero from the first two definite (())(16:46) inner product it follows that x is equal to and so the third part follows ok.

So if B is basis then span B is V is what is being used ok. This is just to develop machinery for a particular problem that I mentioned briefly the problem is the following.

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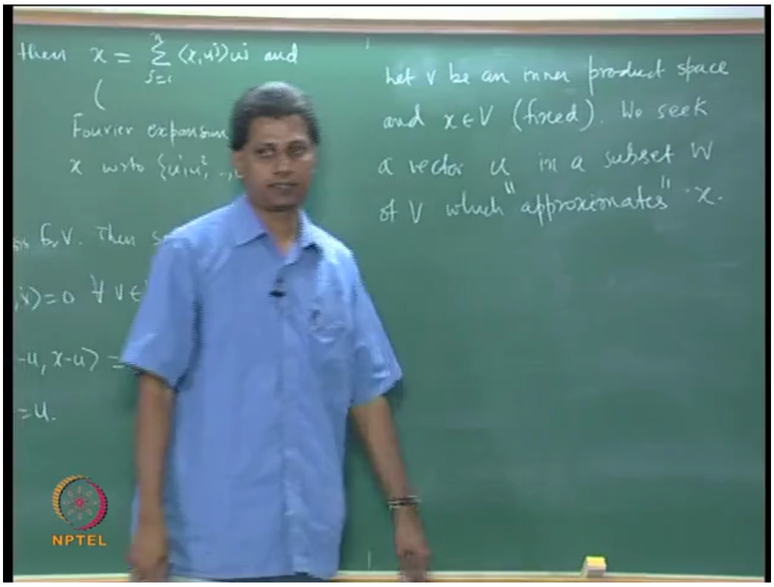

Often we consider the following situation given an inner product space and a fixed vector we seek a vector U in a subset W of V which approximates x. Given a fixed x in an inner product space and a subset W among all the elements from their subsets W I want to choose a particular vector U which satisfies the property that it will serve as an approximation for the x that I started with ok. Let me give one or two examples, one example motivating example from (())(18:53) geometry.

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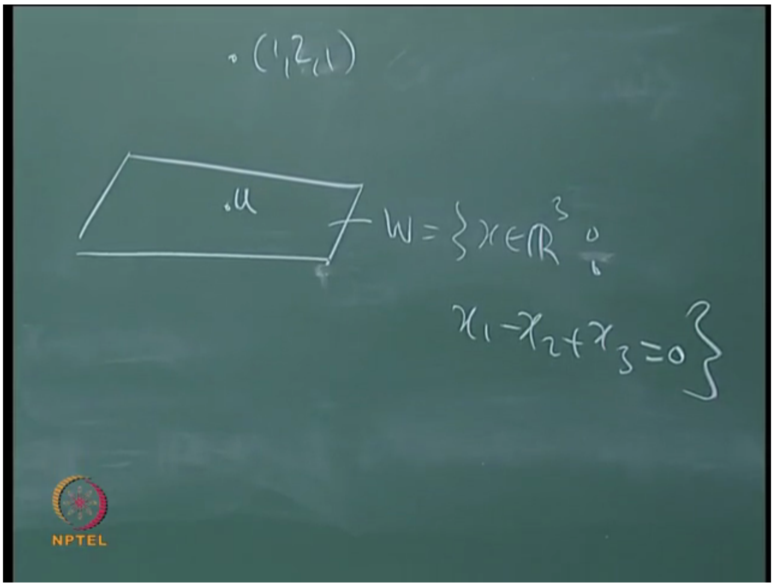

Suppose I am given a vector let us say 12 1 in R3 I want to find a vector on this plane this is W, W is the set of all x and R3 such that it's a x1minus x2 plus x3 is zero. For example we ask this question what is the length of the projection from a point onto a plane? What is the distance of a point from a plane etc? I have a point to this is x this is my subset W in this case it is a subspace the point U can be thought of as an approximation to the point x, 1 2 1 already belongs change the equation 1 2 1 is already there so let us say this is my equation x1 plus x2 plus x3 is zero, the question is, is there a point U in W that will approximate x? From (()) (20:20) geometry of three dimensions we know that you need to just project x onto this plane.

That projection onto W will give the vector U, that projection will give the vector U, that is also observe in this case that the vectors x minus U is perpendicular to the plane, the vector x minus U is perpendicular to the plane ok. Remember what we proved in the first part of the previous result. So these are related, x minus U is perpendicular to the plane because it's a projection ok, this is one motivating example. Sometimes in the space of functions in space of continuous function, let say C 0 1 we would be interested in approximating the exponential function in terms of just writing down as a linear combination of certain polynomials, certain polynomials you look at the subspace span by those polynomials then I would like to write down the exponential function in that space.

So what is the approximation to that exponential function in that subspace this is one another question ok. So let us pose this problem we need to understand this meaning approximation and then see how this problem can be solved ok. See in the finite dimensional case this problem can be solved, in the infinite dimensional case it is not there are examples where the problem does not have a solution ok, but will confine our attention to the finite dimensional case.

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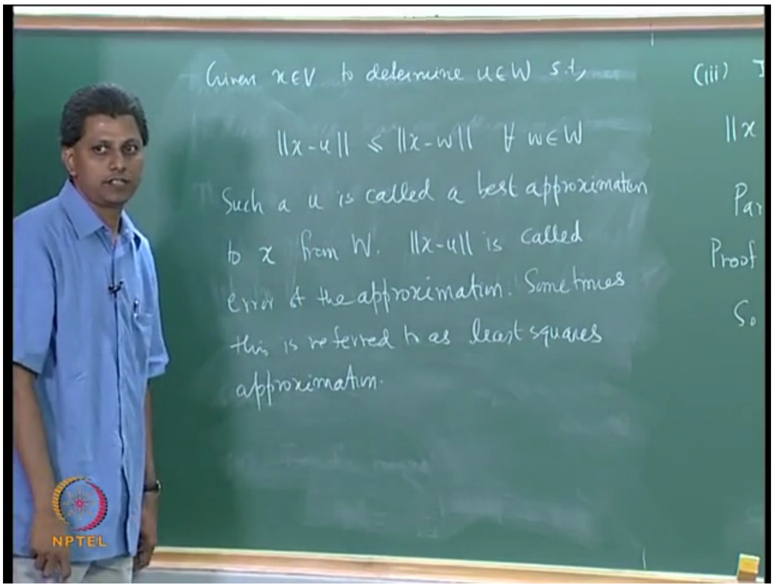

So let me rewrite this problem mathematically, the problem is given x in V we need to determine if possible U in W such that so it is an approximation, approximation is given in terms of a norm, norm induced by an inner product.

Given x in V to determine U such that norm of x minus U, this particular U this norm should not exceed the norm of x minus W for all W, this is the problem. So its just not an approximation it is the best approximation in that sense, such a U is called a best approximation, a best approximation to x it also depends on the subset W so it is a best approximation to x from W, this will obviously change if I change W. So I need to minimize this norm x minus W for all W and W, to see whether there is a U in W that satisfies this condition ok, this is called the error of the approximation.

Norm x minus U is called error of the approximation, now in many practical problems this norm is the two norm that is sum of squares or integrals of square of functions, so this is many instances called the least squares approximation and this vector U sometimes is also referred to as a projection of x onto W, there is a motivation coming from geometry the problem that we considered just now. The vector use a projection of x onto the subspace W so sometimes U is called a projection of x ok that is another terminology.

In general as I told you, in general this U may not exists it may not be unique ok but for in the finite dimensional situation the compute the existence is there with the computational follows by using Gram-Schmidt Process ok. So let me prove that result next and go back to this problem and see how this problem can be solved by this method. So this proof is constructive, in the next theorem tells you how to construct the best approximation in case it is unique.

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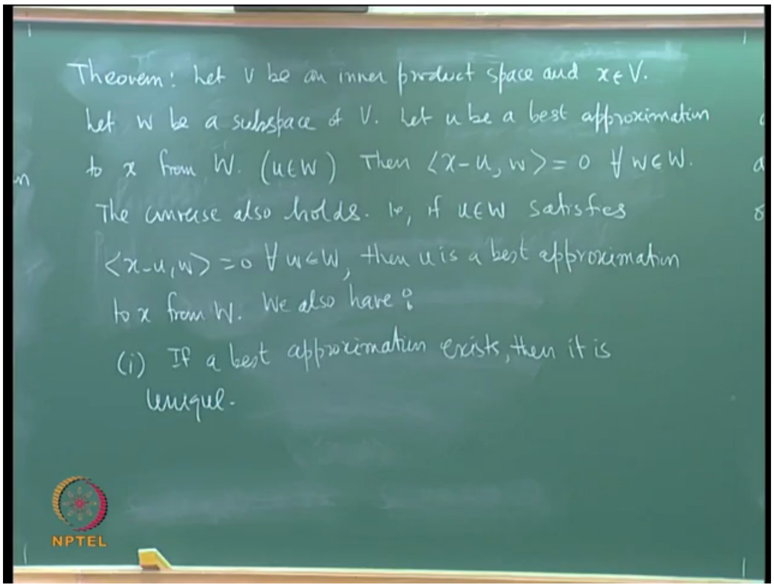

So I have an inner product space and x is a fixed vector we need W to be subspace. Let W be a subspace of V, let U be a best approximation to the vector x from the subspace W. See what this also means is that U belongs to W ok among all the vectors in W U is the one that minimizes norm x minus W ok, let U be a best of approximation x from U then x minus U with I will use W, this is equal to zero x minus U must be perpendicular to W for all W in W. The converse also holds that if U is a vector in W that satisfies this condition then you must be a best approximation to x converse also holds, ok let me write just down that is if U element of W satisfies ok that is the converse.

You also have the following, so I will call this 1, what we would like to have? Ideally if a best approximation exists then it must be unique that is always the case, existence might be a problem but if it exists then it is unique. In the case of an inner product in the case of a see this problem is posed as m, and problem involving norms. In a general norm linear space it may not exist but if it is a norm induced by inner product in the general norm linear space it may not be unique but if it is a norm induced by an inner product if it exists then it is unique ok and second property.

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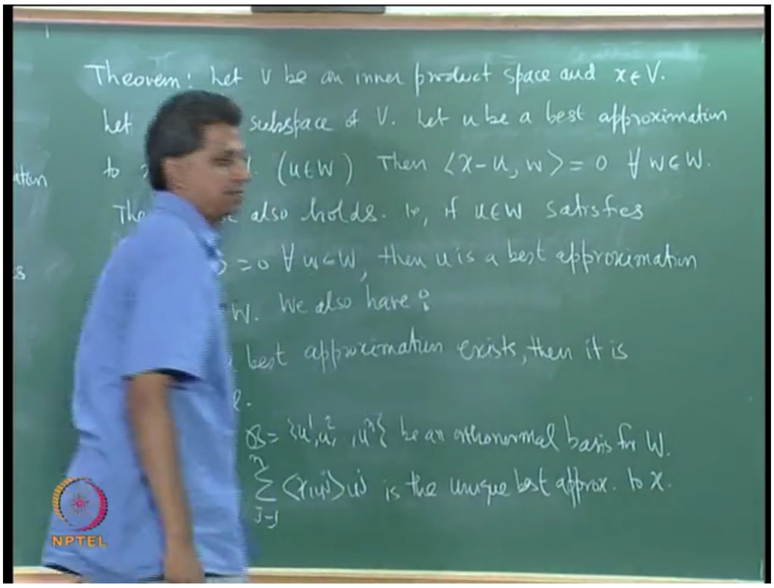

Second part tells you how to construct this U, let this be U1 U2 etc U n be an orthonormal basis for W. So this last part assumes that W is finite dimensional V can be infinite dimensional, there is no condition on V, V can be an possibly infinite dimensional inner product space but if W is finite dimensional subspace then the last part tells you the formula for (best approximation) then U is summation j equals 1 to n x Uj Uj is the unique best approximation to x again this should remind you of the formula for U that we defined earlier ok, so if W is a finite dimensional subspace of V and if you have this orthonormal basis then the best approximation to x is given by this explicit formula.

Ok you know x, you can compute these numbers, so this U can be determined ok. Maybe I will give an example will take the same example and use this construction given in 2 and then solve the problem and then come to the proof of this result. What is U that satisfies norm x minus U less and or equal to norm x minus W for all W in capital W. Probably we could look at the other methods of solving this problem.

For instance how would you solve this problem? Using techniques from (())(32:28) geometry of 3 dimensions. Do you remember? Yes

Student: (())(32:51)

Can you use calculus? Because the minimization problem, so how?

Student: using the calculation x1 plus x2 plus x3 zero, by using Lagrange method we can find.

You must use Lagrange (())(33:43) for functional several variables to determine this minimum ok. You try both those methods and also the method that comes from applied linear algebra ok. What does this tell you? This last part tells you that you must first construct an orthonormal basis for W ok. Let us consider a basis for W, see this is a problem the problem is to determine U that minimizes this distance where x is this vector, W is this subspace ok. I want the basis orthonormal basis for W. What is a dimension of W?

Student: 2

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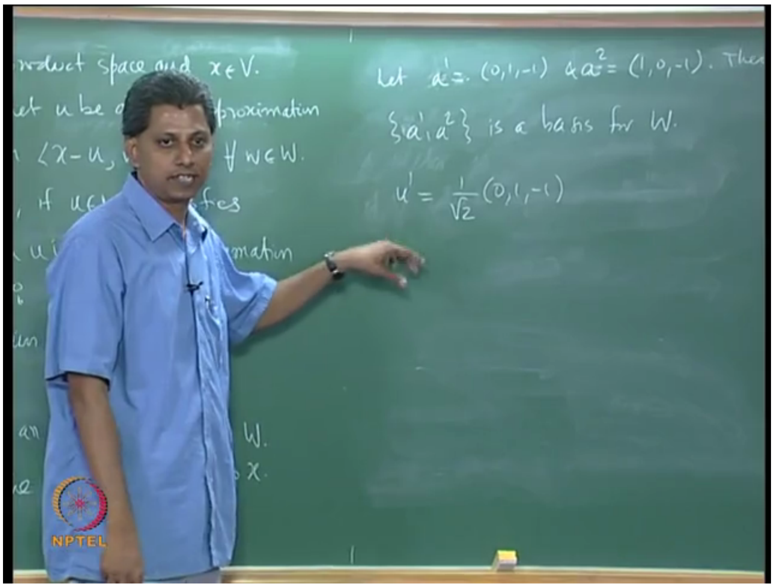

So I need to determine two vectors, so let us call U1 as 0 1 minus 1 and U2 to be 1 0 minus 1 both these vectors belong to W and they are linearly independent, one is not a multiple of the other. Ok I shouldn’t call it U1 U2 I will call it a1 a2, U1 U2 is a notation for an orthonormal basis. So a1 a2 is a basis so we need to apply Gram-Schmidt Process for the first one ok. For me U1 is 1 by root 2 into 0,1 minus 1 again for convenience I am writing the vectors as row vectors, this is U1 and what is U2?

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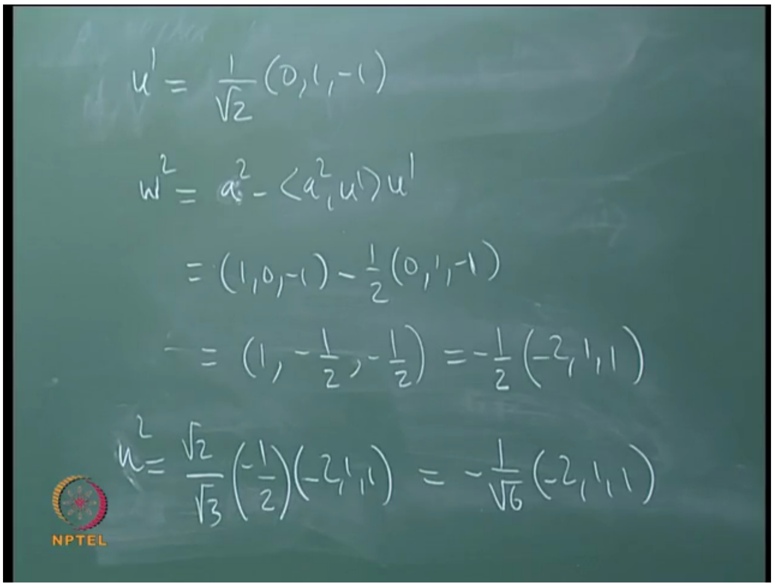

To determine U2 I must use W2, W2 is U2 minus sorry a2 minus a2 U1 U1, a2 is 1 0 minus 1 minus a2 with U1 this two terms are zero 1 by root 2 there will be another 1 by root 2 that is 1 by 2 into U1 0,1 minus 1.

So that is 1,0 minus 1 by 2 minus 1 plus 1 by 2 minus 1 by 2, so this is orthogonal to this and the norm of this so I can write this as I will take minus 1 by 2 outside then it is minus 2,1,1. So that I get 1 minus 1 by 2 minus 1 by 2, V2 is what is the norm of this 1 by 4, 2 by 3 root 2 by 3 into W2 minus 1 by root 6 into minus 2 that is so this I should call it U2, 4 plus 1 plus 1 so norm is 1, this is orthogonal to this vector ok.

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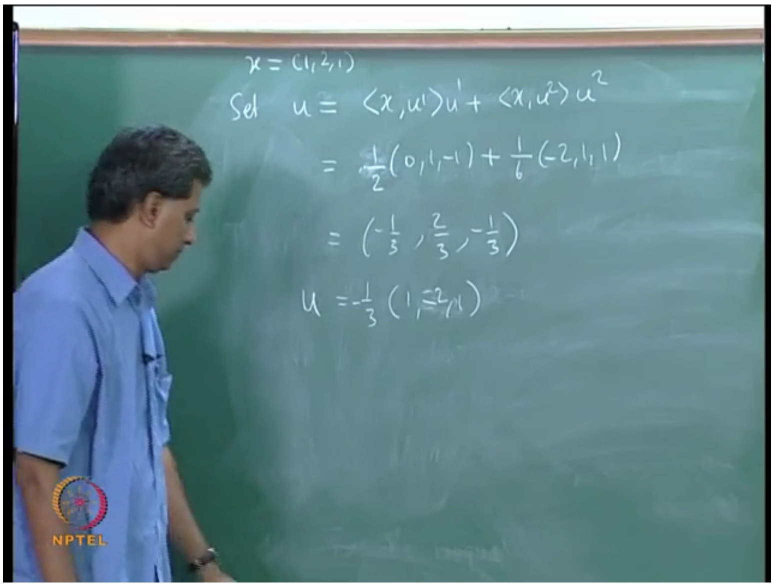

Now determine U from the formula that is given here, x is 1, 2, 1 right, ok call U as summation so that is x, x is 1, 2, 1, x, U1 (1 plus x, U2) x, U1 this two, this goes with the minus 1 just 1, 1 by root 2 into U1, U1 also has a 1 by root 2 plus U2 goes with 1 by 6 inner product x with U2 that is minus 2 1 1.

Minus 2 1 those two gets cancelled I get just a 1 again into U2, U2 is minus 2 1 just check if my sign is correct minus so that is taken care of I should go with a minus 2 1 1 ok. So that is minus 1 by 3 then 1 by 2 plus 1 by 6 2 by 3, minus 1 by 2 plus 1 by 6 minus 2 by 3, the last term is minus 1 by 2 plus 1 by 6 that makes it minus 1 by 2 plus 1 by 6. So this is 3 minus 1 by 3. So I can this as 1 by 3 say minus 1 by 3 into 1, 2, 1 minus 2 1, this is my vector U ok. So you please go back and check that this is the vector that you would get by the Lagrange multiplies method and to the geometry ok.

Lagrange multiplies method is conditional (())(40:06) minimize the distance between x and U subject to U being in a subspace, that is conditional (()()40:16) ok. In advance calculus you must have studied ok, so let me stop here then I will prove this next time ok. Do you have any questions?

Student: (())(40:48)

It may not exist but if it exists it is unique in an inner product space that is in a norm linear space induced by an inner product, in a general norm linear space it may not exist. In a general norm linear space it may exist but it may not be unique ok.