Linear Algebra
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Module 11-Inner Product Spaces
Lecture 40
Norms on Vector Spaces

Ok so let me emphasize what I said yesterday the notion of inner products spaces more generally norm linear spaces this has relevance to practical notions like approximation and convergence, these will be made mathematically precise a little later ok. So we are only trying to develop the background material for that ok.

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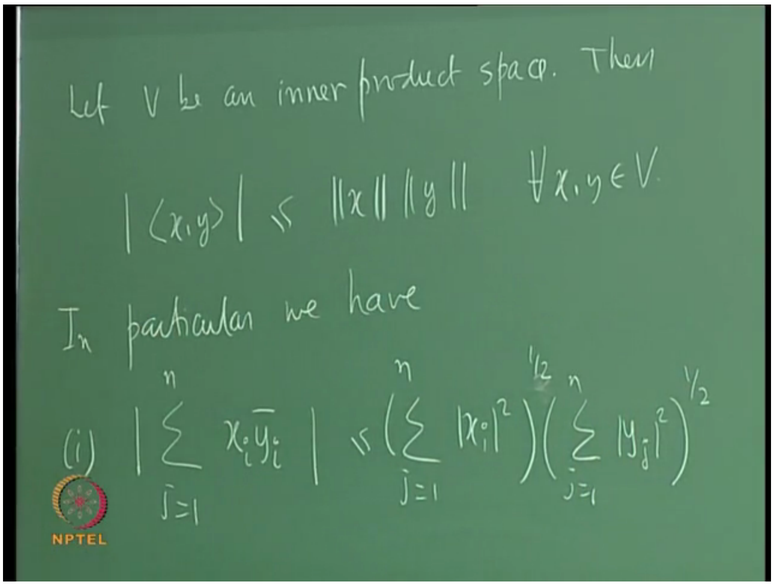

Lets also recall (())(00:44) I want to give two examples, (())(00:48) in an inner product space ok, modulus in a product x y this does not exceed the product of the norm of x and the norm of y this is true for all x y, the right hand norm comes from the inner product ok.

So lets just remember that again, what does this inequality say with regard to the three inner products spaces we have seen before. In particular we have the following, look at C N for instance, I will call it one summation J equals 1 to N I am using xi yi bar and then I take the modulus, this is a inner product of two vectors in C N this does not exceed norm of x into norm of y, norm of x is summation J equals 1 to N mod xi square to the 1 by 2 into norm y a similar expression, summation J equals 1 to N mod yj square to the 1 by 2 ok, this is one particular case.

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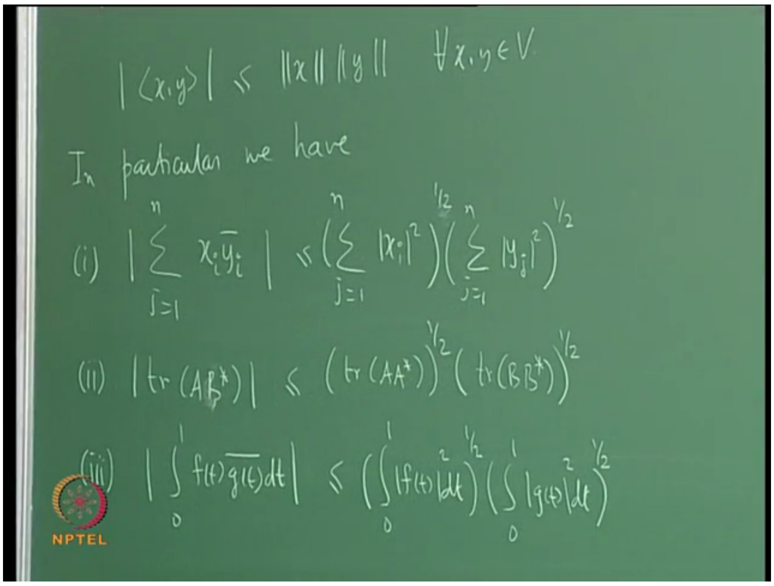

Look at what happens to the inner product space C N cross N with the trace inner product. Trace of A, B star modulus of that this is less and or equal to norm A norm B, norm A is trace of A A star to the half norm B similar expression trace of B B star to the half, finally if you look at the infinite dimensional example, infinite dimensional inner product space C 0 1 we have the following, modulus integral 0 to 1 inner product x y, f of t gt bar dt modulus of this does not exceed the product of norm f norm g, what is norm f? Norm f is integral 0 to 1 mod f t square dt to the 1 by 2, the second factor 0 to 1 mod g t square dt to 1 by 2 ok.

Inequalities are important when you discuss notions of approximation convergence etc, so you will encounter these if not in this course some other course. So this are specific instances of the (())(4:15) ok.

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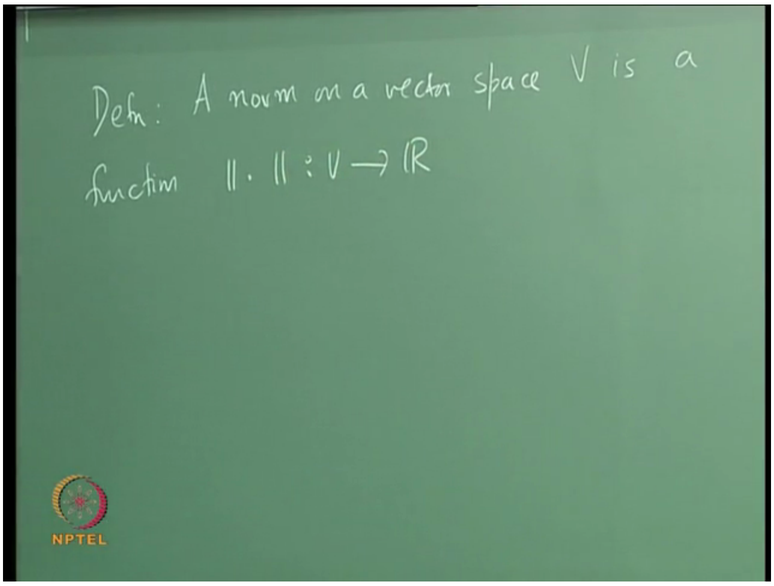

We have seen yesterday that the notion of a norm can be introduced for a vector inner product space, more generally we have the following, that is a norm need not be induced through inner product. One can have a general norm linear space. A norm on a vector space V is a function it will be denoted by these two parallel lines, I am sure you must have encountered this. it is already there in inner product spaces.

It is a function from V to R unlike the inner product which can be a complex number so this is a function from V to R such that the following conditions are satisfied.

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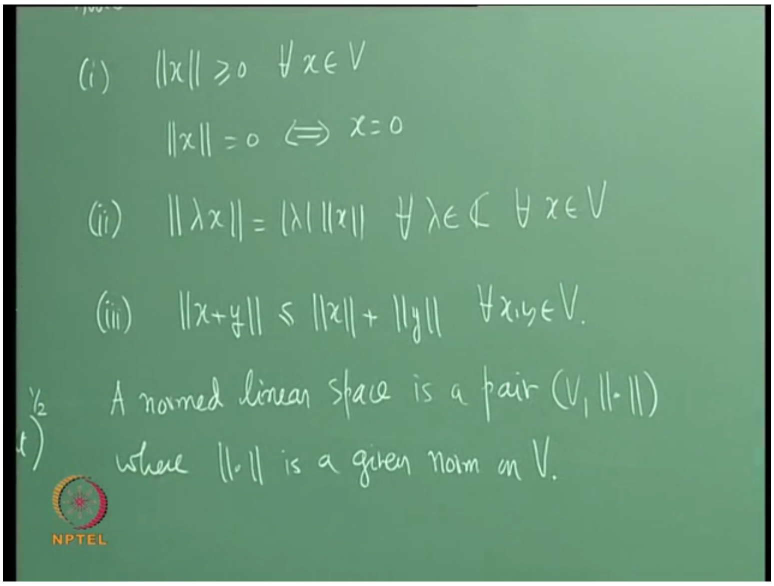

Norm of X is greater than or equal to 0 for all x and V and this goes along with this norm x equal to 0 if and only if x is equal to 0. Condition 2 is again a condition that we have seen in the context of inner product spaces. Norm of lambda x is mod lambda norm x for all lambda in lets restrict lets look at the case of a complex vector space. So I will take the scalars from C for all x and V, second condition. Third condition is just a triangle inequality norm x plus y must be less nor equal to norm x plus norm y.

A norm on a vector space is a function that satisfies this conditions a vector space together with tis norm with a given norm is called a norm linear space norm vector space. A norm linear space or a norm vector space, vector spaces are also called linear spaces, a norm linear space is a pair, it is a pair V, some norm where this is a given norm, norm V ok and so every inner product space is a norm linear space, every inner space is a example of a norm linear space it is a sub class.

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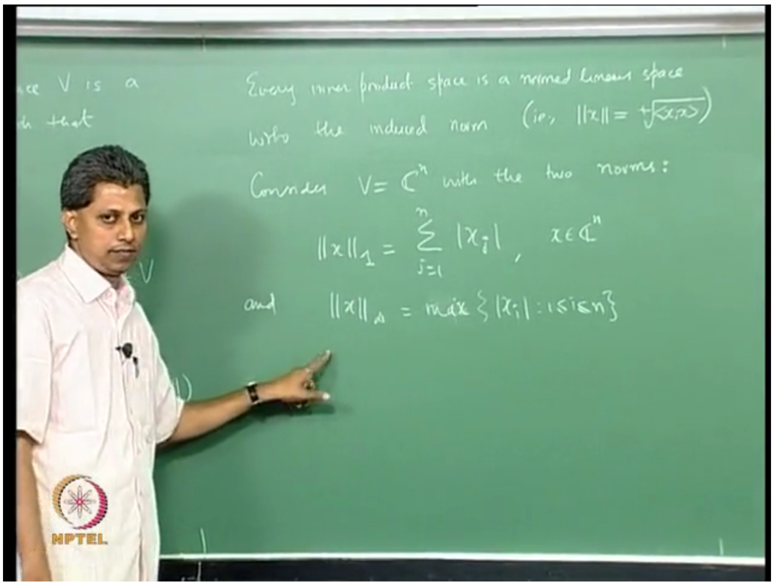

This normed linear space with respect to the induced norm. Just to recall norm x is the positive square root of inner product of x with itself ok.

What is also important is to observe that on a given vector space you can define several norms and what can be shown is that not all norms are induced by inner product ok. Now let me give you atleast two different on norm on C n for instance, this will also serve as examples of norm linear spaces. Consider V to be C n with the two norms defined as follows, I will define two norms with respect to which C n becomes norm linear space. One is called as the one norm sometimes called the absolute value norm. So it is it goes with the subscript 1 norm x goes with 1 this is summation J equals 1 to n mod x j where the usual convention is that x1 x2 etc xn are the coordinates of x, this is called the 1 norm.

This is 1 norm and yeah all that I am saying is that this satisfies you can verify that this satisfies this conditions ok. So C n with this norm is a norm linear space there is also called the so called supremum norm or sometimes a maximum norm, this is equal to the supremum of actually it is maximum ok, there are only finitely many numbers here. Maximum of mod xi 1 less (and) or equal to I less (and) or equal n. Maximum of the mod I of the coordinates of the vector x, this is called the infinite norm or supremum norm maximum norm. C n is a norm linear space with respect to both these norms ok.

C n already has a norm with respect to the standard inner induced by the standard inner product ok. In numerical linear algebra one would like to ask questions like whether this norms are equivalent ok, will not deal with those but it is in that context you would like to know whether norms are equivalent. If norms are equivalent then see a norm linear space can be shown to be a metric space D x y equals norm x minus y then with respect to this metri we ask questions about conversions, then the question is if its if a sequence xn is convergent with respect to one norm, should it be convergent with respect to another norm? And this is related to the question as to whether two given norms are equivalent ok.

That is why it is of interest to know different norms on the same space and different norms have different, different norms are suitable for different applications. For example, when we do calculus it is the stand, it is a norm induced by the standard inner product, it is called the two norm or the Euclidian norm ok.

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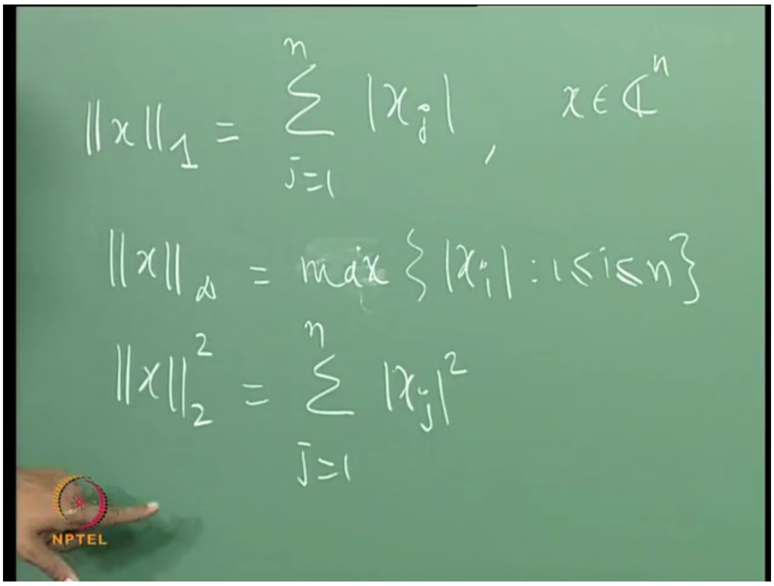

Let me write that also, it is called the Euclidian norm or the two norm that is from the standard inner product so can you tell me what the two norm is? J equals 1 to n so norm x square ok, that is mod x j square, this is called the Euclidian norm or the two norm. In calculus it is a two norm which is important whereas in a robot trajectory planning etc it is a infinite norm that is used ok.

So different applications ask for different norms. The question however is we need to go back to this question, the two norm is induced by the standard inner product what happens this two? The claim is that, this two are not induced by any inner product ok, how do you prove it? In order to prove it the following result is useful, it is called the parallelogram law, which holds in a vector space. So let me state and prove that and then I will leave it for you to verify that this two norms are not induced by inner products by any inner product ok, parallelogram law let me state that here and prove it there in an inner product space.

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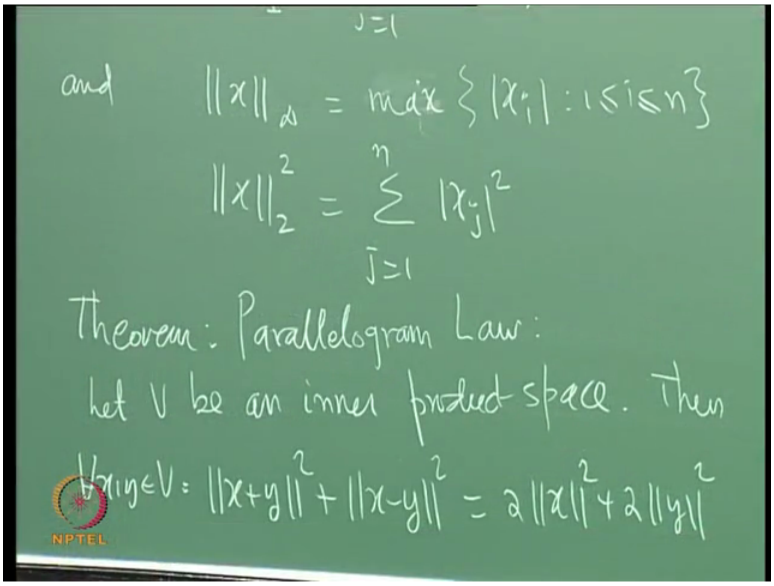

Let the V be an inner product space then we have the following, this the role that I am going to write is motivated by what we have seen in two dimensions even three dimensions. Norm of x plus y the whole square plus norm x minus y the whole square. You can think of x and y as two dimensional vectors on the plane then if then x plus y is the length of one of the diagonals x minus y is the length of the other diagonal, the sum of the squares of the diagonals must be two sides two times the sum of the square of the sides. Two times norm x square plus two times norm y square for all x y this law holds, this is the parallelogram law.

In an inner product space this holds where the norm is offcourse the norm induced by the inner product ok. So if V is an inner product and this is the norm induced by the inner product parallelogram law holds. If I have a norm linear space where the parallelogram law does not hold then it cannot be the norm cannot be induced by any inner product, that is what you should use to prove that these two are not induced by any inner product. You have to take sample vectors x and y calculate this numbers and verify that this law does not hold for this two norms. I am going to leave that as an exercise this two are not induced by any inner product that is an exercise. But let me prove the parallelogram law, the rather straight forward.

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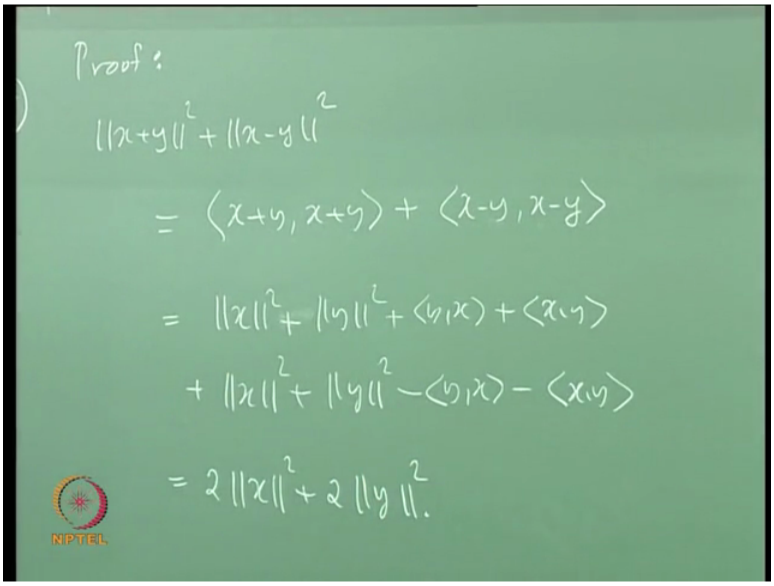

You simply look at see this is induced by an inner product so you need to use that so look at norm x plus y the whole square plus norm x minus y the whole square, this is inner product of x plus y with itself plus inner product x minus y with itself. Just expand and simplify x with x is norm x square y goes with y for norm y square and you have a y x and an x y the second term gives you norm x square plus norm y square minus y with x minus x with y, so you get right answer. This is two times norm x square plus two times norm y square ok, so that proves the parallelogram law straight forward but it is still powerful in showing that certain norms are not induced by any inner product.

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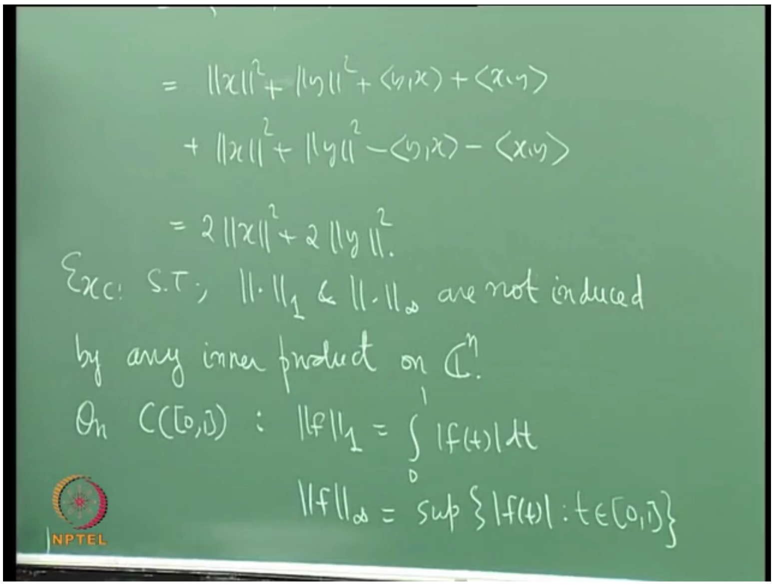

So the exercise for you is show that the one norm and the infinite norm are not induced by any inner product, the context is C N you also have similar results for the space of continuous functions over 0 1, whose space of continuous functions over 0 1 there is a two norm induced by the inner product which I have given there but there are other norms that can be defined on C 0 1 so let me also mention on C 0 1 I will define two norms similar to the one norm and the infinite norm on C 0 1 norm f the one norm is any guesses about what this is, f is a continuous function over 0 1. This is similar to the one norm integral mod, integral 0 to 1 mod f t dt, f is continuous modulus is continuous so the integral exists.

Similarly the infinite norm, what is a infinite norm? Supremum, supremum of modulus of F of t, tn 0 1, the supremum exists because f is continuous mod is continuous composition of two continuous functions. So there is a maximum and a minimum, I want the maximum. So infcat I can replace supremum by the maximum. So with respect with this two norms 1 and infinity C 0 1 is a norm linear space it can again be shown using the parallelogram law that this are not induced by any inner product ok. So lets move on, this are some of the basic notions one of the motivations for an inner product spaces is that it should allow us to generalize a notions of the usual dot product the notions of angle between in vectors in particular orthogonality ok.

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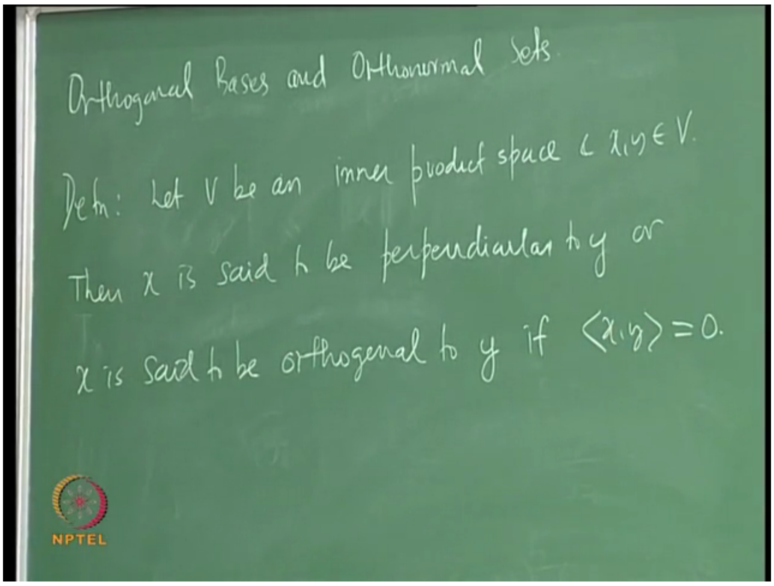

Lets look at this notions, so in particular I want to look at the concept of an orthogonal basis and orthonormal basis I will simply say orthonormal sets ok a notion of orthogonality. See it is done through the inner product so this definition is natural. Let V be an inner product space take two at vector x or y then x is said to be perpendicular to y or x is said to be orthogonal to y, if the inner product of x with y in this fashion is zero ok. If x if the inner product of x with y taken in this manner is zero then the inner product of y with x, y first x next that is also zero because of the conjugates symmetry ok.

So then we can say that x and y are orthogonal, can say that x and y are orthogonal to each other ok.

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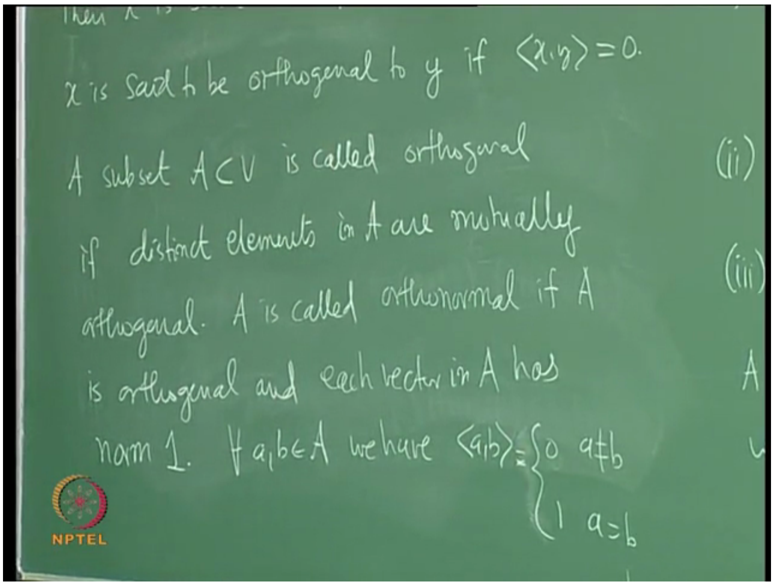

For a subset A contained in V is called orthogonal a set is called an orthogonal set if distinct elements are orthogonal distinct (vectors) distinct elements in A are mutually orthogonal if distinct elements in A are mutually orthogonal. Zero vector is the only vector that is orthogonal to itself that is if A, A is equal to zero then A is zero that comes from the first two (())(22:23) inner product. Orthogonal we need something more A is called orthonormal if A is orthogonal and its vector in A has norm 1, which vector A has norm 1 so such a set is called an orthonormal set that is for every a b in A we me must have the inner product of a b is zero if a is not equal to b it is one if a is equal to b.

So we write like this, distinct vectors are orthogonal and each vector has norm 1 so such a set is called an orthonormal set, do you have examples of orthonormal sets.

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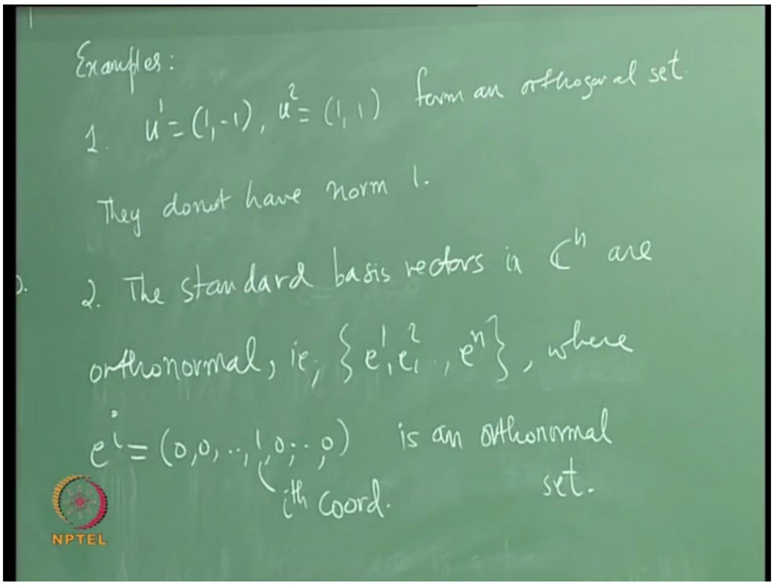

Look at the vectors that belong to standard basis but before that I will give another example, consider the following vectors, U1 is 1 minus 1, U2 is 1 1 these two vectors form an orthogonal set not orthonormal, these two form an orthogonal set not orthonormal because they do not have norm 1. Norm of U1 or U2 is infact 1 by root 2 sorry just root 2, U1 U2 both have norm root 2. On the other hand if you look at the standard basis vectors.

The standard basis vectors in C n are orthonormal infact just emphasize what the standard basis is, look at e1 e2 etc e n where ei is 0, 0 etc 1, 0, 0 where this occurs in the ith coordinate, this is an orthonormal set ok. This is this probably the simplest orthonormal set one would encounter. I want to explain a procedure, the question is the following. Given a linearly independent set can we get can we construct an orthonormal set out of it? Ok, the answer is yes, but before that we must understand that orthonormal (vect) orthogonal vectors are linearly independent ok. But even before that I want to prove Pythagoras theorem then I will come to this.

Pythagoras theorem which holds we have seen in the plane holds in a general normally general in a product space.

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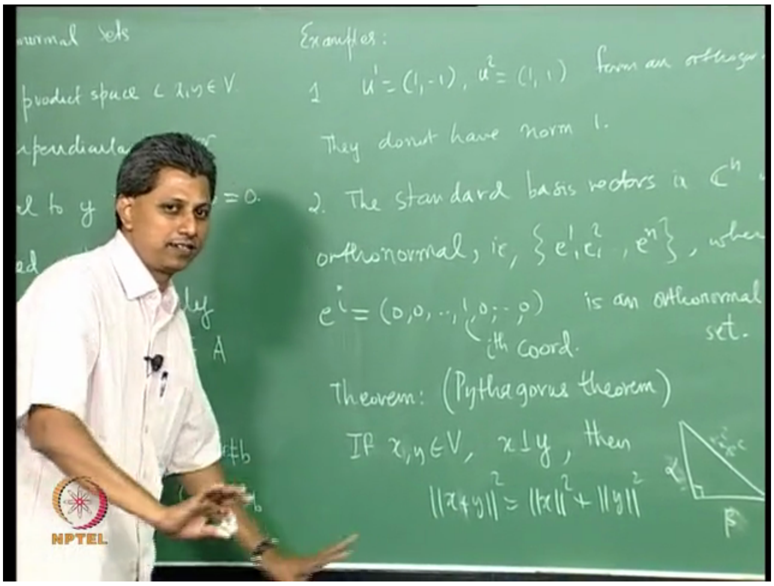

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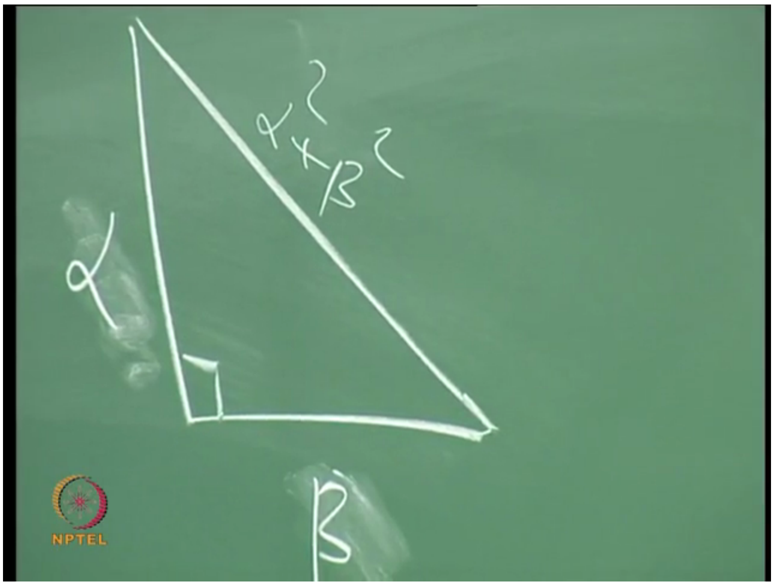

So I want to prove this and then look at the process of constructing orthonormal vectors from an independent set linearly independent set. Pythagoras theorem, the setting is an inner product space so if x y belong to inner product space such that x is perpendicular to y then the inner right triangle there is a hypotenuse there are other two sides look at the square of the lengths of the other two sides, that sum is equal to the length of the hypotenuse.

Norm x plus y square is norm x square plus norm y square ok. Just to recall if this is 90, this is x and this is y this are the lengths ok, I think I should use alpha beta numbers then this is alpha square plus beta square, alpha and beta are the lengths norm x norm y are the lengths of the side this holds in a general inner product space, I will leave the proof ok. You have to as before start with norm x plus y whole square use the inner product and one line the proof ok. So the high school notion of Pythagoras theorem you see holds in a abstract inner product space ok.

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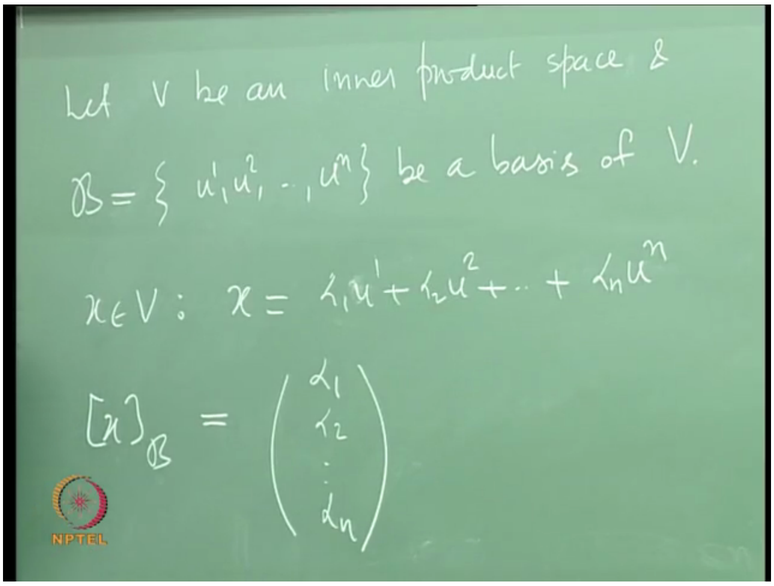

I told you that orthogonal vectors are linearly independent, let me prove that lets recall the following. Let V be an inner product space and let me take this as a basis, so I am considering a finite dimensional inner product space, B equals lets call the vectors U1 U2 etc Un let this B a basis of V ok so it is a finite dimensional inner product space. Given a its as before it's an ordered basis ok, that is U1 is a first vector, U2 so second vector etc Un is the last vector. So that when we write down the matrix of a linear transformation or the matrix of a vector we know what is the first component, second component etc.

So this is an ordered basis it means that any x in V can be written as ordered basis. See when you write down the matrix of a vector then it is always done with respect to a basis, that is this x there is a representation, this x can be written given this basis this x has the following (representation) unique representation alpha 1 U1 plus alpha 2 U2 etc plus alpha n Un, where the numbers alpha 1 alpha 2 etc alpha n are unique for this x ok and we always deal with standard basis for the reason that we will have occasion to talk about the first coordinate of x second coordinate of x etc when we do matrix operations.

So it is natural to call alpha 1 as a first coordinate of x, alpha 2 as the second coordinate of x etc. Do you remember this we used to write the matrix of x relative to this basis and then that is a column vector coming from the first term coefficient of the first term, coefficient of the second term etc. now what is to be understood is that, the sum does not change if you alter the first and the second term for instance but when you write down the matrix of the vector corresponding to the basis, it does make a difference ok. So we will always have in mind that there is an ordered basis.

So there is a first coordinate, second coordinate etc. So this an ordered basis I have this representation as I told you these numbers alpha 1 etc alpha n are unique for the particular x that we started with ok. How do you compute this numbers? Given a vector x do you remember how we compute this numbers alpha 1 etc alpha n, in a general vector space. See U1 U2 etc they do not form an orthogonal basis orthonormal basis they form just a basis, ordinary ordered basis. So how do you find this numbers?

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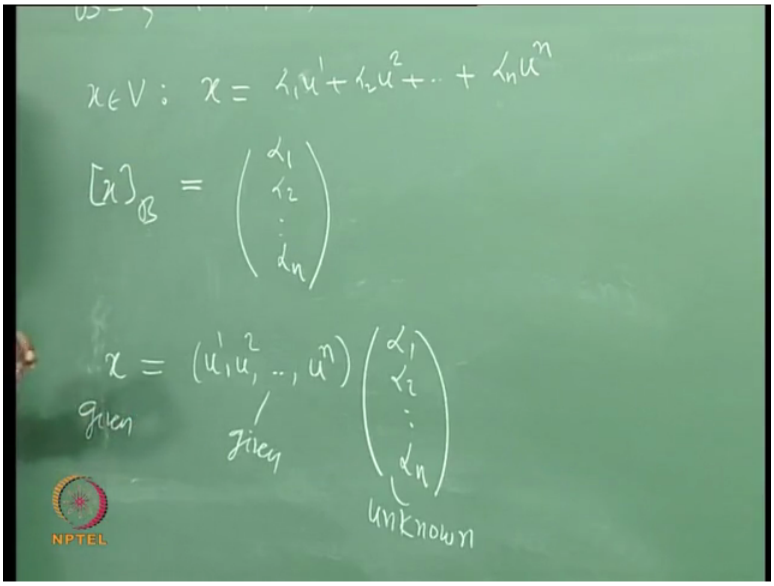

Solving a system, you can write this as ok see x is given, I need to find this numbers so what we do is look at the matrix whose columns are U1 U2 etc Un and then I want to determine the numbers alpha 1 alpha 2 etc.

This left hand side is given I know what x is, I want to determine the numbers alpha and the coefficients of x relative to this basis, I know what this is, this is also given the basis is also given I need to determine this, this is unknown. So this is essentially solving a linear system of equation. So you need to do elementary raw operations and then determine the unknowns from the system of equations ok. Now that’s we know that takes a little effort. In the case of a so to determine the coefficients of a vector x you need to solve a system of linear equations but if this is not just a basis but an orthonormal basis then this is very easy.

That is a advantage of an orthonormal set an orthonormal basis. By the way what is an orthonormal basis? A basis which has a property that the vectors are mutually orthogonal and have each have norm 1 is an orthonormal basis.

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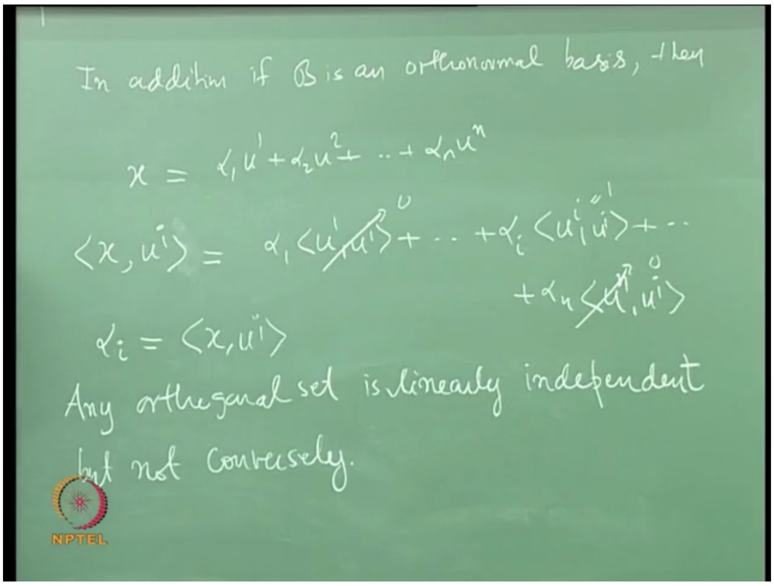

If this is an orthonormal basis, so in addition if B is an orthonormal basis then this computation immediate, there is no computation involved, it is immediate, then we have the following. Ok I will go back to this equation alpha 1 U1 plus alpha 2 U2 etc alpha n Un I take the inner product of x with Ui, I runs from 1 to n then this is alpha 1 U1 Ui plus etc alpha I Ui Ui etc plus alpha n Ui ok how do I write the terms? U1 you consider the first I am taking on the right yeah so Un Ui.

Since is orthonormal basis all terms cancel except this one of this as zero this is one and so this is alpha I and so the coefficient alpha I is determined as inner product of x with Ui. So the coefficients can be computed by multiplication the dot product by the dot product immediate. But the price you have to pay is the computation of an orthonormal basis from a linearly independent set. It is just a basis ordinary basis it is a linearly independent set, there is some effort involved in going from a linearly independent set to an orthonormal basis there is a (())(35:41) process Gram Schmidt procedure, numerically it can be modified but will simply locate the Gram Schmidt procedure.

That tells us how to go from a linearly independent set to an orthonormal set, so once you do that certain computations become easier ok. I told you that orthogonal vectors are linearly independent, can you see that to happening here immediately? In general orthogonal, if x is zero that is if I take a linear combination of the vectors U1 etc Un equate that to zero then is it clear that you comeback see that the coefficients must be zero. So an orthogonal set is linearly independent not conversely. Any orthogonal set there is no orthonormality that we are using here any orthogonal set is linearly independent but not conversely.

That the converse is not true has been exhibited already you look at those vectors 1 minus 1 11 I am sorry they are linearly they are orthogonal ok. You give an example ok that is easy, linear independent vector 1, 1 1, 2 they are linear independent but not orthogonal ok. So this means we need to look at the procedure that takes linearly independent to an orthonormal basis, this is called the Gram Schmidt procedure let me discuss that next.

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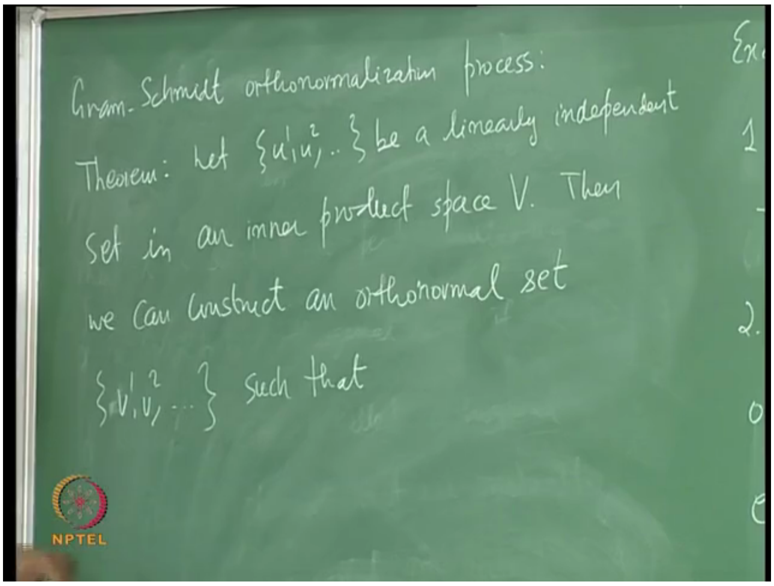

So we have what is called as the Gram Schmidt orthonormalization process ok, what is this process? I will state that as a result let U1 U2 etc be linearly independent set in an inner product space V so I start with a linearly independent set in an inner product space then I can construct then we can construct an orthonormal set I will denote that by V1 V2 etc. Remember this can be an infinite set so you can apply this to an infinite dimensional space C01 for instance. We can construct an orthonormal set V1 V2 etc which satisfies the following such that see for one thing it is an orthonormal set they are mutually orthogonal and the norm of each vector is one.

There is another thing it satisfies such that the following holds.

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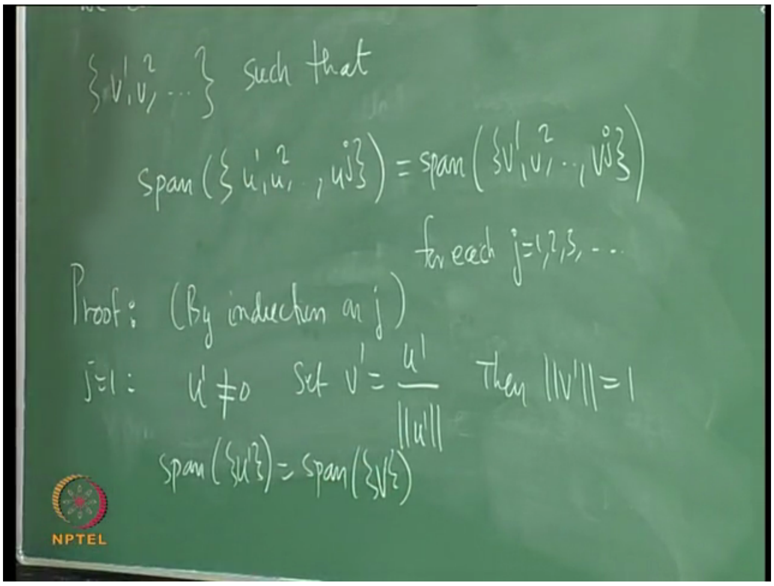

Look at this span of U1 U2 etc Uj for any j you can show that this span is a same as the span of V1 V2 etc Vj for each j ok. Step by step that is look at a span of U1 that is the same as this span of V1, span of U1, U2 is equal to span of V1, V2 etc for every j, this two sub-spaces of V are the same ok, the proof I will complete the proof today the proof is by induction. The proof is by induction on j, ok to apply the induction principle you need a base step and then an inductive step ok. Base step take the case of equal to 1, j equals 1. I have the vector U1 I must show how to construct V1 such that span of U1 equals span of U ok.

But remember that we start with a linearly independent set so this U1 cannot be zero any vector that contains a zero vector is linearly dependent ok. So none of this vectors is zero, U1 is not zero so I can divide by norm U1. So I will call V1 as the vector U1 by norm U1, norm U1 is not zero because U1 is not zero then this V1 satisfies the requirements, for one thing norm V1 is 1 and you don't have to take another vector to take the dot product etc there is this is a basis step there is only one vector plus also clear is its span of U1 is span of V1 that is because V1 is a multiple of U1.

Anything that is in the span of U1 is a multiple of V1 that is obviously a multiple of V1.

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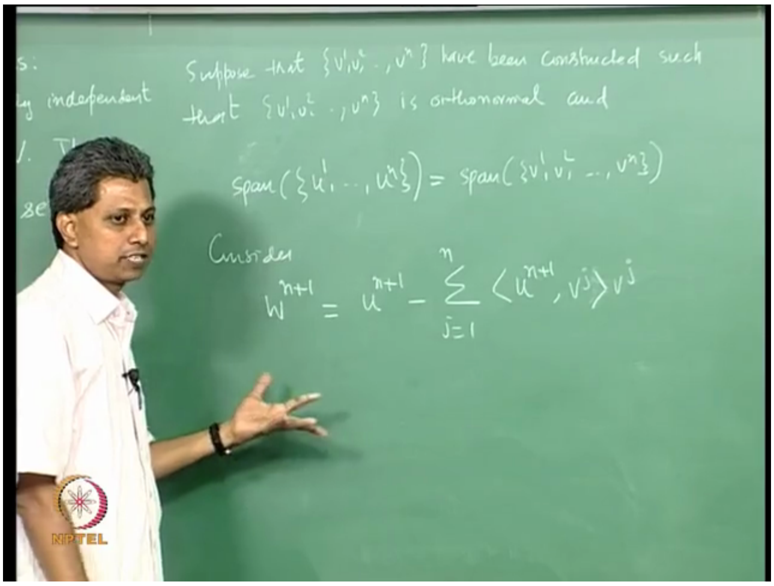

So this two subspaces coincide.so the basis step holds so we apply the inductors suppose that V1 V2 etc Vn have been constructed such that span of ok such that for one thing this is orthonormal ok, set ok I want to write that again. Suppose this is orthonormal and this condition must also hold span of U1 etc Un is equal to span of V1 V2 etc Vn. So you assume that you are able to construct n vectors then you must show that you can do it for N plus 1 vectors then by the induction principle it follows that this can be done forever indefinitely ok.

I need to give a formula for Vn plus 1 then we are done ok, given V1 etc Vn I must tell how to construct Vn plus 1 that is done as follows. Consider the vector it is a new vector that I will define I have n vectors V1 etc Vn I define new vector Wn plus 1 as lets take Un plus 1 the one that we started with and then subtract the following sum, J equals 1 to n take the inner product of Un plus 1 with each of the vectors that we have constructed each of the vectors V1 etc Vn that we have constructed Vj and then take dot product of that with Vj.

There is a geometric significance to this but this can be explained only a little later ok you remember U1 U2 etc that infinite set is given to us so I know what Un plus 1 are I have computed V1 upto Vn only those I am using here so I delete this from the vector Un plus 1. The first observation is that to this is not Wn plus 1 is not the zero vector, can you see that? If you can see that then I can skip that step, Wn plus 1 is not the zero vector, how do you prove?

As usual by contradiction, if Wn plus 1 is zero then this vector is zero, so what is the contradiction? If Wn plus 1 is zero then, yes Un plus 1 can be written as this sum it’s a it is in the linear span of V1 etc Vn but V1 etc Vn the span that is equal to this, which mean Un plus 1 is our linear combination of this contradiction to the fact that we started with this as a linearly independent set so no vector can be written as a linear combination of the previous vectors.

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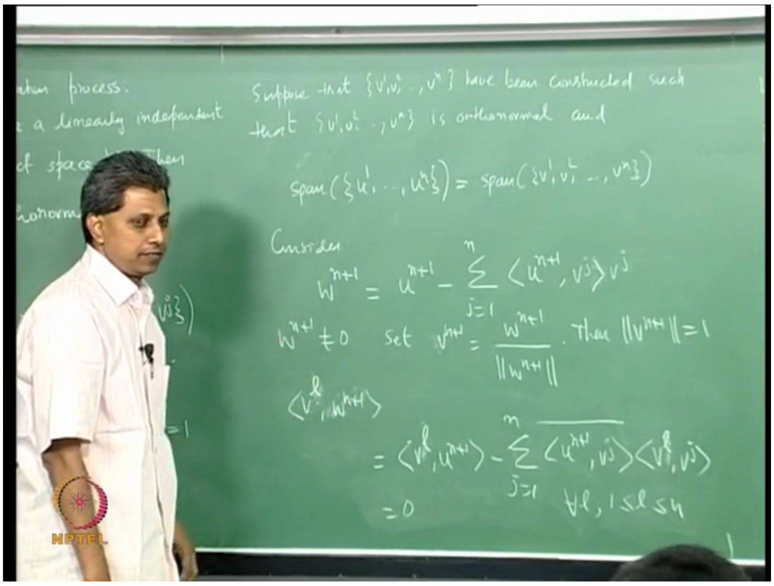

So this can't be zero, so Wn plus 1 is not zero it makes sense to talk about the norm W1 and then divide that by divide a vector by that so I will do something that similar to the first step call Vn plus 1 as Wn plus 1 by norm Wn plus 1, this is well defined because denominator is not zero the claim is that this V n plus 1 obviously has norm 1 but the claim is this is orthogonal to the vectors V1 etc Vn ok then we are through almost. For one thing norm Vn plus 1 is 1 however the orthogonal how was Vn plus 1 orthogonal to the previous n vectors, that is that follows from this formula. Simply look at Vj, Vn plus 1 ok, look at Vj, Wn plus 1, so this is I am doing it for the first argument.

So using this formula it is Vj, Un plus 1 minus summation j equals 1 to n that is the first argument so this will go with a complex conjugates so can you see that this is what we have. Summation j equals 1 to n the complex conjugate Un plus 1 Vj oh this is I need to change this. The summation index is j I will call this L so this is L this is j this with respect to this so that is be L Vj is that ok? The summation index is j I do it for all L, L is fixed L runs between 1 and n I am looking at the inner product Wn plus 1 with V L, Vl in the first argument, so V l with Un plus 1 V l with this now this will go out with a complex conjugate, V L, V j is that clear?

J is running index L is fixed this is zero if j is different from L so all terms are gone except the term corresponding to j equals L when J is equal to L this is 1, this becomes Un plus 1 V L with a conjugate that is V L Un plus 1 that's get cancel with this so this is zero ok. So V L is orthogonal to W n plus 1 for all L so how do I choose L? This is true for all L such that one less and or equal to L less and or equal to n. So Wn plus 1 the new vector is orthogonal to V L alright but since V n plus1 is just a multiple of Wn plus 1 it is also orthogonal to the vectors V1 etc Vn.

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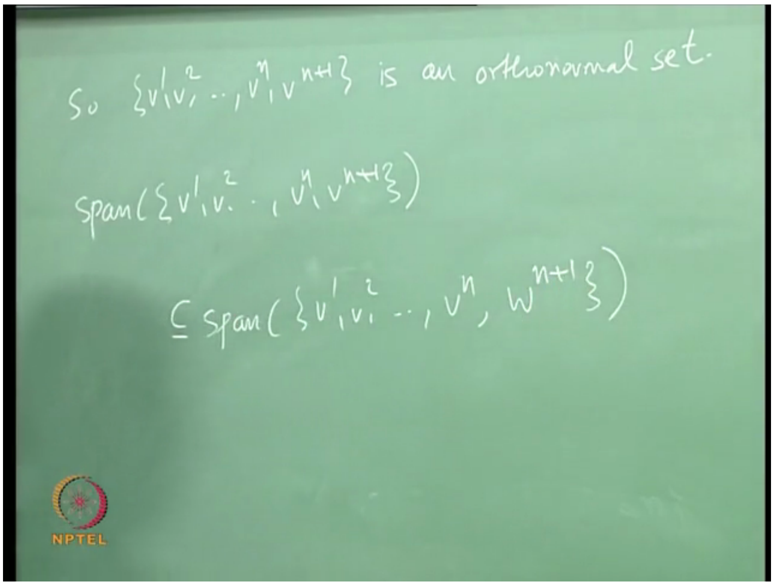

So V1 V2 etc Vn together with Vn plus 1 is an orthonormal set. The last point is to verify that the span of this is equal to the span of U1 etc Un plus 1 ok just consider that. I will prove one inclusion the other one is similar, look at span of V1 V2 etc Vn Vn plus 1, this span is contained in I will keep the first n vectors V1 V2 etc Vn and observe that Vn is a multiple of Wn plus 1 so instead of Vn plus 1 I can use Wn plus 1 ok. But when I write Wn plus 1 I observe just go back to the formula Wn plus 1 is a linear combination of Un plus 1 and the other V1 etc but that is a linear combination of U1 etc which means Wn plus 1 is a linear combination of U1 U2 etc U n Un plus 1 agree?

Yes, Vj Un plus 1 or j equals L because all terms are gone except j equals to L, minus, no real number it is a complex number. See take the conjugate but then inner product x, y bar is y x so this becomes V L Un plus 1 this is V L Un plus 1 so it gets cancelled ok, is this step clear? Vn see this step is obvious I am sure because Vn plus 1 is a multiple of Wn plus 1 on the other hand Wn plus 1 ok, you tell me if this is clear.

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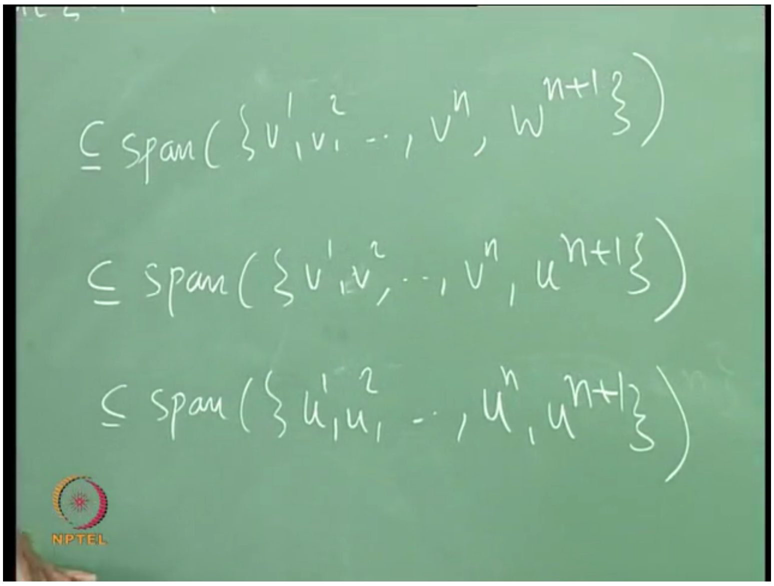

What I am trying to explain is this is a linear combination of V1 V2 I am again writing V1 V2 etc Vn Un plus 1 do you agree?

That is because Wn plus 1 is with regard to you can write it in terms of V1 V2 etc Vn and together with that you append Un plus 1 so this is fine. But V1 V2 etc Vn span of this vectors is equal to span of U1 etc so this is again contained in span of U1 U2 etc U n Un plus 1, ok so this is one inclusion I want to show that the span of this two sets are the same, this is one inclusion the other inclusion is similar. You can simply retrace the steps, ok that completes the proof. We will look at some examples next time and also applications of the Gram-Schmidt process in certain atomization problem ok, so let me stop here.