Linear Algebra

By Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology, Madras

Module 1
Lecture 3b
Equivalent Systems of Linear Equations 2

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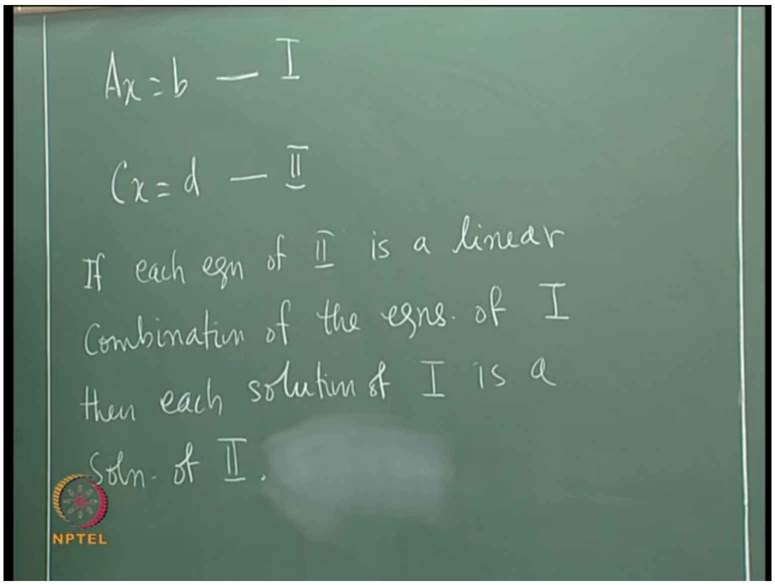

Okay, we have two systems we have first Ax equal to b I am calling that a system 1 and Cx equal to d calling that system 2, okay then what we have observed yesterday is the following, if each equation of 2 each equation of system 2 is a linear combination of the equations of system 1 then each solution of 1 is a solution of 2 and conversely, okay let me make it more precise this statement for the you can one can interchange the roles of systems 1 and 2 and say that if any equation of system 1 is a linear combination of the equations of system 2 then any solution of system 2 is a solution of 1, okay.

In this case the solution set of these two systems is the same, okay the solution set for these two systems is the same that is the first theorem.

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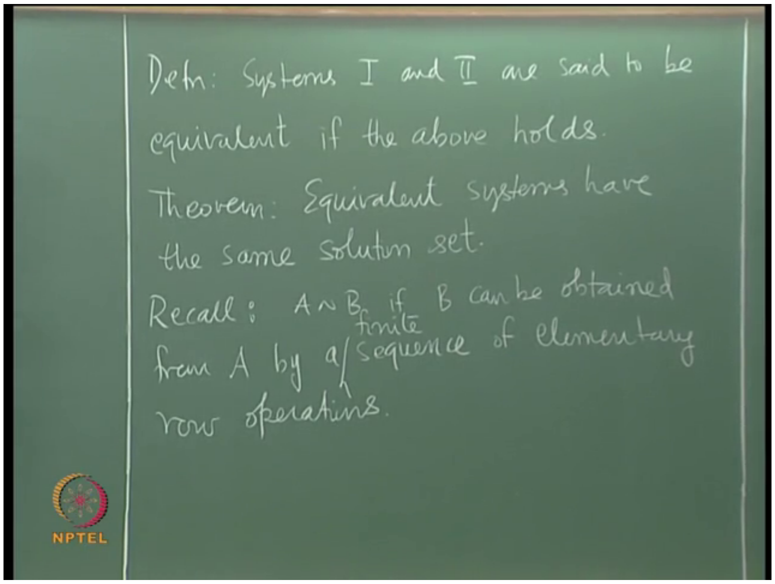

Let me give this definition before I state this theorem systems 1 and 2 are said to be equivalent if the above holds the statement that I made just now, okay I will not write down all the details here systems 1 and 2 are said to be equivalent if this statement and the corresponding statement for solutions of two being solutions of 1, okay together with that statement these two statements if these two hold then we say that systems 1 and 2 are equivalent, okay then we have the following theorem equivalent systems have the same solution set, okay.

See there are two apparently two different notions that we have discussed till now one is the row equivalence of matrices that is one is doing elementary row operations on a matrix the other one is linear combinations of equations of two systems these are two different notions that we have discussed these two are related that is what we will discuss today these two notions are related and let us see how these two are related.

So recall the definition of row equivalent matrices A is row equivalent to B this is the notation for that A is row equivalent to B if B can be obtained from A by a sequence of by a finite sequence of elementary row operations, okay and remember that we had seen that this is an equivalence relation.

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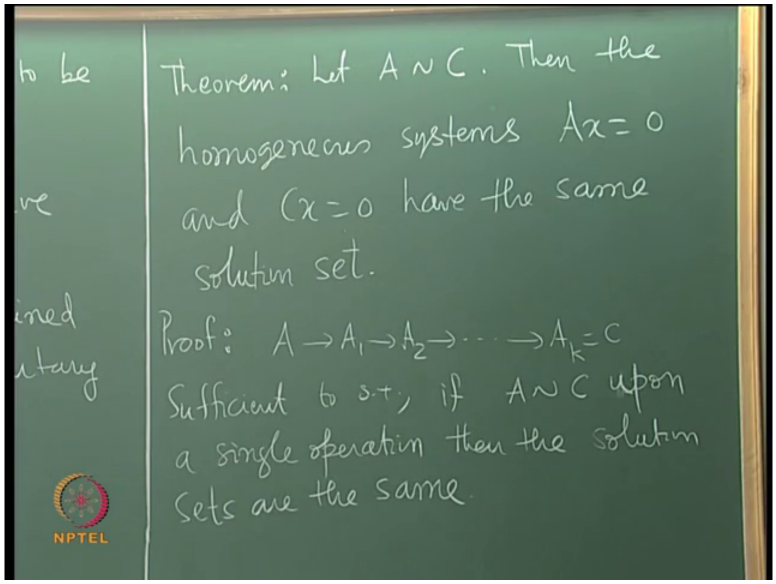

So let us now combine these two notions and prove the following theorem let A be row equivalent to C then the homogeneous systems Ax equal to 0 and Cx equal to 0 have the same solution set, okay if A is row equivalent to C then the homogeneous systems Ax equal to 0 and Cx equal to 0 homogeneous system by which I am in the right hand side requirement vector is a 0 vector these two systems have the same solution set, okay.

So can you now see that these two notions are related one is solutions sets being the same the other one is doing elementary row operations on a matrix to get another matrix row equivalent matrix, okay let us see how the proof goes A is row equivalent to C so these is a finite sequence of elementary row operations that one does on A to get C, okay. So let us say I have A going to A1, going to A2 etcetera going to Ak I will call this as the matrix C so this is my finite sequence of elementary row operations that I have performed on A to get the matrix C I must show that the systems Ax equal to 0 and Cx equal to 0 have the same set of solutions, okay one does not have to consider all the terms of the sequence it is enough if we prove the statement for one reduction, can you see why?

Sufficient to show that if A is row equivalent to C upon a single operation A is row equivalent to C upon a single elementary row operation if I am able to show that the solutions sets are the same I do not have to consider each term of the finite sequence I am claiming that it is enough to show that it is enough to show the following, suppose A is obtained C is obtained from A by a single elementary row operation I show that the systems Ax equal to 0 and Cx equal to 0 have the same solution set, I hope this is clear.

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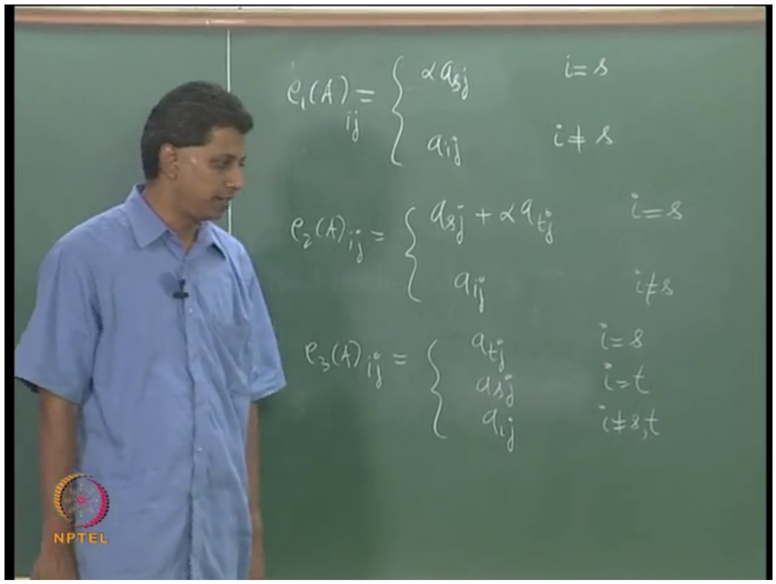

Instead of writing down the proof let me just tell you orally you can fill up the details you derive C from A by a single elementary row operation look at each of the three row operations that we have written down e1 of A is the ijth term of e1 of A is I am looking at the first operation alpha aij for alpha not 0 asj if i is equal to s and its aij if i is equal to s this is the first operation the second operation is replacing the sth row by the tth row sth row by sth row plus alpha times the tth row replace the sth row by alpha times the tth row so this is an operation performed only on the sth row all the other entries are the same all the other rows remain the same finally interchanging of any two rows, rows s and t so that is atj if i is equal to s it is asj if i is equal to t it is aij if so the other rows are left as they are, what is to be observed is that each of these operations can you see that it is a linear combination that is being performed on the rows of A each of these operations if amounts to performing linear combination on the rows of A, okay.

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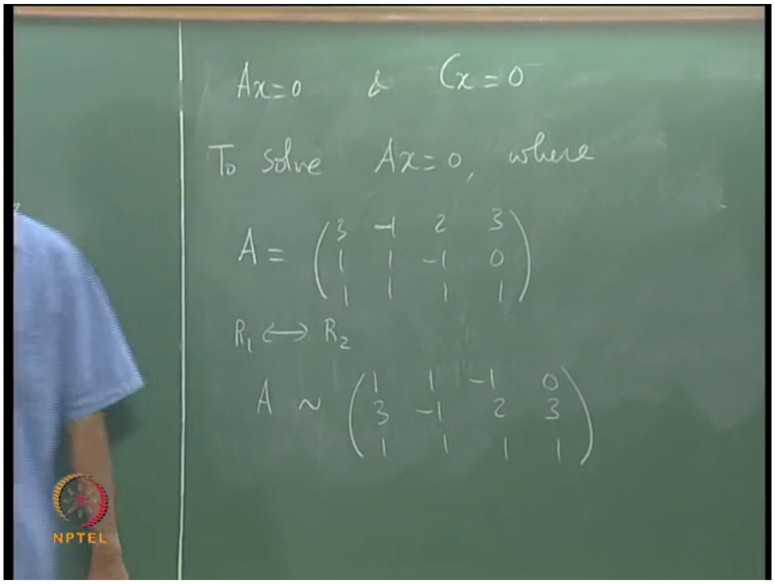

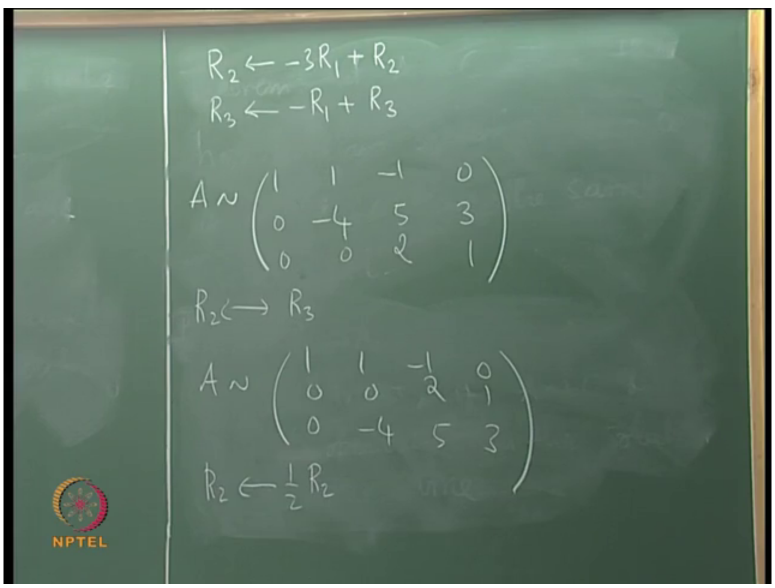

So consider Ax equal to 0 and Cx equal to 0 each equation in Cx equal to 0 is a linear combination of certain equations of Ax equal to 0 because of the fact that an elementary row operation on A is a linear combination of the rows of A, okay. And so the solutions of Ax equal to 0 satisfy the system Cx equal to 0, okay that is the first observation. The second part I must show that every solution of Cx equal to 0 satisfies Ax equal to 0 but we had seen that each of these elementary row operations has an inverse operation and each of the inverse operation is an elementary row operation of the same type and so one could go from Cx equal to 0 to Ax equal to 0 that is any solution of Cx equal to 0 is a solution of Ax equal to 0 and hence the systems are equivalent they have the same set of solutions, okay.

So write down the details but I have told you essentially what are the steps involved in the proof, okay. So this is the connection between elementary row operations and linear combination of equations between two systems, okay. Let us look at one or two numerical examples, okay I want to look at the problem of deriving a solutions of homogeneous system how it is done by using elementary row operations, okay let us look at the first problem.

Let us say we need to solve the system Ax equal to 0 where the coefficient matrix A has say let us 3 rows and 4 columns, okay so this is my matrix A I am now seeking solutions of the system Ax equal to 0 I will do it by using the elementary row operations what I know is that by the theorem that we have seen just now what I know is that if I get a matrix C upon doing elementary row operations with a particular structure of C in mind then the solution set of Ax equal to 0 is the same as solution of Cx equal to 0 C must be simple in order for me to write down the solutions probably immediately, okay.

So let us do the elementary row operations with a certain structure of C in mind and then we will formalize why this structure of C we need in the form of what are called row reduced echelon matrices, okay. So let us now proceed we will let us do one elementary row operations at a time, first I will interchange row 1 and row 2, okay I will denote that by R1 double arrow R2 interchange row 1 and row 2 it will be clear in the next step as to why we are doing this, then A is equivalent to so I will use this symbol A is row equivalent to the matrix the first row is 1, 1 minus 1, 0 second row is 3, minus 1, 2, 3 the last row remains the same, then the second operation I would like to make these two entries 0, okay why I would like to make these entries 0 that will be made clear a little later.

So I will do these two operations now, row 2 I am replacing that by minus 3 times row 1 plus row 2, okay this is one operation I will do a similar operation for row 3 also, row 3 the entry is 1 so I will replace row 3 by minus 1 times row 1 plus row 3, then the row reduced matrix row equivalent to A is the first row remains as it is 1, 1 minus 1, 0 the second row is minus 3 times this plus this 0 minus 3 minus 1 minus 4, 3 plus 2 5 this remains as 3 that is the second row, okay please check the calculations minus 3 times this this is 0 minus 3 times 1 that is minus 3 minus 1 minus 4 minus 3 times this is 3 the plus 2 is 5 this is 0 so this will remain as it is the next operation is minus first row plus the third row that is 0, 0, 2 and 1 so this is what I get after performing 3 elementary row operations, the next step would be we could proceed taking several different steps but I would like to proceed in this example as follows, I will interchange row 2 and row 3 to get the following row equivalent matrix row 1 remains the same 1, 1, minus 1, 0 0, 0, 2, 1 0, minus 4, 5, 3, okay this is my latest row equivalent matrix the next step is clear I divide the second row by the constant 2, okay.

So row 2 will be replaced by 1 by 2 times row 2 and I will keep row 3 as it is then I get the following row equivalent matrix, see the objective right now may not be to do a computationally efficient apply a computationally efficient procedure I am trying to arrive at a particular structure of C, okay.

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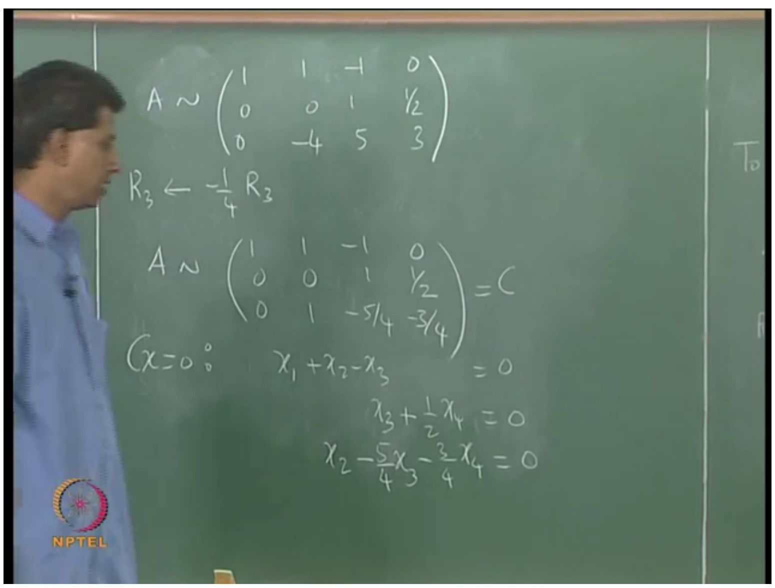

Then A is row equivalent to 1, 1, minus 1, 0 second row is 0, 0, 1, 1 by 2 third row remains as it is 0, minus 4, 5, 3 the next step will be to divide the third row by minus 4, okay and so third row is minus 1 by 4 times the third row, then A is equivalent row equivalent to 1, 1, minus 1, 0 0, 0, 1 half 0, 1, minus 5 by 4 minus 3 by 4, okay actually I could stop here to write down the solutions or do one more elementary row operation, okay let us say I stop here and write down the solution set what are the equations corresponding to Ax equal to 0, now A has been this is the matrix C I want to look at Cx equal to 0 what are the three equations that give me Cx equal to 0.

The first equation gives me x1 plus x2 minus x3 plus 0 times x4 equal to 0 so probably I will remove that, second equation gives me see remember A is 3 by 4 matrix so the number of unknowns is 4 the second equation gives me x3 plus 1 by 2 x4 that is equal to 0 third equation gives me x2 minus 5 by 4 x3 minus 3 by 4 x4 equal to 0, okay I am multiplying C on the right by the column matrix x1, x2, x3, x4 the vector of unknowns.

So I get these three equations so what is clear is that if I fix x4 the solution set can be determined that is x1, x2, x3 all three can be determined in terms of x4, okay.

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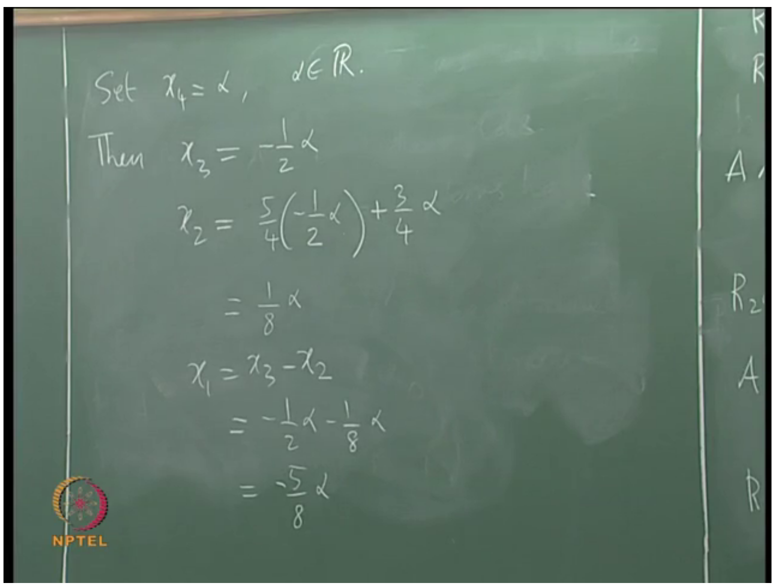

So let us say I fix x4 let us call x4 as alpha for some alpha arbitrary then x3 can be determined from the second equation x3 is minus 1 by 2 alpha, x2 can be determined from the last equation x2 is 5 by 4 x3 plus 3 by 4 x4 so this is minus 5 by 8 plus 6 by 8 1 by 8 alpha, is it okay? Minus 5 by 8 plus 6 by 8 alpha that is 1 by 8 alpha finally x3 can be determined from the first equation x3 is x1 plus x2 x1 I want x1 x1 is x3 minus x2 so that is minus 1 by 2 alpha minus 1 by 8 alpha so that is minus 5 by 8 alpha minus 4 minus 1 minus 5 by alpha so that gives me the solution set for this system.

So I am sure you will now agree that this system Cx equal to 0 is much easier to handle then the original system Ax equal to 0, okay but of course you need to do this to take this effort of reducing A to a row reduced to a row equivalent matrix.

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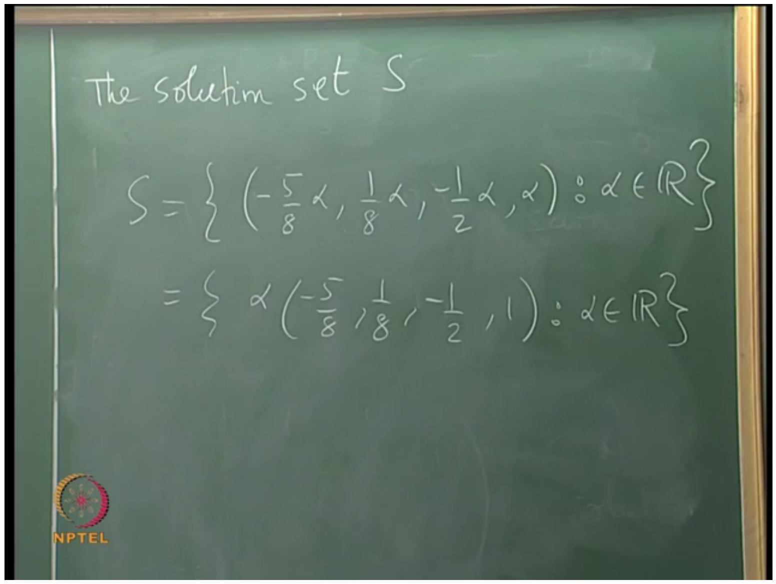

So let me write down the solution set for this example the solution set S is given by x1 is minus 5 by 8 alpha, x2 is 1 by 8 alpha, x3 is minus 1 by 2 alpha and x4 is alpha where alpha is an arbitrary real number so what is first clear is that there are infinitely many solutions and please observe that the number of equations is less than strictly less than the number of unknowns, okay this will be precursor to what we are going to prove a little later that is if you have a rectangular system of homogeneous equations where the number of equations is strictly less than the number of unknowns it always has a non-trivial solution where are going to prove this this is an example which already sort of gives a trailer I can take alpha outside and write this as set of all alpha times minus 5 by 8 1 by 8 minus 1 by 2 and 1 alpha belongs to R this gives me the set of all solutions of the system homogeneous Ax equal to 0.

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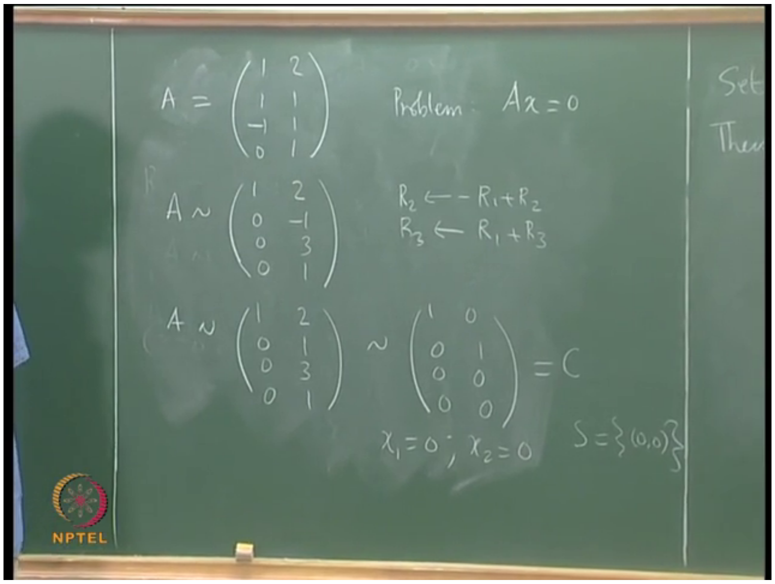

Let us look at another example a little simpler than this, okay there are four equations in two unknowns the objective is to determine completely the set of all solutions of the homogeneous equation Ax equal to 0, okay so let us do again elementary row operations on this A, A is equivalent to first row is kept as it is the second row is okay so this time I am not writing down what are the operations, okay maybe I will do that here row 2 is minus row 1 plus row 2, row 3 is row 1 plus row 3 and row 4 is kept as it is because the (())(26:11) is already 0.

So I get minus this plus this 0 minus 1 this plus this 0, 3, 0, 1 the next step will be to keep the first row as it is the second row I perform this operation multiply by minus 1, okay so I will do not think I need to write that now multiply by minus 1 I get 0, 1 third and fourth are kept as they are the next step will be so I write that here by the side I will keep the second row as it is and then do the operations based on the second row. So I will keep the second row as it is first row will be minus 2 times the second row plus the first row the new first row is minus 2 times the second row plus the first row so that gives me 1 and 0 third row is minus 3 times the second row plus the third row 0, 0 fourth row is minus times minus 1 times the second row plus the fourth row 0, 0 and I should stop here, okay that is clear by looking at the entries of the matrix which we call C.

So can you tell me what is the solution set? There are two unknowns for equations, so what this says is that the last three equations are redundant they are unnecessary the solution set is given by the first two equations first equation gives me x1 is 0, second equation gives me x2 is 0 there are only two unknowns so the solution set for this problem I will again call that as S is just 0, 0 so this system has only one solution and that is a 0 solution, okay.

So this system has only the trivial solution, okay so these two numerical examples I have been given in order to consolidate what we have learnt till now that is the idea of performing elementary row operations on a matrix and also see how it is related to solutions of homogeneous equations so we are right now concerned with solutions of homogeneous equations non-homogeneous equations will come a little later we need the notion of row reduced echelon matrix for that and the other difference between a homogeneous system and non-homogeneous system is that a homogeneous system always has a solution a non-homogeneous system may not have a solution, okay.

Homogeneous system always has a solution follows from the fact that 0 is a solution, a non-homogeneous equation system in general may not have a solution, okay so that needs a different treatment so we will come to that later let us now look at particular structure of the final matrix C that we are arriving at, okay this has a specific structure let us formalize that we will discuss what is called as a row reduced echelon matrix and then formalize what we have done till now, okay.

So the next topic is row reduced echelon matrices and what does one do with row reduced echelon matrices you will see that it gives the solution set completely for a system homogeneous or non-homogeneous, okay.

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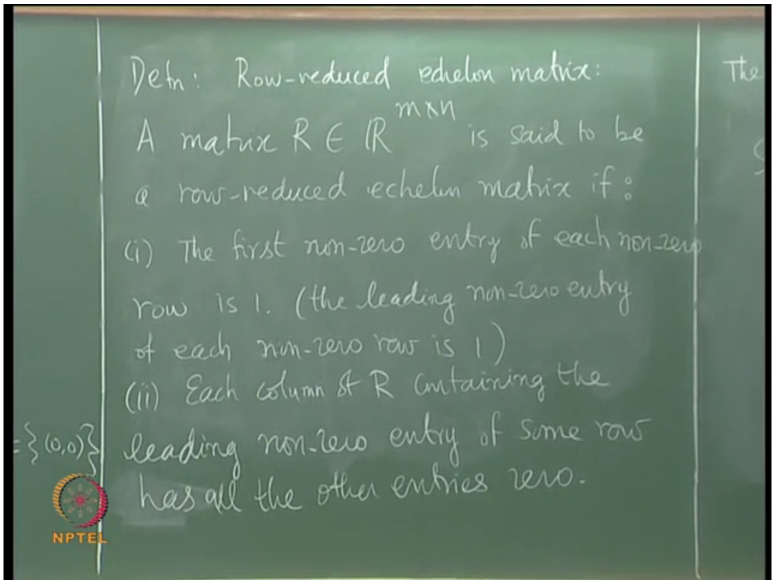

So I would like to discuss the notion of row reduced echelon matrix, a matrix R I will assume that it is of order m by n m rows n columns a matrix R is said to be a row reduced echelon matrix if it satisfies the following conditions, the first condition is the first non-zero entry of each row the first non-zero entry of each row is 1 that is the first problem the first non-zero entry of each row is 1 we will call this as the leading non-zero entry this will be called the leading non-zero entry that is the first non-zero entry will be called the leading non-zero entry so we require that the leading non-zero entry of each row each non-zero row is 1, okay.

So one could include this here also the first non-zero entry of each non-zero row the first non-zero entry of each non-zero row of R that must be 1 so the leading non-zero entry of each non-zero is 1 we will use this terminology that is the first condition for row reduced echelon matrix the second condition each column of R containing a leading non-zero entry of some row if I have a column which has a leading non-zero entry corresponding to some row then all the other entries must be 0 let me say has all the other entries 0 each column has the other entries 0 what is non-zero the only non-zero entry is the one that corresponds to the leading non-zero entry of a particular that is a second condition.

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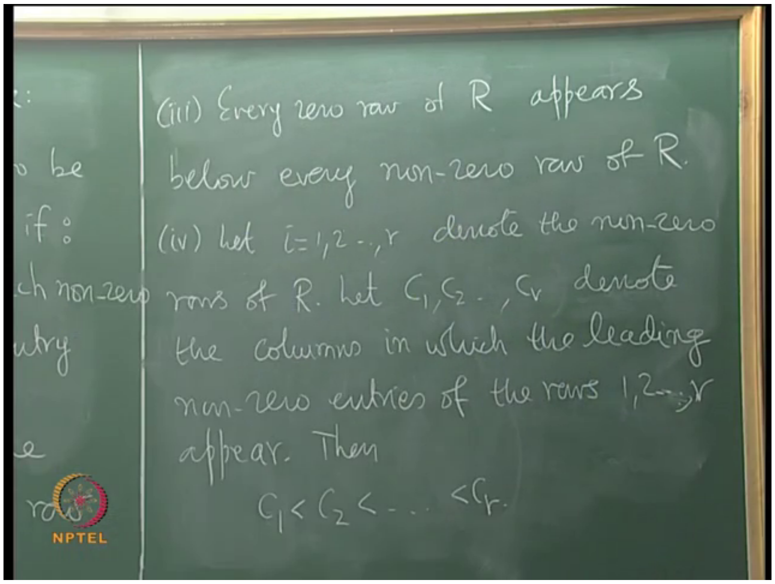

The third condition every 0 row of R appears below every non-zero row of R every 0 row of R appears below every non-zero row of R, okay just to clarify it if it not been made clear to earlier a row is called a 0 row if all its entries are 0 it is called a non-zero if it has at least one non-zero entry, okay. So condition three says that the 0 rows are stacked at the bottom 0 rows are stacked at the bottom something like what has happened in the second example. The final condition that must be satisfied by a row reduced echelon matrix is condition 4 let i equal to 1, 2, 3 etcetera r denote the non-zero rows of R, R has non-zero rows at the top 0 rows at the bottom.

Let us say that there are R non-zero rows now each non-zero row has a leading non-zero entry appearing in a certain column, okay each non-zero row has the leading non-zero entry appearing in a certain column let us call these as columns 1, 2, 3 are c1, c2, c3, cr let us c1, c2 etcetera cr denote, okay c standing for column denote the columns in which the leading non-zero entries of the rows 1, 2, 3 etcetera r appear, okay there are R rows each row has a leading non-zero entry I look at the column in which these entries appear column c1, column c2 etcetera column cr, okay then what is the condition that must be satisfied for the matrix R to be a row reduced echelon matrix this condition must be satisfied c1 strictly less than c2 less than c3 etcetera less than c3, okay these are the four conditions that a row reduced echelon matrix must satisfy.

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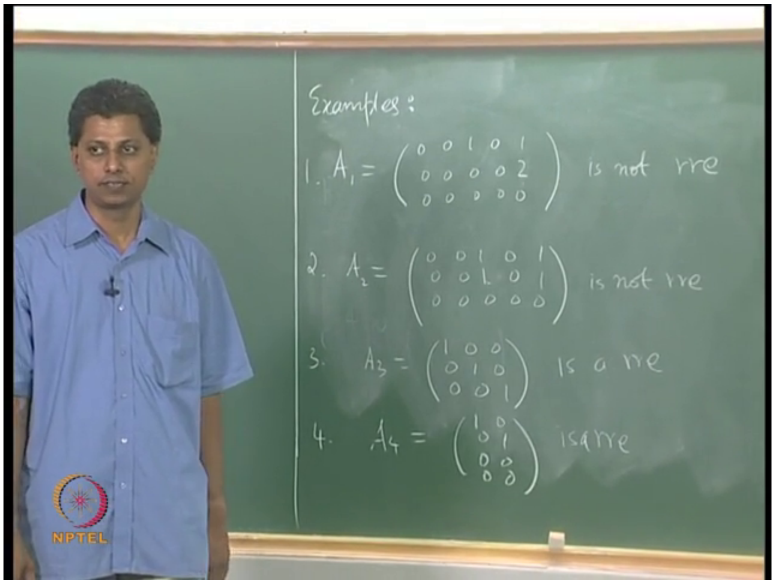

Let me conclude with two or three examples, let us look at the following, okay to serve as a means to consolidate so let us look at the first example let us say A is 0, 0, 1, 0, 1 0, 0, 0, 0, 2 0, 0, 0, 0, 0 this is not a row reduced echelon matrix because the leading non-zero entry of the second row is not 1, okay. So this is not row reduced echelon rre this is not a row reduced echelon matrix the leading non-zero entry of each row must be 1, let us look at another example I will call this A1, this is A2 0, 0, 1, 0, 1 0, 0, 0, okay 1, 0, 1 0, 0, 0, 0, 0 very similar to the previous example two entries have been changed this has the first property that the leading non-zero entry is 1 but it does not have the second property.

The third column has the leading non-zero entry of the first row, okay we must have this entry 0 in order for this to be a row reduced echelon matrix that is not the case, so this is not a row reduced echelon matrix, okay. Example 3 I will call it A3 simplest row reduced echelon matrix, okay this is a row reduced echelon matrix a non-trivial example one could look at example 2 if you want A4 that is 10010000 is a row reduced echelon matrix, okay we will discuss further properties of row reduced echelon matrices and how they help in solving a non-homogeneous system if it has a solution in the next lecture, okay you have any questions?

C1 corresponds to let us say the second column here C1 corresponds to C1 is C2 is 2, C3 is 3 here C1 is 1, C2 is 2 C1 is the column in which the leading non-zero entry of that particular row appears. So C1, C2 etcetera they are numbers, okay other questions? Strictly less than in this definition we require that C1 strictly less than C2 strictly less than C3 etcetera strictly less than Cr not equal to, in the first example? Fourth example is this one, yeah C1 is 1 see for the fourth example C1 is 1, C2 is 2 the column number where, yeah so what is the problem with the first example? What is the context?

This is not a row reduced echelon matrix because the leading non-zero entry of second row is 2, not 1 this is not a row reduced echelon matrix because the leading non-zero entry of the first row appears in the third column the third column the other entries must be 0 that is your condition 2 that is not the case the third column the leading non-zero entry is 1 the other entries must be 0 that is not the case so for this reason it is not row reduced echelon matrix, okay.

Any other question? So C1, C2 etcetera Cr are the column numbers are the columns numbers, okay yes, it is not the leading non-zero entry of the second row is 2it is not 1, so it is not row reduced echelon matrix, leading non-zero entry must be 1 about other entries they are not necessarily 0 maybe I will give other examples next time other see it does not mean all the other entries are 0,

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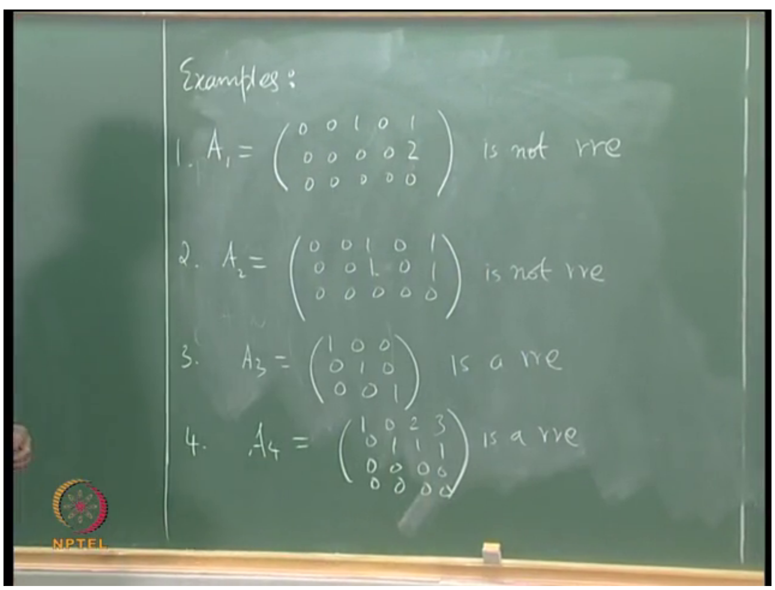

this is possible this is also a row reduced echelon matrix, okay so let me stop here.