Linear Algebra

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Module 9 Direct Sum Decompositions
Lecture 34
Direct Sum Decompositions and Projection Operators 2

Okay, we are looking at invariant direct sum decomposition characterisation of diagonalizability. We are to proof the converse part right let me write down that statement of the converse there is a slight modification to the statement that I gave in the last lecture.

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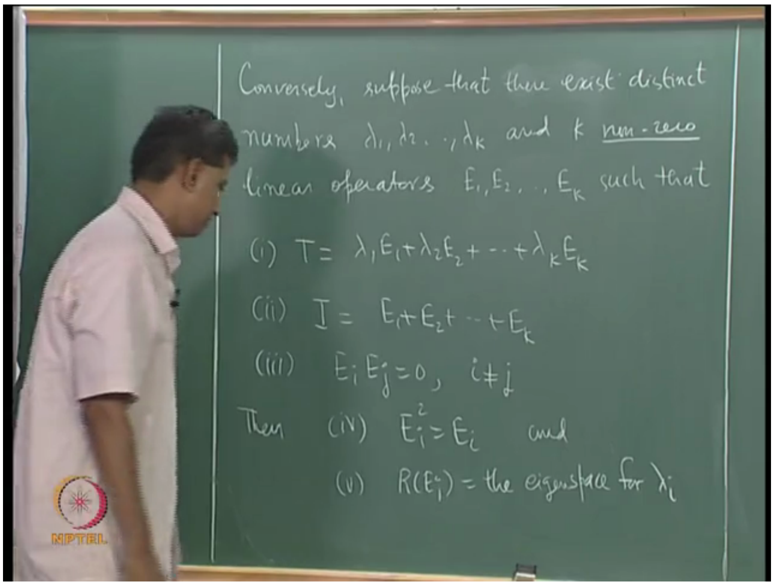

So let me write down all the conditions conversely suppose that suppose that the numbers suppose that there exist I want to say suppose that there exist distinct numbers lambda 1, lambda 2, etc lambda k and k non-zero linear operators k non-zero linear operators I am calling them E 1, E 2, etc E k these are operators on V such that such that the following conditions are satisfied.

First condition is T is lambda 1 E 1 plus lambda 2 E 2 etc plus lambda k E k, T is a specific linear combination of these operators. Condition 2 identity is just a sum of these k operators that is the second condition, third condition I will write E i E j equal to 0 for i not equal to j this is not the third condition that I gave you yesterday, third condition I gave yesterday is E i square is E i.

Suppose I have these numbers lambda 1, etc lambda k and operators k operator non-zero operators that is important such that these conditions satisfied then E i square equals E i this is really a consequence not an assumption is a consequence of these three conditions this happens and 5 also holds. Now what is 5? 5 is range of E i equals the eigenspace corresponding to the eigenvalue lambda i.

So I will simply say eigenspace for lambda i this is another consequence, condition 4 which was assumed yesterday is really a consequence of 1, 2 and 3. Also what is more important is that really this statement I want to prof earlier than 4 and 5.

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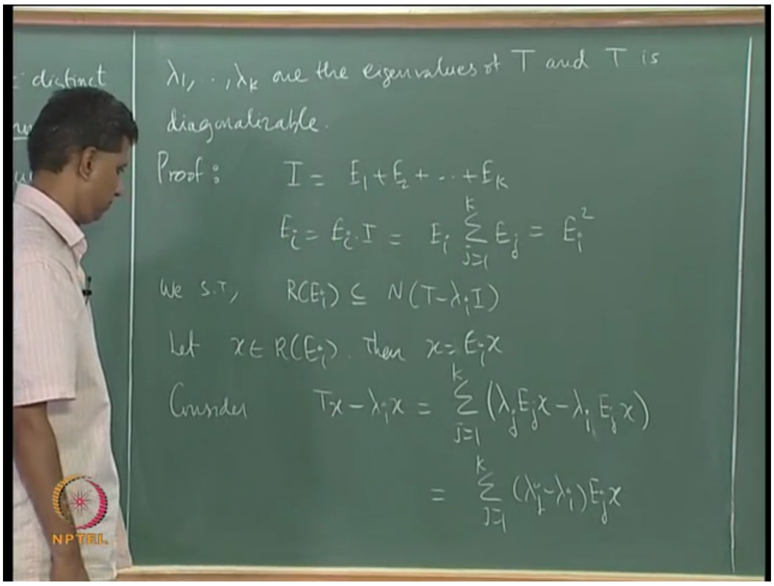

Lambda 1, etc lambda k are the eigenvalues of T these are precisely the eigenvalues of T that is what this means and T is diagonalizable okay these are the 4 consequences. We will first show that lambda 1, etc lambda k are the eigenvalues and then show that E i square is E i then show that T is diagonalizable and finally show that range spaces are the eigenspaces, okay.

So this is slight that is a slight change from what I have given yesterday, okay first thing is this that is easy I is E 1 plus E 2 etc plus E k so I multiply by E i E i is E i into I that is E i summation j equals 1 to k E j i is fixed the subscript i is fixed, the running index is j when j takes the value I have condition 3 I have condition 3 that the product is 0 when the subscripts are different.

So this goes inside and I have the product summation j equal to 1 to k E i E j when j equals i it is E i square all the other terms are 0. So this is just E i square started with E i I have E i square so condition 4 holds that is an immediate consequence of 1 and 3 sorry consequence of 2 and 3 that is condition 4. I will show that range of E i is eigenspace for lambda i not entirely I will show that range of E i is contained in the eigenspace corresponding to lambda i later I will show that the eigenspace for lambda i is contained in range of E i, okay so this shows 4.

Next we show that okay let me write this we show that range of E i is contained in the eigenspace for lambda i which is null space of T minus lambda i this is the eigenspace of the eigenvalue lambda i, okay I will show that range of E i is contained in this. Let x belongs to range of E i then x equals E i x. Consider Tx minus lambda i x I want to show that this is 0 I want to show that this is 0, right. If I show this is 0 then it follows that x belongs to null space of T minus lambda i I started with x in range of i so that will proof this, okay.

This can be written as T I will use representation 1 and then lambda i identity x I use representation 2. So I can write this as summation j equals 1 to k, for T I will have lambda j E j x for T I have summation lambda j E j. For the second term minus so I will use a bracket here, second term goes with an i i is fixed i is fixed so this goes with lambda i minus lambda i into identity x identity is summation E j identity is summation E j so this is E j x, do you agree lambda j E j x lambda i E j x lambda j E j minus lambda i so this is lambda i. I can write this as okay okay this is 0 I want to show this is 0, okay.

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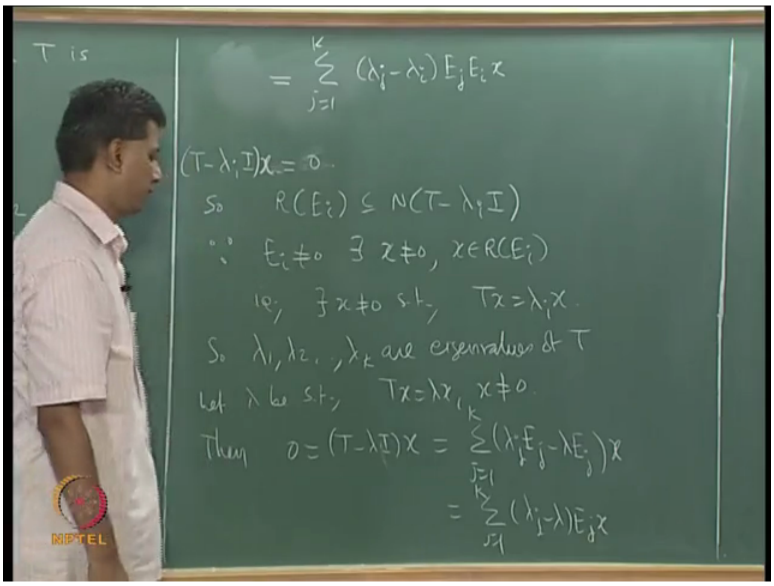

Okay, let me write down this is equal to summation j equals 1 to k lambda j minus lambda i E j x is what I wrote x is E i x, okay. So when again j is running index, i is fixed j is running index, i is fixed whenever j is not equal to i these terms are 0 whenever ya the only term that is left is when j takes the value i when j takes the value i this is 0 so this is 0, okay. So what we have shown is that T minus lambda i x we have shown that this is 0 yes and so this holds.

So range of E i is contained in null space of T minus lambda i what also what this also means is that see each of these operators is non-zero each of these operators is non-zero which means range must have at least one non-zero element, one non-zero vector which means that what is the consequence? Since E i is not equal to 0 there exist x not equal to 0, x element of range of E i that is from what we have shown just now there exist x not equal to 0 such that Tx equals lambda i x that is these numbers are eigenvalues of this operator T, right.

So lambda 1, lambda 2, etc lambda k these are not the eigenvalues these are eigenvalues presently these are eigenvalues of T the question is does it exhaust all the eigenvalues is it possible that we missed some eigenvalue which is not counted because of the way we have done the way we have got this inclusion. Have we missed other eigenvalues, have we missed other numbers which are possible eigenvalues of T.

Then we will show that these are the only eigenvalues so it will follow that these are the eigenvalues is the problem clear, these are eigenvalues alright are there other eigenvalues which are not accounted here. Let me prove that that will proof this statement then diagonalizability will follow after that we will prove (condition 4) condition 5. So let me take a number which is an eigenvalue and then show that this number must be one of these one of these lambda i's yes why am I using I suppose that is clear, okay.

I want to show that if lambda is a number if lambda is a number for which T minus lambda x is 0 for some x not equal to 0 then lambda is one of these okay. So let lambda be such that Tx equals lambda x with x not equal to 0, I want to show that this lambda is one of these lambda i's then it will follow that this is the complete list of eigenvalues till now we have not shown that this exhaust all the eigenvalues of T, we have shown that these numbers are eigenvalues we will show that any other number that satisfy this equation must be one of these, okay. So that this statement would be proved, okay.

Lambda such that Tx equals lambda x, x not equal to 0, so look at this equation this means 0 equals T minus lambda I of x I will now use the representation once again. Summation j equals 1 to k lambda j E j minus lambda E j of x lambda j E j is T, I is summation lambda I is lambda summation E j this is summation j equals 1 to k lambda j minus lambda into E j x lambda j minus lambda E j x this is 0. I will use this part, okay now I will have to make use of the fact that this is an equation like u 1 plus u 2 plus etc u k equal to 0, where u 1 is in range of E 1, u 2 is in range of E 2 etc but we know that these are these are eigenspaces because range we have shown range of E i is contained in null space of T minus lambda I. So this is an equation involving eigenvectors really is that clear?

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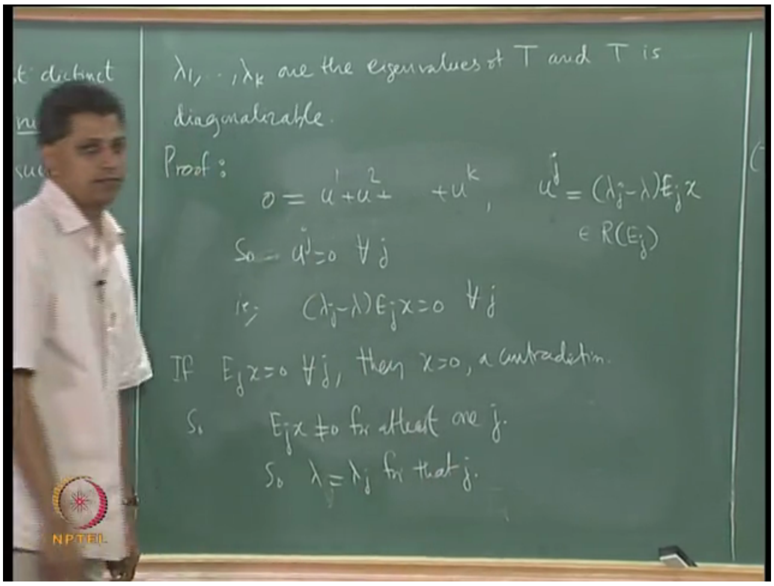

What I have is 0 equals u 1 plus u 2 plus etc u k where what is u i u j maybe u j is the jth term that is lambda j minus lambda E j x I am calling the first term u 1, second term u 2 etc this is 0. Now this u j belongs to range of E j and range of E j we have just now shown any vector in range of E j if it is not 0 then it is an eigenvector so these spaces are independent range of E 1, range of E 2, etc are independent because they corresponds to distinct numbers lambda 1, lambda 2, etc lambda k this is an equation involving a sum where the each term comes from a subspace the subspaces are independent.

So this means each must be 0 u j equal to 0 for all j that is lambda j minus lambda E j x equals 0 for all j. Now is this clear the fact that the fact that u j equal to 0 from this equation follows because I am now really looking at eigenvectors corresponding to distinct eigenvalues I am looking at subspaces which are independent that is the reason why each u j is 0. I have this equation for all j suppose E j x is 0 for all j then what do you know about x from this E j x equal to 0 for all j what do you know about x? x must be 0 because this equation tells me that x is equal to E 1 x plus E 2 x is the E k x so if each E j x is 0 then x is 0.

But we started with x not equal to 0, we have x not equal to 0. So if E j x equal to 0 then x equal to 0 a contradiction and so E j x cannot be the 0 vector for at least one j this cannot be 0 for at least one j I go back to this equation this number into this vector is 0 for at least one j I am sorry this is true for all j for at least one j this vector cannot be 0 so this number must be 0 for that j.

So lambda I want to really write lambda equals lambda j for that j so what we have shown is that these numbers exhaust all the eigenvalues lambda 1, etc lambda k exhaust all the eigenvalues of the operator T, is that clear? We started with lambda being an eigenvalue so the equation Tx equals lambda x with x not equals to 0 must be satisfied the rest is consistent with throughout we have as assumptions, okay.

So what we have shown is that these are precisely the eigenvalues now lambda 1, etc lambda k are precisely the eigenvalues of T, why is T diagonalizable that is almost a tautology it does not need much explanation probably. Any vector x in the space V can be written as sum of vectors x 1, x 2, etc x k, x i coming from E i but I know that range of E i okay I know that anything in the range of E is an eigenvector so long as it is non-zero.

All that I will do is take a basis for range of E 1, take a basis for range of E 2, etc combine them take the union that will be a basis for V this basis for V has a property that each vector is an eigenvector. So from this equation and from the fact that anything in range of E i belongs to the null space of T minus lambda I it follows that T is diagonalizable, is that clear? What do I want to show? I want to show that I want to show T is diagonalizable so I want to show that there exist a basis for V each of whose vector is an equation, okay all that I do is take a basis for range of E 1, take a basis for range of E 2, etc take a basis for range of E k the fact that the sum is equal to identity means that I have exhausted the the number of elements in the basis for E 1 plus a number of elements in a basis for E 2, etc the number of elements in a basis for E k that number must be equal to the dimension of the space because of this condition so T is diagonalizable, okay. So it is a immediate consequence of 2.

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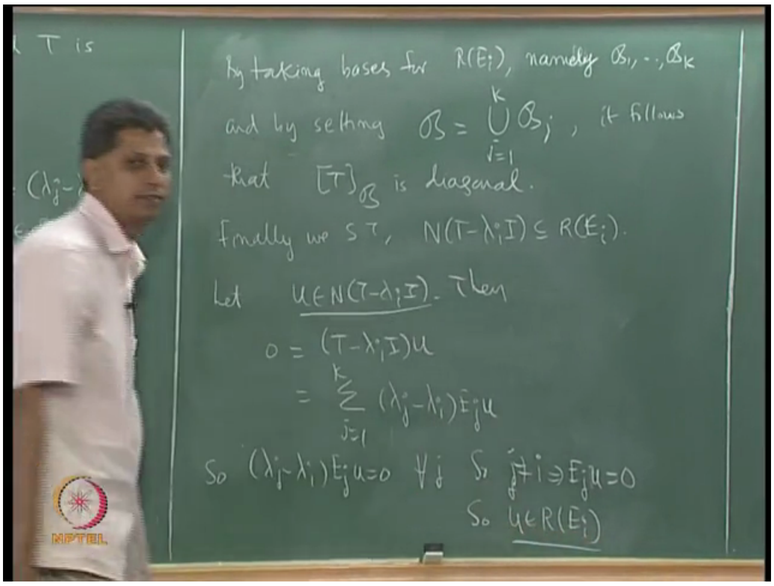

I will just write that by taking basis for range of (E 1) range of E i okay let me just give it for the sake of completeness namely B 1, etc B k and by setting script B to be the union of these basis it follows that the matrix of T relative to B is diagonal that is each element in this basis is an eigenvector for T each element in the basis B is an eigenvector for T because each element in the basis B must belong to one of the B i's anything in B i is basically a subset of range of E i anything in range of E i these are basis vectors so they are non-zero vectors so each of them is an eigenvector, is that clear so T is diagonalizable that is the most important part but it follows as an easy consequence of what we have done earlier.

The final part is to show that this eigenspace is contained in range of E i that is the last part all other things have been proved, okay we have shown that range of E i is contained in the eigenspace we must show that the eigenspace is contained in the range of E i, null space of T minus lambda I will show that this is contained in range of E i, okay let us take an element I will call it u let u belongs to null space of T minus lambda I then u is T minus lambda I T minus lambda I u is 0 I will again use that representation for T and I. Summation j equals 1 to k this time I will write down the simplified expression this will be lambda j minus lambda i E j u lambda j E j u that is T u lambda i u is lambda i E j u i is summation E j so I have this.

Now this is 0 again this is an equation involving vectors that are in range of E i range of E j so each term must be 0 because these are independent subspaces. So I have lambda j minus lambda i E j u this is 0 for all j not equal to i because when this is true for all j is okay we will exploit this for all j this is true all you have to do is ya conclude from this that when j is not equal to i look at this equation when j is not equal to i it will tell you that E j u so just tell me if this is clear so j not equal to i implies E j u is 0 when j is not equal to i see I have this product to be 0 about the ith equation I do not know about the ith equation I cannot make any conclusions but about all other equations what I know is that this is not 0 so this must be 0, okay I have equation from 1 to k, i occur somewhere that ith equation is lambda i minus lambda i into E i u equals 0 so there I cannot conclude E i u equal to 0.

While if I look at the other equations it follows that E j u is 0 for all j not equal to i, okay what is the meaning of this E j u equals 0 for all j not equal to i it means again any any vector can be written as any vector x can be written as E 1 x plus E 2 x etc all terms are 0 except the ith term so x belongs to range of E i ith term comes from range of E i so sorry not x u u belongs to range of E i.

So from this the last step is u belongs to range of E i i is fixed. So I started with u and null space of T minus lambda i I have shown that u belongs to range of E i already we have shown that range of E i is contained in null space of T minus lambda i so these subspaces are the same that is range of E i is the eigenspace corresponding to the eigenvalue lambda i, okay now that is the complete theorem with its proof probably we look at an example numerical example, okay I want to look at a numerical example where we will calculate all these an example where the matrix is diagonalizable the operator is diagonalizable.

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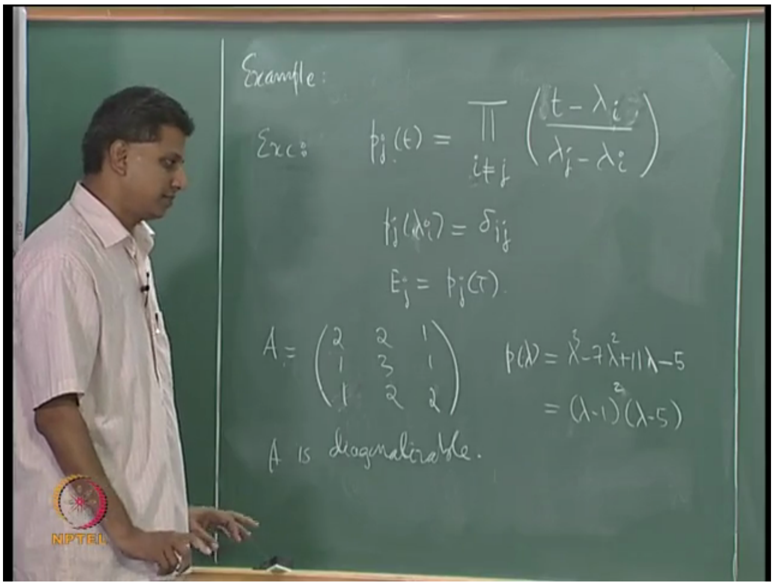

So let me quickly see if this this works by the way before I work out this example there is a fact that I will state without proving we can discuss it if you want the proof later outside the class. It can be shown that this the operators, the projections, the E j's are in fact polynomials in T in particular may be I will give this as an exercise. Let me define p j of t to be the product of all these polynomials p j of t so the product is over i let us say i does not take the value j look at t minus lambda i by lambda j minus lambda i. I am defining a polynomial in this manner this polynomial for one thing has the property that p j of t i is delta i j this polynomial has a property that p j of t i is delta i j, p j of lambda i is delta i j. I am defining for the numbers lambda 1, etc lambda k that are distinct.

For every fixed j here the product the index is i for the product index is i these are these are really lagrange interpolating polynomials we have encountered this before these are lagrange interpolating polynomials what follows what can be shown is that E j is p j of T, what is the meaning of this? It says there is a the formula for E j's can be obtained from the formula for the polynomials here, okay why this is true is an exercise for your real I will freely use this in the example, okay. What is that example?

Let me take the matrix A as there are relationships between these are called Newton's formulas there are relationships between the entries of the matrix and the eigenvalues. For example the sum of the eigenvalues is the trace of the matrix the sum of the product of 2 eigenvalues taken at a time is the sum of the 2 by 2 principle minors the product of the eigenvalues is the determinant of the matrix, okay I will use these three properties I will not prove any of these again these are exercises.

I want to write down the characteristic polynomial for this matrix I use t right for that so p of t I claim is 2 plus 3 5 plus 2 7 lambda cube minus 7 lambda square I want to look at the 2 by 2 determinants 6 minus 2 is 4 6 minus 2 is 4 what I want is really 4 minus 1 is 3 that is 11 it goes with a minus this goes with a plus the final term is a determinant what is the determinant? Determinant is 5 plus 7 plus 11 lambda minus 5 this is the characteristic polynomial, okay this can be shown.

What are the factors? 12 minus 12 1 is a factor verify that this is lambda minus 1 the whole square into lambda minus 5 5 is also a factor p of lambda, is this diagonalizable, how?

Student is answering: lambda equal to first get a eigenvector.

Can you repeat for this.

Student is answering: For lambda equals to 1 than null space of t minus i is 2.

Did you see this for lambda equals 1 the substitute there is only one equation 1 2 1, 1 2 1, 1 2 1 only one row the rank of that is 1 so nullity is 2 so this matrix is diagonalizable, A is diagonalizable A is diagonalizable. I want to compute see all that I want to do is to illustrate that this previous theorem by means of this example. So I will compute E j's and then verify sum of E j equals identity and then t can the product is 0, t can be written as lambda 1 E 1 plus lambda 2 E 2 that is enough really, okay.

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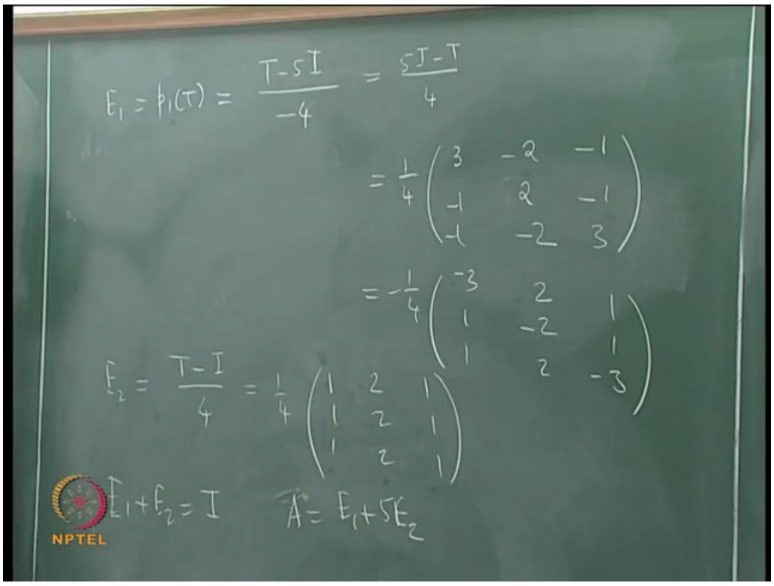

E 1 is p 1 of T p 1 of T there is only one factor left out p 1 corresponds to the first eigenvalue T minus 5 identity by what is fixed is 1 is fixed so 1 minus 5 minus 4 this is E 1, 5 I minus T by 4 let me write this maybe 1 by 4 to be taken outside 5 I minus T 5 minus 2 3 5 I minus T minus 2 minus 1 minus 1 minus 1 minus 1 minus 2 5 minus 3 is 2 5 minus 2 is 3 there are many minuses let us take minus 1 by 4 outside minus 3 2 1 1 minus 2 1 1 2 minus 3 please check that my calculations are correct.

What is E 2? E 2 will be T minus I by 4 I am sorry I minus T minus I is correct yes. So this is 1 by 4 T minus I 1 2 1 1 2 1 1 2 1 is that okay yes when you add do you get what we want okay this one goes with a plus sign 3 plus 1 4 4 by 4 1 ya this goes with a minus sign the previous one would have been better I will do it with the previous one 0 this is 0 this is 0, this is 4 by 4 is 1, this sum is 0 this is 0, this is 0, this is 3 plus 1 by 4 so sum is identity.

E 1 plus E 2 equals identity. See I got a once I got E 1 I could have got E 2 by using the fact that E 2 is I minus E 1 but I wanted to really verify that so I calculated E 2 independently by this formula, okay finally you check that T that is A in this case lambda 1 E 1 plus lambda 2 E

2, okay that is 5 by 4 plus 3 by 4 8 by 4 2 that is the first entry please check this is satisfied A is E 1 plus 5 E 2, please also write down the range range spaces of these two operators they will be precisely the eigenspaces.

For example it is immediately clear that the range of E 2 range of E 2 is 1 1 1 (())(36:07) by 1 1 1 range of E 2 (())(36:10) by 1 1 1 and E 2 range of E 2 is one dimensional, range of E 1 two dimensional because lambda for lambda equals to 1 the null space has two independent vectors, okay so this is just to illustrate. As I told you there are certain properties I have just stated I have not used the relationship between the roots of the equation p of lambda equal to 0 that is eigenvalues and the entries and why is this formula correct? You please check this there is also another result which at least I will state may be.

Let me state this result and then prove it in the next lecture. Let us it is kind of a summary of what we have done is the following. We discussed the notion of diagonalizability, we characterized diagonalizability in terms of the minimal polynomial, okay an operator is diagonalizable if and only if the minimal polynomial is a product of distinct linear factors. There is another characterisation if the dimensions of the eigenspaces sum upto the dimension of the space then the operator is diagonalizable.

There is really another characterisation that we have got just now this involves invariant direct sums invariant direct sum decompositions. If T is diagonalizable and if I can find out certain certain operators then I know that operators will satisfy these conditions and conversely if there are certain operators and certain numbers related in a specific manner then T is diagonalizable.

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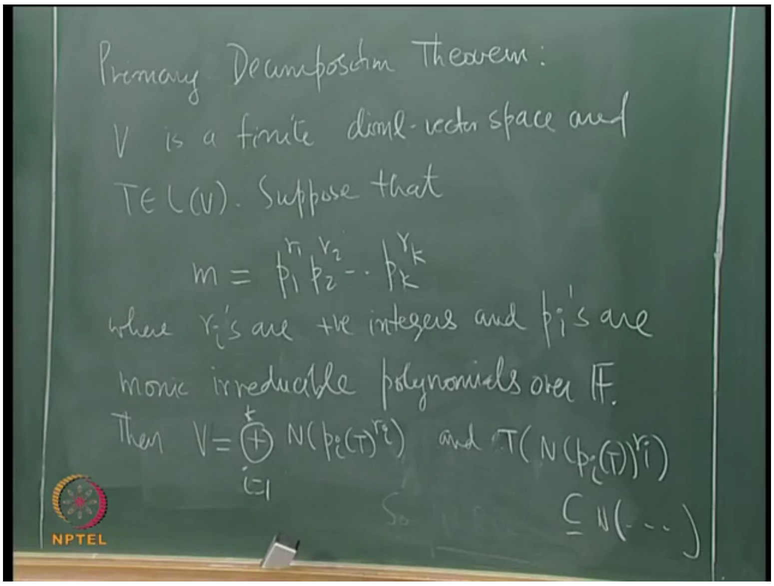

If you look at the general problem then in all these cases what is important is to realize that the minimal polynomial can be written as okay where if T is diagonalizable then r 1, r 2, etc r k they are all 1 if T is not diagonalizable then sum of these will be at least 2 at least one of them will be 2 for instance but if T is not diagonalizable we know that it is always t triangulable if the minimal polynomial is of this form any operator is triangulable provided all its eigenvalues lie in the underlying field which is a same as saying that the minimal polynomial is of this form, okay.

So this kind of encompasses all these both the case of diagonalizability and triangulability, what happens if in in the case of r for instance the irreducible polynomial the degree the degree of an irreducible polynomial can be 2 in the case of r, in the case of c the degree of an irreducible polynomial is precisely 1 for instance t square plus 1 is an irreducible polynomial in the field in the space of polynomials over r in the case of c these are linear polynomials, okay.

What happens to a direct sum decomposition in this case that is what we will refer to as a primary decomposition theorem. In this general case that is in the case when the eigenvalues lie in the underlying field what is the representation? What is the direct sum decomposition? Okay I will write down the statement and prove it tomorrow this is called the primary decomposition theorem in other words this is kind of a most general theorem that one has all the other results are particular cases all the other results that we have proofed can be shown to be particular cases of this.

V is finite dimensional. Suppose that the minimal polynomial is written as p 1 to the r 1, p 2 to the r 2, etc p k to the r k I am writing an expression which is more general then what I wrote down here. Suppose if the minimal polynomial can be written in this manner where remember little m is minimal polynomial for us that is a notation where r i's are positive integers what are these p i’s p i’s are monic irreducible polynomials over the underlying field F.

Now this F is general r c rationals it could be a finite field, could be any other field then we have the following. Let me write it like this let me use this notation this is what I want to say V is the direct sum of the null space of p i T power r i V is the direct sum of the null space of p i T power r i each of these subspaces is invariant under T each of these subspaces I could have call them W i each of these subspaces W i is invariant under T T W i is contained in W i this is called the primary decomposition theorem this is the most general that one could do for any operator.

Why is it call primary? Primary decomposition is clear it is called primary decomposition theorem because see it is monic polynomial the degree of the coefficient of the highest degree is 1 that is monic irreducible, what is irreducibility? It is because of this irreducibility that this is called the primary decomposition theorem. If a polynomial is irreducible over a field then it is called a prime polynomial that is if a polynomial is irreducible when do you say the polynomial is reducible if it can be written as a product of two or more polynomials each has degree at least 1 each has degree at least 1, okay x square minus 1 is (x min) lambda square minus 1 is lambda plus 1 into lambda minus 1 so lambda square minus 1 is not irreducible over r lambda square plus 1 cannot be written as lambda minus lambda 1 into lambda minus lambda 2 where lambda 1 and lambda 2 are real numbers.

So lambda square plus 1 is an irreducible polynomial over r lambda square plus k where k is a positive number is an irreducible polynomial over r such polynomials are called prime polynomials. If a field is such that the prime polynomials are only linear such a field is called an algebraically closed field this is the same as saying that any polynomial equation has all its zeros in the field any polynomial equation has all its zeros in the field is a same as saying that the polynomial can be factored into a product of linear factors, okay.

So in general if F is not r of c, an irreducible polynomial is called a prime polynomial. So these are prime polynomials they cannot be factored into a product of polynomials of lesser degree just like how you have composite numbers and prime numbers. So this is called primary decomposition theorem for any operator this is the most general that one could have, okay the decomposition corresponds to null space of p i T r i, I will prove this theorem in the next lecture let me stop here today.