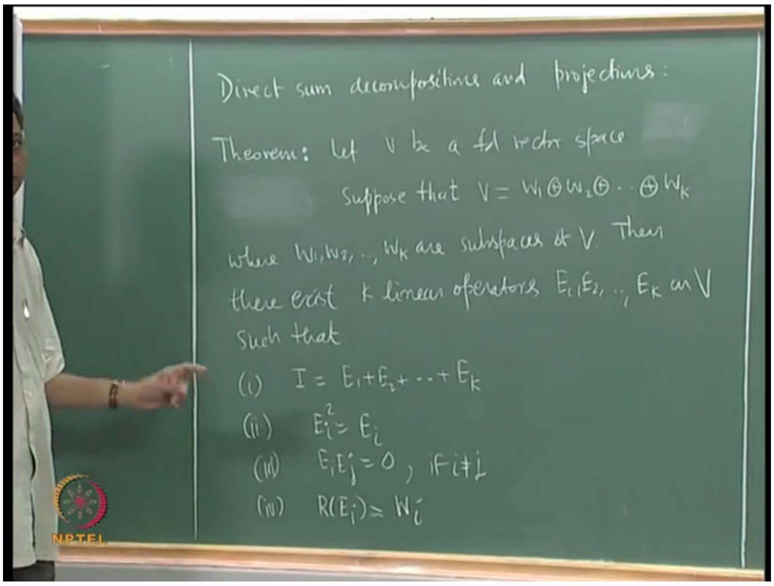
Linear Algebra
Professor K.C Sivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Module 9 Direct Sum Decompositions
Lecture 33
Direct Sum Decompositions and Projection Operators 1

Okay, we are discussing direct sum decompositions and projections, okay. I gave you some of the preliminary ideas of the proof of the following theorem let us prove it today completely.

(Refer Slide Time: 0:33)

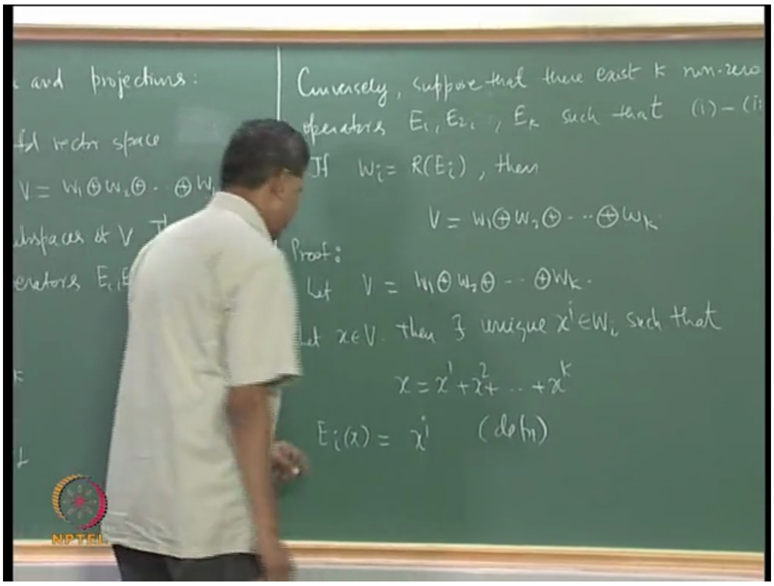


Let V be a finite dimensional vector space and T is a linear operator on V. Suppose that V is the direct sum of the subspaces W 1, etc W k. Suppose I have a direct sum decomposition of the vector space V, okay I must mention that there is no operator to begin with operator connection will come later. So I have just just this the operator connection will come later that will be the next result.

Let us first discuss the relationship between direct sum decompositions and projections that is what we will first discuss. So V is the direct sum decomposition of these subspaces the first part is there exist k linear operators there exist k linear operators E 1, E 2, etc E k defined on V such that the following conditions are satisfied by these operators. The first condition is that identity can be written as E 1 plus E 2 plus etc plus E k identity can be written as a sum of these projections we are going to prove these are projections. E i square is E i that is E i is a projection you take different E i's and take the product that is 0 operator.

Finally the relationship between E i's and W i's is given by the last condition which is that range of E i is W range of E i is W. So that is the first three are properties of these operators E 1, etc E k the last one connects the projections connects these operators to the subspaces W i that we started with which form a direct sum decomposition of the vector space V, okay. Now what is important the converse is also true.

(Refer Slide Time: 3:58)



Conversely suppose that there exist k non-zero operators E 1, E 2, etc E k such that the such that conditions 1 to 3 hold conversely suppose that I have operators E 1 etc E k such that conditions 1 to 3 are satisfied by these operators. If we set W i to be the range of E i call these call the range spaces of these operators as W i then the converse statement says that we have this decomposition, okay.

So this is really the connection you can go from one to the other you can if you have a direct sum decomposition of a vector space into independent subspaces then you can define projections on those subspaces, conversely if you have projections on subspaces then you can use that to obtain a direct sum decomposition of the vector space, okay. We will need this result we will do a little further perhaps in today's lecture itself when we will also discuss how these projections are related to operators the projections must be related to operators so the subspaces must be related to operators the natural connection between subspaces and operators is that subspaces must be invariant, okay. We will also try to discuss invariant direct sum decompositions today but first direct sum decompositions and projections.

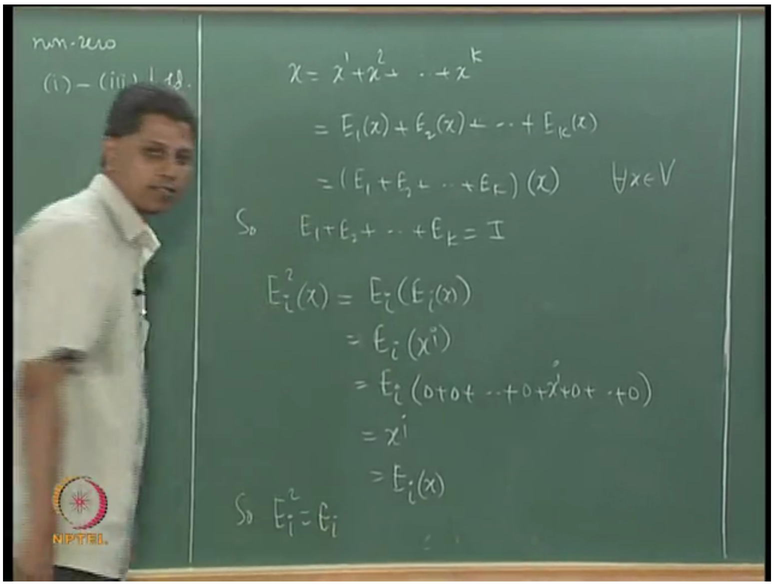
Proof I have already given the first few steps in my last lecture there are two parts, first part is if there is a decomposition then we can define projections which satisfy certain properties and the relationship between the projections and the subspaces given by the last formula. So let us assume that V has this direct sum decomposition if this happens then we know that any vector in V has a unique decomposition, okay x equals x 1 plus x 2 etc x k there is unique decomposition and when I write this decomposition I mean that the first term comes from W 1, the second term comes from W 2, etc the last term comes from W k there is an order in which write that I write the terms.

Remember that these are additional vectors so I can write this as x 2 plus x 1 etc but I will not do it I want to look at a decomposition which corresponds to the direct sum decomposition that is given to me. So this is what I will write, given a vector x I will look for its unique decomposition in this manner that is always possible because there is a direct sum decomposition, okay.

I want to define the operators this was done last time let me do this quickly. For each E i I define E i of x to be the ith term in this. Now that is why I need to know what is the ith term when I say it is ith term I must be careful about the order in which write the terms that order has already been fixed. I will define E i of x to be x i the ith term on the right hand side, the claim is that these E i's for 1 to k satisfy these conditions first three conditions and that the range space of these E i's equal W i that is what we will proof but all these are very straight forward.

For one thing look at this is my definition okay this is a definition of E i's whenever I define a function I must know that it is well defined for an x is this well-defined, is this x i unique it is unique because this representation is unique and I am following this order first term coming from the subspace W 1, second from W 2 etc last term coming from W k. So this is unique, this is well defined, linearity, easy to see. Is this item potent that is okay first thing is identity can be written as a sum of these that is almost straight forward.

(Refer Slide Time: 9:02)

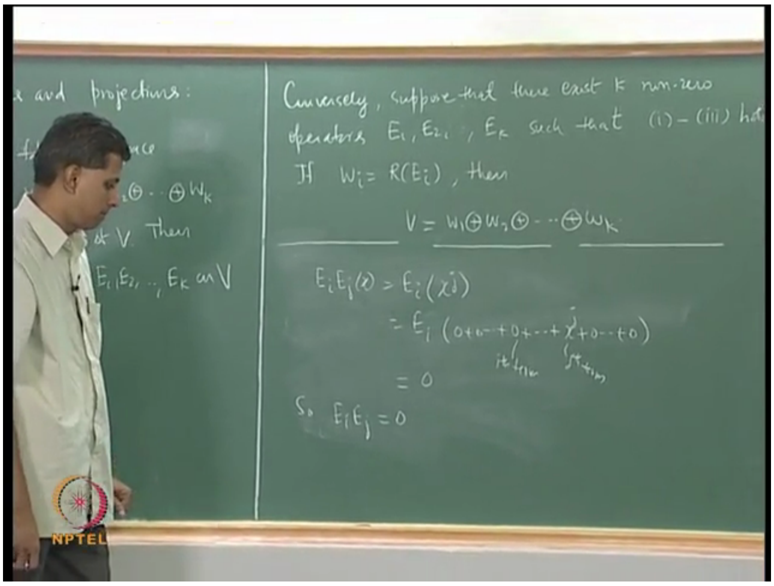


Let us do that here x is by definition x 1 plus x 2 etc x k where x 1 by definition is E 1 x, x 2 is E 2 x, etc I have define this k operators by means of that formula this is E k x each is linear so I can take the sum E 1 plus E 2 etc E k operating on x, I have started with x I have written that as t times t of x so this t must be equal to identity this is true for all x for each x I have to look at this representation unique representation. So it follows that E 1 plus E 2 plus etc plus E k is a identity operator that is the first condition that these operators must satisfy.

Second condition E i square let us look at E i square of a vector x the vector x has this representation E i square x by definition is E i operating on E i of x, E i operating on E i of x it is ith term so it is E i of x i. Now I will rewrite this as E i of 0 plus 0 plus etc plus 0 plus x i plus 0 plus etc plus 0 where all the zeros comes from the other subspace W 1, W 2, etc W i minus 1 W i plus 1 etc W k this is the unique representation of x i as a sum of vectors from the subspaces W 1, etc W k E i of any such vector will be the ith term the ith term here is x i so this is x i but x i by definition is E i of x, okay.

And so we have shown E i square of x is E i of x for all x. So second condition is satisfied E i square equals E i, second condition is satisfied.

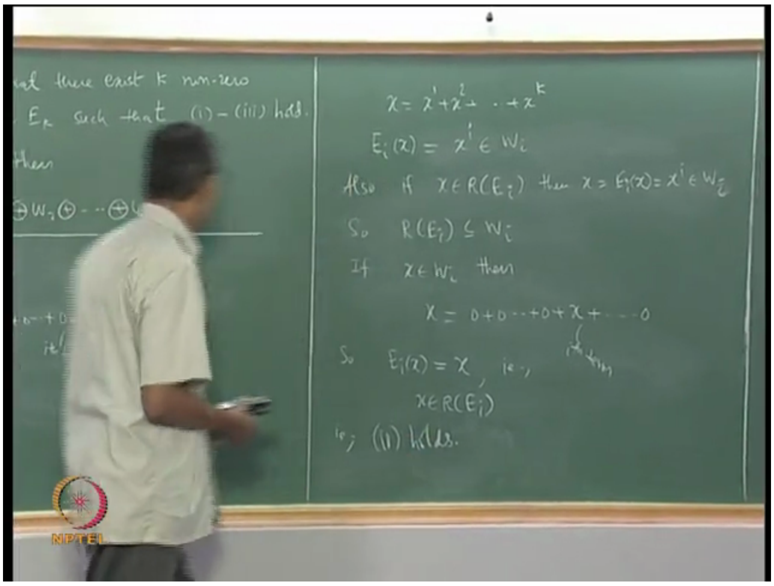
(Refer Slide Time: 11:22)



Let me go back and verify the third condition I will do it here this is the statement of the theorem I want t verify this condition E i, E j is equal to 0 and finally range of E i is W i, okay that is also an immediate consequence of the definition. Look at E i, E j of x this is E i of E j of x is the jth term x j, E i of x j again I will do a similar thing this will be 0 plus 0 etc plus ith term is 0, jth term comes somewhere here I am assuming i less than j there is no loss of generality if i is greater than j this will come later jth term here plus 0 etc plus 0 this is the jth term that is this is a unique representation of the vector x j as a sum of elements of W 1 etc W k, E i of this representation I know is ith term that is 0, okay. If i comes after j then also it is 0 there is no loss of generality.

So I have shown that E i E j x is 0 for all x so this operator must be the 0 operator, E i E j is 0. Range of E i equals W i how does that follow what is E i of x.

(Refer Slide Time: 13:06)



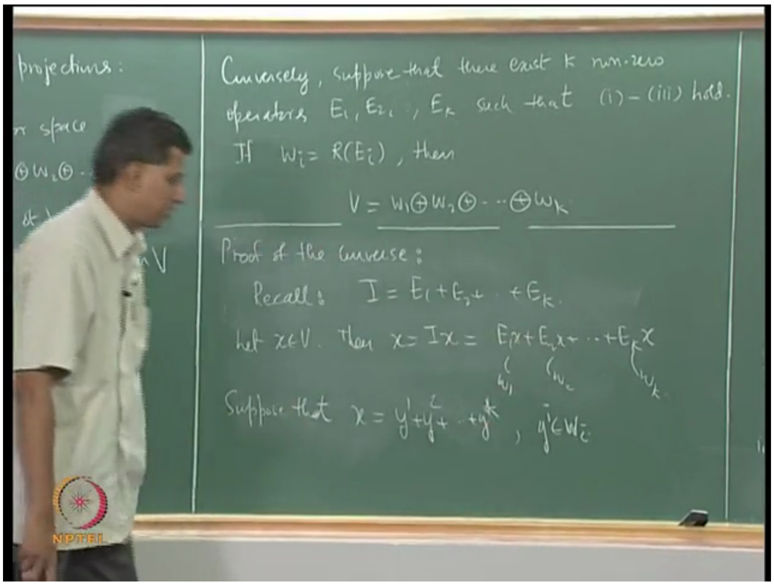
So I need to go back to this I will keep this if x is this then E i of x is equal to x i this by definition belongs to W i, okay and remember that if x belongs to range of E i then then what you have an any you have an idempotent operator then the operator acts like identity on its range that is what we saw last time. So if x belongs to range of E i then range of E i then x is equal to E i x E i x for me is x i that belongs to W 2, so what have I shown?

I have shown that range of E i is contained in W i sorry this is W i range of E i is contained in W i take any idempotent operator it acts like identity on its range. So if x belongs to range of E i then x is equal to E i x, x is equal to E i x but E i x we have by definition is x i x i by definition comes from W i. So range of E i is contained in W i on the other hand on the other hand if x belongs to W i then if x belongs to W i then I want to know what is E i of x to know E i of x I must know what is the representation of this as a sum the unique representation.

I know that this is in W i all the other terms must be 0 this is the ith term all the other terms are 0 I must know this representation in order to write E i of x then this so E i of x is equal to the ith term that is x so I have shown that if x belongs to W i then I have shown x belongs to range of E i that is W i contained in range of E i combine with this condition 4 holds that is the connection between the projections and the subspaces.

Otherwise if you look at the first three conditions they talk only about this collection E 1, etc E k, what are the properties that these satisfy so this is the first part okay really simple if you just follow this follow the logical steps coming from the definition converse I need these three conditions.

(Refer Slide Time: 16:06)



So I will try to proof the converse again in this part. I am assure that conversely if I have these subspaces these operators satisfying the first three conditions and if I call the range of these subspaces as W i then this gives rise to a direct sum decomposition of the vector space V, okay okay. The only thing that I will do is first I will show that any vector in V can be written as a sum of vectors coming from the subspaces and then show that this representation is unique it would then follow that it is a direct sum decomposition.

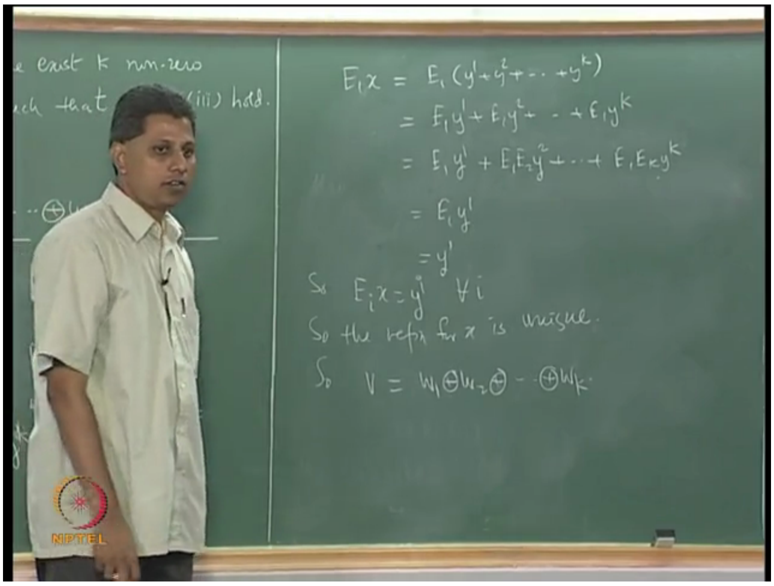
Proof of the converse part proof of the converse, I am assure that any x in V has can be written as a sum of vectors coming from the subspaces but that is straight forward. If you look at the condition that these operators satisfy what is the first condition? Identity is a sum of these operators E 1 plus E 2 etc plus E k. So I take x in V then x can be written as I x that is E 1 x plus E 2 x plus etc plus E k x where okay where each of these this belongs to W 1, this belongs to W 2, this belongs to W k.

So for one thing I have written any vector as a sum of vectors coming from W 1 etc W k. So V is contained in this sum if I show that these subspaces are independent then it is a direct sum, it is same as showing that this representation is unique it is same as showing that this representation is unique, okay that is the definition of independent subspaces, is that correct? Subspaces W 1, W 2, etc W k are independent if the equation u 1 plus u 2 plus etc plus u k equal to 0 u i coming from W i if this equation implies each u i is 0.

This also gives rise to the fact that representation in terms of the sum is unique. So we will proof that this representation is unique. Suppose I have another representation if possible suppose that by the way I must mention that E 1 x belongs to W 1, E 2 x belongs to W 2, etc E k x belongs to W k because of this and this is just a notation it is not an assumption just a notation it is not an assumption.

See this if this holds then we want to show this holds, this is just a notation the ranges of these operators are called W 1 etc W k if the operator satisfy condition 1, 2, 3 then that must give rise to a direct sum decomposition is what the claim is, okay. So range of this is in range of E 1 that is W 1 and so I have this representation, okay. I want to show uniqueness suppose that x is written as let us say y 1 plus y 2 etc y k where each y i belongs to W i I must show that y 1 is E 1 x, y 2 is E 2 x, etc I will show that y 1 is E 1 x, y 2 is E 2 x, etc y k is E k x, okay.

(Refer Slide Time: 20:00)

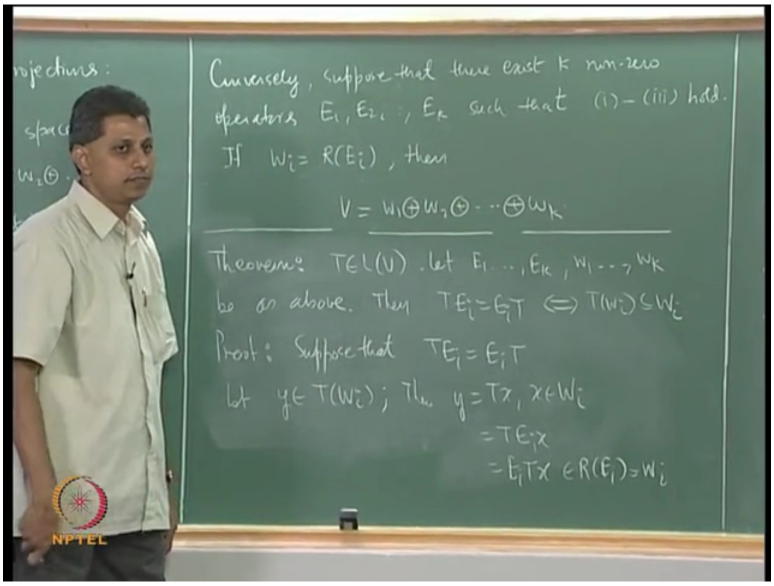


I will just look at E 1 x E 1 x by definition is E 1 of y 1 plus y 2 etc y k E 1 is a linear operator it is E 1 of y 1 plus E 1 y 2 plus etc E 1 y k E 1 y 1 is E 1 y 1 is okay I will keep this as it is for the moment. Look at the other terms E 1 y 2 y 2 is y i is in W i y 2 is in W 2, W 2 is range of V 2. So this is this is E 2 y 2 really right again I am using the fact that any element if E is a projection then E acts like identity on its range, y 2 belongs to range of E 2 so E 2 y 2 was y 2 each term E 1 E k y k.

Look at the terms from the second onwards I will use property 3, property 3 is satisfied by these operators so all these terms are 0. So I will have just E 1 y 1 but where does y 1 come from y 1 comes from W 1 that is range of E 1 so E 1 y 1 is y 1 this is what we wanted to proof. So what we have shown is that E 1 x is y 1 this E 1 x is y 1 similarly E 2 x is y 2, etc E k x is y k so the representation is unique.

So it can be similarly shown that E i x equals y i for all i so the representation for x is unique that is the subspaces are independent which is same as saying that the sum is not just ordinary sum it is a direct sum decomposition. So V is W 1 direct sum W 2, etc direct sum W k, okay that is the converse part, okay let us is that clear. We need to as I mention we need to also look at operators and their relationships with these projections, okay what is the relationship and how is that relationship given in terms of the subspaces.

(Refer Slide Time: 23:12)



I will state the next result which will make use of this theorem again so I will keep this whole thing once again this statement we need now I will look at T let T be a linear operator on this vector space V that is finite dimensional. I have the subspaces W 1, etc W k satisfying the conditions of this theorem, okay let me just mention let E 1, etc E k, W 1, etc W k be as above, what is the meaning? The meaning is that even E 1, etc E k, are operators that satisfy the first condition of the previous theorem W 1, W 2, etc W k are the subspaces that are defined as range of E i equals W i.

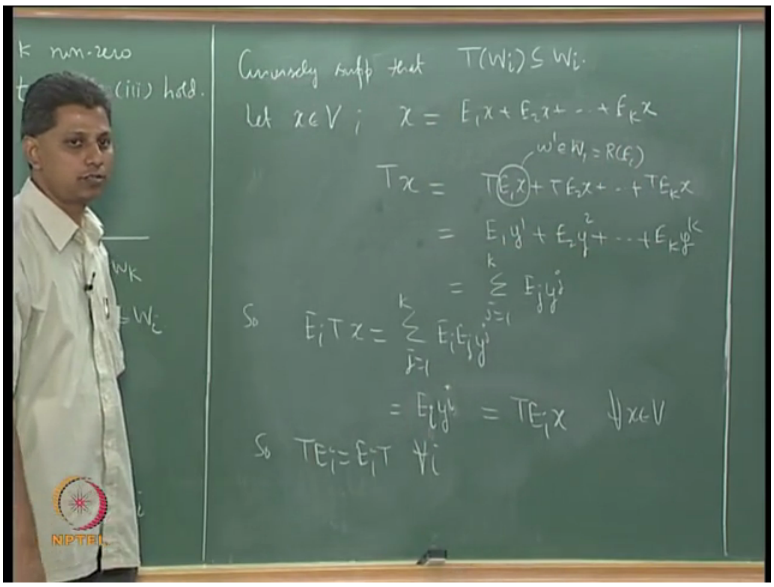
So I already have a direct sum decomposition of the vector space V, how do you get T into the picture? The first thing we must observe is that T E i equals E i T if and only if T of W i is contained in W. So the answer is in terms of invariant subspaces that is why this notion was introduced some time ago this is a natural notion to connect a linear operator and projections associated with the linear operator, we will show how these projections are associated with the linear operator T but this is already one relationship.

If each W i is invariant under T then each E i will commute with T each W i is invariant under T then each E i will commute under T and converse it, okay using this we will look at diagonalization in a different language. For diagonalization we have seen two characterisations we look at another characterisation diagonalization using invariant direct sum that is the objective of this of this topic, okay. Let us first proof this and then look at that theorem.

Proof I can freely use the properties of E i's and W i's that have been defined earlier, okay let us first show that if this condition holds then each W i is invariant. Suppose that suppose that T E i is equal to E i T for all i I look at W i is invariant under T let me take y in T of W i that y belong to T of W i I must show that y belongs to W i then I can write y as Tx for some x in W i by definition.

Now x is in W i, W i and d i are related by the condition fourth condition, so this x can be written as E i x W i is range of E i so this y is T E i x instead of x I have written E i x E i acts like identity on its range T E i I know is E i T I am assuming this. So this is E i Tx. Now whatever be Tx this is E i of something that belongs to range of E i which is in which is W i. So this belongs to range of E i which by definition is W i this is what I wanted to proof if y belongs to T of W i I must show that y belongs to W i I have done that. So if T E i is E i T then T W i is contained in W i we must establish the converse.

(Refer Slide Time: 27:39)



Conversely let us assume that W i is invariant under T, I must show that T and E i commute, okay so we need to look at representation. Let us take x in V then I know that x has this representation E 1 x plus E 2 x etc E k x this comes from the first equation identity is E 1 plus E 2 etc E k. What is Tx? Tx will then be T of this T E 1 x plus T E 2 x each of these is invariant under T.

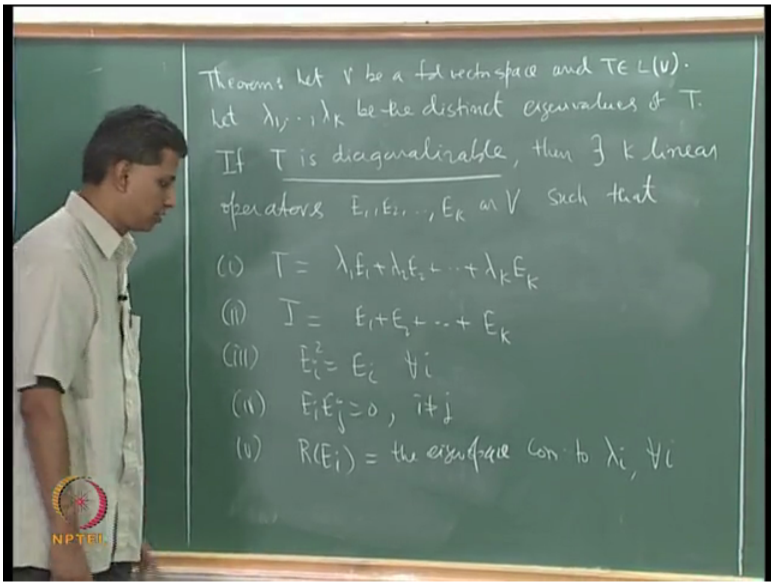
So I can write this as T E 1 T of anything T of something in W i is contained in W i. So this can be written as this is in W i and W i is range of E i so do you agree that I can write this as let us say E 1 y 1 plus E 2 y 2 E k y k I am assuming that each W i is invariant under T I will show that each of the E i's commute with T each W i is invariant under T I will show that each E i commutes with T.

So T E 1 this is T of a vector in W 1 that is in W 1 but anything in W 1 is W 1 is range of E i W 1 is range of E 1 so this is E 1 of some vector I am calling that as y 1, E 2 y 2 etc E k y k, okay what do I want to look at I want to look at E i of T I can write this as summation j equals 1 to k E j y j and then I will look at E i T. So E i T of x is summation j equals 1 to k E i E j y j, i is fixed, j is running index, i is fixed, j is running index so when when j takes the value i I get E i square all other terms are 0 because the properties that these projections satisfy E i into E j equal to 0 when i is not equal to j, when j is not equal to i, j is running index.

So this is simply E j y j I am sorry E i y i, j is running index j takes the value i it is non-zero all the other terms are 0. So E i Tx is E i y i but E i y i by definition from here is I just look at the ith term T E i x E i y i is ith term coming from this representation that is T E i of x. So I have shown E i Tx equals T E i of x for all x so okay that is the second part. See the reason here is T of T of this vector that vector is in (W i) W 1 this is in W 1, so I have T of some vector let us say W 1 T of W 1 but T of W 1 is in W 1 and W 1 on the other hand is range of E 1 so this little w 1 is in range of E 1 so it is E 1 y 1 for some y is that okay this vector let us call it w 1 w 1 belongs to capital W 1 that is range of E 1, if a vector belongs to range of this then w 1 can be written as E 1 of some vector I am calling that y 1 I do that for each term. So w 1 is E 1 y 1, w 2 is E 2 y 2 etc, okay.

So this relationship holds between all the projections and the operator T if and only if each of these subspaces that is the range of the projections must be invariant under T, okay. Let us now connect all these with diagonalizability.

(Refer Slide Time: 33:43)

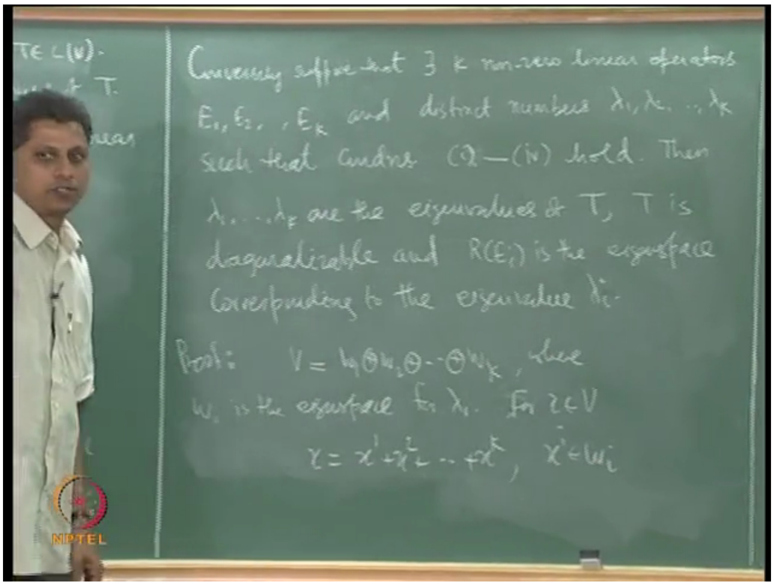


V is a finite dimensional vector space and T is an operator on V. Let lambda 1, etc lambda k be the distinct eigenvalues of T. If T is diagonalizable if T is diagonalizable, then there exist k linear operators E 1, E 2, etc E k such that the following conditions are satisfied. The first condition is T is a linear combination of these operators and the coefficients in fact come from the eigenvalues lambda 1 E 1 plus lambda 2 E 2 plus etc plus lambda k E k this is one relationship between T and the operators the projections E 1 etc E k.

Second formula, third formula, fourth formula we have seen before. Identity is E 1 plus E 2 etc E k each of these is a projection that is E i square is E i for all i they are kind of perpendicular the product is 0, E i E j is 0 whenever i is not equal to j the final condition is that the range spaces are some subspaces these subspaces give rise to a I am recalling what we proved just now these E i's are such that range of E i's are certain subspaces which give rise to a direct sum decomposition. Now I also have the operator T and (eigenvectors) eigenvalues.

So what do you expect these subspaces to be? Eigenspaces condition 5 range of E i equals the eigenspace corresponding to the eigenvalue lambda i for all i range of E 1 is eigenspace corresponding to lambda 1, etc since T is diagonalizable what it also means is that V is direct sum of these subspaces that is diagonalizability, converse also holds what is the converse?

(Refer Slide Time: 37:25)



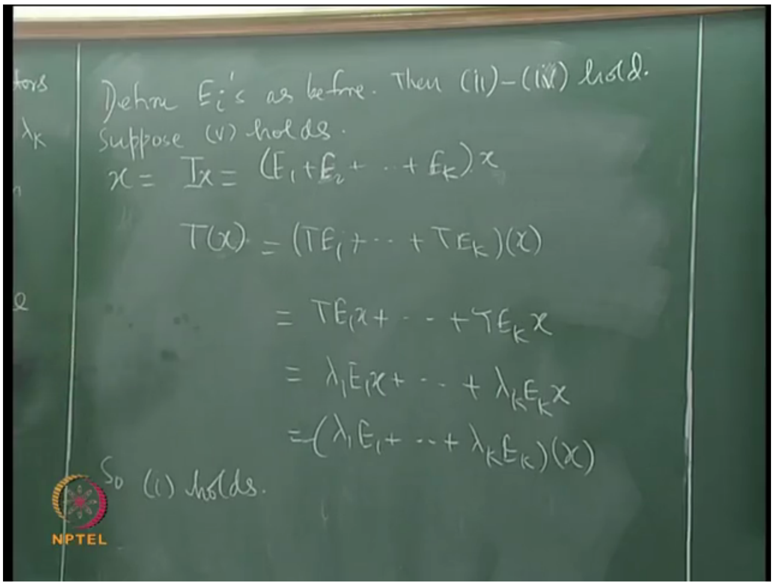
Conversely suppose that there exist k non-zero linear operators E 1, E 2, etc E k and distinct numbers lambda 1, lambda 2, etc lambda k such that conditions 1 to 4 hold conditions 1 to 4 hold conversely I have k non-zero linear operators E 1, etc E k and distinct numbers lambda 1, etc lambda k such that such that T is this particular linear combination, i is this particular linear combination, E i each E i is a projection the product of any two projections is 0 any two distinct projections is 0.

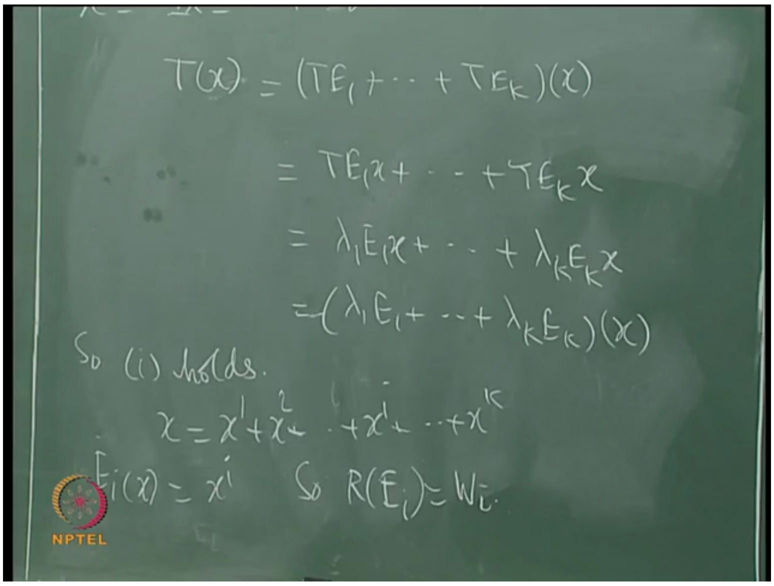
Then lambda 1, etc lambda k are the eigenvalues of T that is the first thing, the operator T is diagonalizable and and the range of E i is the is the eigenspace corresponding to eigenvalue lambda i this is the converse part. So this is a kind of a necessary sufficient condition for T to be diagonalizable.

Again this has two parts first part is relatively easy so maybe I will proof the first part today todays lecture. Proof first part is this if T is diagonalizable then I must show that there exist k linear operators that satisfy these conditions together with the condition that the range is an eigenspace range of E i is the eigenspace corresponding to lambda i, okay of which condition 2, 3 and 4 have been verified already I will simply appeal to that definition I am given that T is diagonalizable.

So I can write V as W 1 direct sum W 2 etc direct sum W k because T is diagonalizable it has a basis such that each basis vector is an eigenvector so these are eigenspaces, where where W i is the eigenspace corresponding to the eigenvalue lambda i this I can write because T is diagonalizable, okay that is the first part if T is diagonalizable I am assure that these conditions are satisfied. Condition 2 to 4 will follow from what we did earlier only we need to verify condition (1 and 4) 1 and 5 define E i as before, okay to do that you need a representation for x in V, I have this representation x equals x 1 plus x 2, etc where this x i comes from W i this representation I know is unique using this representation I can define E 1, E 2, etc.

(Refer Slide Time: 42:24)





Define E i's as before. Then we do not have to do it again I will simply mention that (2 to 3 conditions 2 to 3) I am sorry 2 to 4 conditions 2 to 4 hold, identity is E 1 plus E 2 etc E k E i square is E I, E i E j is 0 when I use not equal to 0 of which I will take this second condition identity is E 1 plus E 2 etc plus E k. So if you look at T T is have a look at T x T x is T acting on this out of x. So let me say x equals Ix operating on x. So T x is T E 1, T E 2, etc okay.

This is T E 1 x etc T E k x now look at E 1 x E 1 x is in range of range of E 1, can we for the moment assume 5 and proceed 5 is immediate. Let us assume 5 for the moment range of E i is W i I will proof 5 next that is immediate suppose 5 holds range of E i is W i, W i's have the property that V is W 1 plus W 2 etc W k that is anything in W i so long as it is non-zero it is an eigenvector corresponding to the eigenvalue lambda i in particular anything in W i will satisfy if if Z belongs to W i then Z satisfies T Z equals lambda I Z that is what I will use.

This is in W 1, W 1 is an eigenspace for lambda 1 so this is lambda 1 E 1x do you agree E 1 is in the eigenspace W 1 W 1 is eigenspace corresponding to lambda 1 so T of that must be that number lambda 1 into that vector that vector here is E 1 x so lambda 1 E 1 x etc plus lambda k E k x take x outside so this is lambda 1 E 1 etc plus lambda k E k operating on x. So now you see that T of x has been shown to be this operator acting on x so this operator must be equal to T.

So 1 holds so 1 holds T is this specific linear combination, lambda 1 E 1 plus lambda 2 E 2 etc lambda k E k. I must proof 5 so that this argument is valid but 5 is really straight forward, what is range of E i okay what is condition 5? Condition 5 I must show that range of E i is the ith eigenspace is eigenspace corresponding to the eigenvalue lambda i but how is E i defined? E i is defined as the ith term in that representation, right, okay.

Let us look at x equals to x 1 plus x 2 etc x i plus etc x k then I know that E i of x is the ith term that is X i this x i is ith term that comes from W i so range of E i is W i but what is W i W i is the eigenspace. We started with this representation where W i is the ith eigenspace corresponds to eigenvalue lambda i and so this is really what we have done earlier the only difference this time is it is an eigenspace corresponding to the eigenspace for the operator T that is the only difference.

Just use this to simply say range of E i is W i that is the argument for the 5th property is as before the only extra thing that we have now is that W i is the eigenspace corresponding to the eigenvalue lambda i and so 5 holds and so this argument is valid, okay that proves the first part. Second part I will proof in the next lecture that will take some time.