Linear Algebra
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Module 8 Invariant Subspaces And Triangulability
Lecture 32
Independent Subspaces and Projection Operators

We will continue to analyze a single linear operator we looked at the matrix formulation for diagonalizability. We will now look at a formulation in terms of subspaces, okay not matrices. So we have discussed a notion of direct sum decompositions and projection operators.

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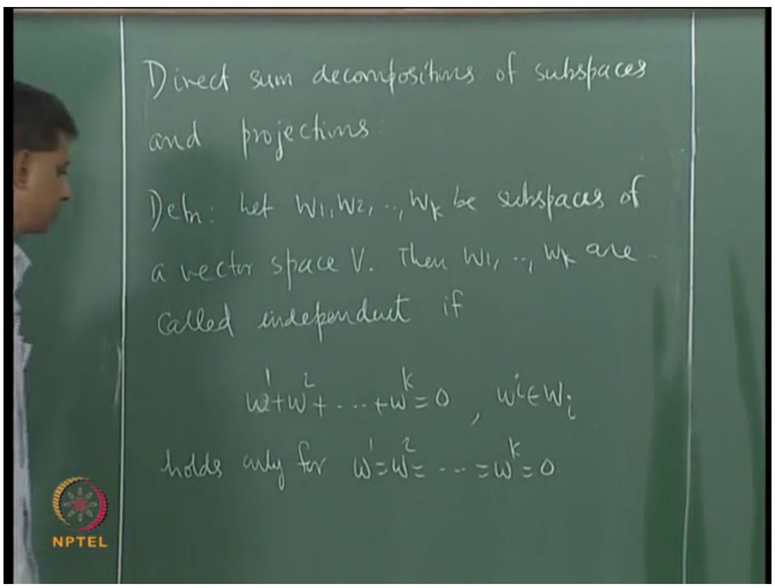

Direct sum decompositions of subspaces and the relationship with projections, okay so this is what we will discuss and see how this is relate to the problem of diagonalizability of an operator, okay we will first discuss a notion of direct sum decomposition we need the notion of independent subspaces. So I have these subspaces k subspaces W 1 etc W k these are called independent these are called independent subspaces if the following implication holds, if W 1 plus W 2 plus etc plus W k equal to 0 with small w i belonging to capital W i, if this equation has 0 as the only solution this is independent of subspaces.

I take any linear combination of vectors in W 1, W 2, etc W k equate that to 0 then each coefficient, each scalar must be 0 that is what this means, okay subspaces are called independent if this condition is satisfied.

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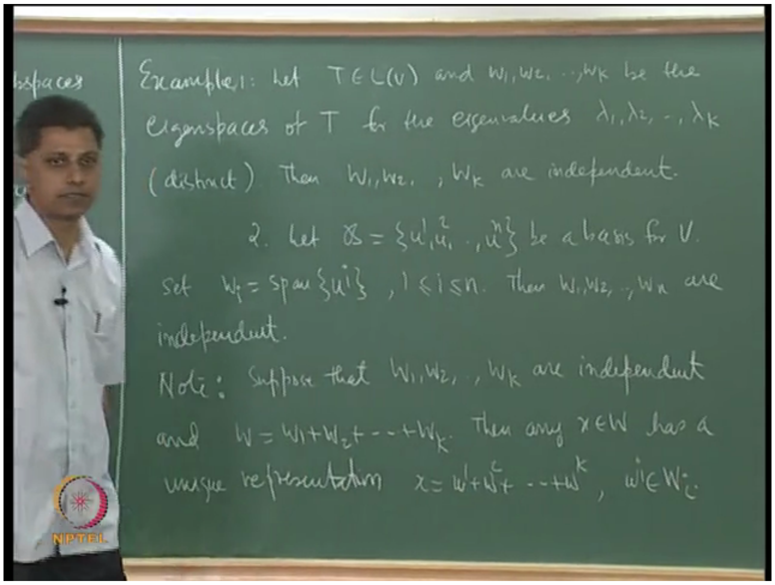

Example, I will just discuss one example for the moment because we have encountered this before. Let T be an operator on V and W 1, W 2, etc W k be the Eigenspaces of T corresponding to the Eigenvalues these are distinct, I have k distinct eigenvalues of the operator T, look at the eigenspaces then we have seen before that this eigenspaces are independent then W 1, W 2, etc W k are independent subspaces, okay we have seen this result.

In fact we have even seen something more if B 1 is a basis for W 1, B 2 basis for W 2, etc B k is a basis for W k then B 1 union B 2 etc union B k is a basis for the sum for the sum W, okay okay this is one example of independent subspaces maybe one more example. Take B to be u 1 u 2 etc u n basis for V, look at the one dimensional subspaces spanned by each of these vectors then this is obviously a set of independent subspaces, okay so two examples of independent subspaces, let us prove a quick characterization look at maybe two more examples and then look at the notion of projections so I want to characterize when the subspaces are independent.

So we have the following result okay before I state that result let us also make the following observation. Suppose that subspaces W 1, W 2, etc W k are independent, I have independent subspaces I look at the sum of these subspaces I will call that W sum of these subspaces W was the sum then we have the following easy consequence any x in W has a unique representation any x in W has a unique representation, x equals W 1 plus W 2 plus etc plus W k, every x and W has a unique representation in this form.

So in some sense this is easy to see given this in some sense you can think of W 1, etc W k as coordinates of a vector in the sum, okay you can associate unique numbers to the vector x belonging to W, you can think W 1 the coefficient of W 1 as a first coordinate, coefficient of W 2 as a second coordinate etc okay why is this unique?

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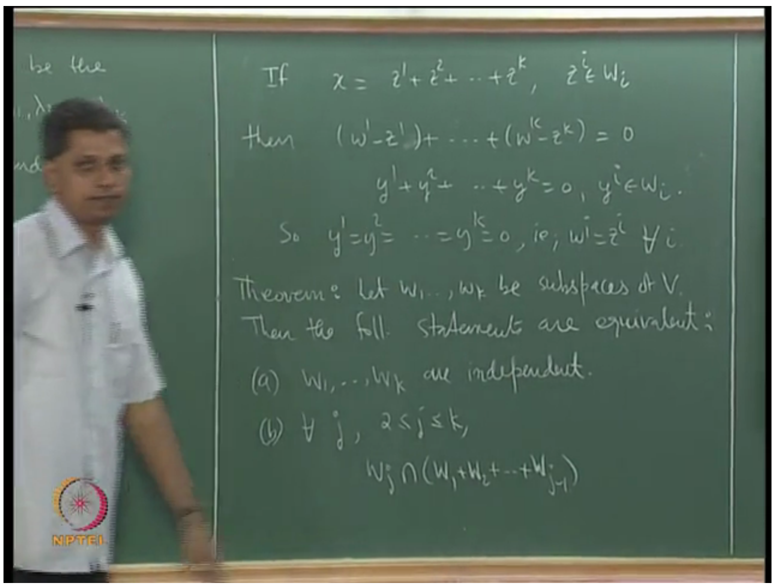

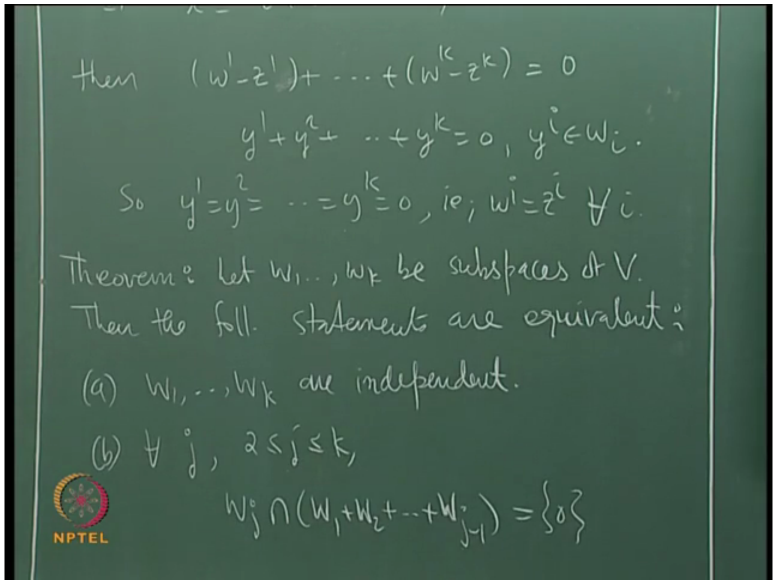

If x can be written as Z 1 plus Z 2 etc plus Z k with Z i in W i, if there is another representation we will show that these two are the same. If that happens then equate the right hand side you will get W 1 minus Z 1 plus etc W k minus Z k equals 0, W is a subspace so this belongs to so I will call it y 1 plus y 2 plus etc plus y k equal to 0 with y i in W i, sum difference etc they belong to the subspace, now this equation tells me that each of the terms must be 0 because the subspaces are independent and so W i equals Z i for all this so this representation must be unique if the subspaces are independent representation in the sum not in the entire space V, the representation in the sum W representation in this sum that must be unique, okay.

Let us look at characterization of independent subspaces, I will write down two statements that are equivalent to the subspaces being independent. So first statement is W 1, etc W k are independent, second statement remember that we are working only finite dimensional vector spaces. For every j 2 less than or equal to j, less than or equal to k the following condition holds you look at W j intersection W 1 plus W 2 etc plus W j minus 1.

Look at the sum of for every j look at the sum of W 1 etc W j minus 1, you take the intersection with W j that must be single term 0 this is the second condition for every j, so for instance (W 1 intersection W j is single term) W 1 intersection W 2 is single term 0, W 1 plus W 2 intersection W 3 is single term 0, etc W 1 plus W 2 etc plus W k minus 1 intersection W k is single term 0, okay all these equations must hold that’s condition b.

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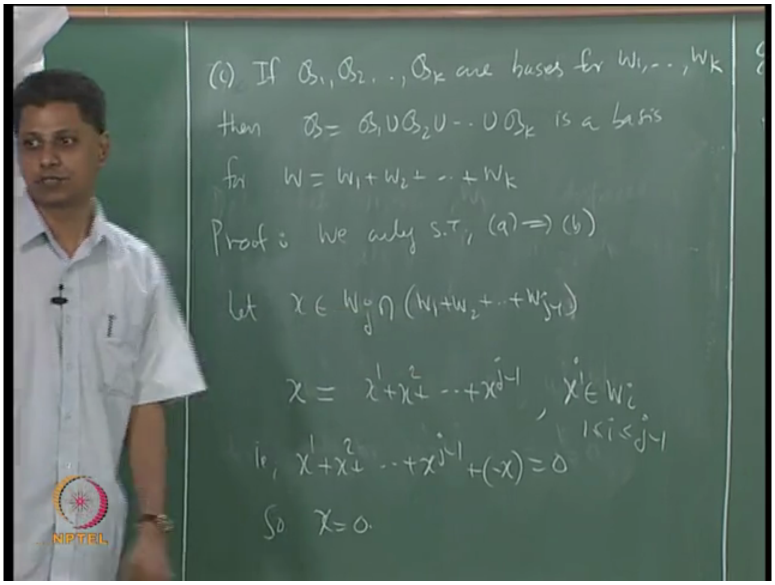

Condition c if B 1, B 2 etc B k are basis ordered basis for W 1, W 2 etc W k then B equals B 1 union B 2 etc union B k, this is a basis for this sum, these statements are equivalent. See for this theorem I will prove only a implies b and leave the rest for you the rest of the statements are almost tautologies, okay.

So I am not going to prove those I will just prove a implies b, we only show that a implies b in fact a if and only if b also follows but I will skip that part, okay. If the subspaces are independent we must show that this intersection is single term 0 for all j, okay. So take x in the intersection W j intersection W 1 plus W 2 etc plus W j minus 1, I am assure that this is 0 this is a 0 vector I am assure that x is equal to 0.

This means what I can write x as for one thing it is in this subspace so I can write x as let us say x 1 plus x 2 etc x j minus 1 because it belongs to this subspace x i belongs to W i, 1 less than or equal to I less than or equal to j minus 1, x belongs to this so x can be written in this manner yes, okay are we through? Look at bring this to the left hand side this is x 1 plus x 2 plus etc x j minus 1 plus minus x equals 0 with the first vector in W 1 second vector in W 2 etc j minus 1th vector is in W j minus 1, this is now in W j I will make use of that this x minus x that belongs to W j, so I have a combination here equated to 0, I am assuming condition a, assuming the surfaces are independent so it follows at each must be 0 what I only want is that x is 0, each must be 0 so this must be 0 so x is 0 that is what I wanted to show if x is in this intersection I have shown that x is

0. So if the subspaces are independent then these intersections are trivial, as I mentioned the other equivalences are exercises okay for you.

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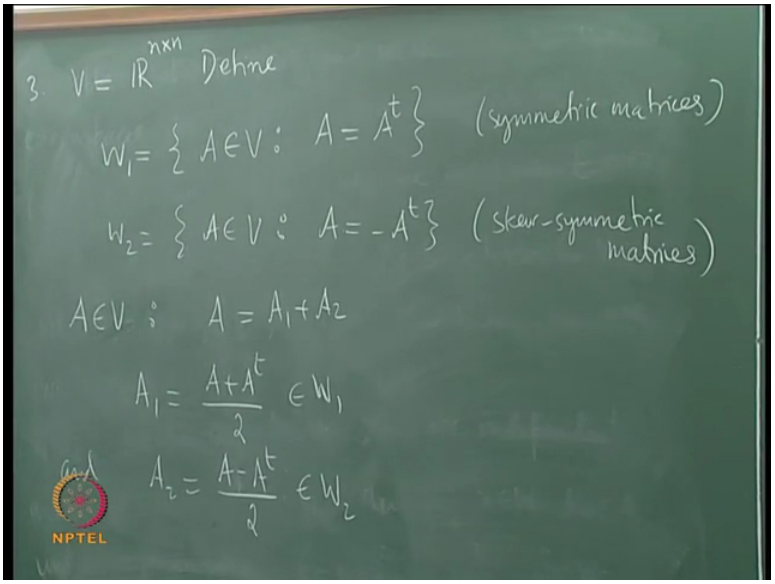

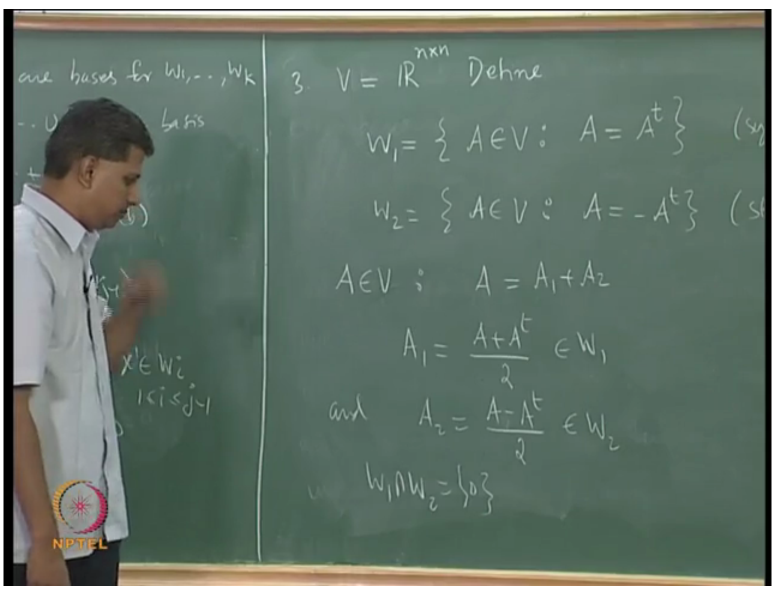

Let us now look at maybe a couple of more examples of independent subspaces. Let us consider so I am looking at example 3. Let us say I have the vector space to be R n cross n the set of all square matrices with real entries, let me take W to be the subspace of all A such that A equals A transpose the subspace of symmetric matrices, the subspace of all real symmetric matrices I will call it W 1 and define W 2 as the set of all A in V such that A equals minus A transpose, skew symmetric matrices, W 2 is the subspace of all skew symmetric matrices, I have already mentioned that these are subspaces so I wanted to verify that these are indeed subspaces, some of two symmetric matrices symmetric constant multiple of a symmetric matrix is symmetric.

Similarly for this, now what can be shown is that, okay that is probably easy to see. Any A in V can be written as a sum of two matrices A 1 and A 2, A 1 belongs to W 1, A 2 belongs to W 2. A can be written as A 1 plus A 2 where A 1 is a symmetric part A 2 is a skew symmetric part so A 1 is A plus A transpose by 2 this belongs to W 1 and A 2 is A minus A transpose by 2, this belongs to W 2 and that this representation is unique okay.

But before that can you see that this is obviously symmetric you take the transpose you get the same thing, this is skew symmetric take the transpose this goes with A with a minus sign and so this must be skew symmetric, okay and that the intersection of these two subspaces is single term 0, the only matrix that is both symmetric and skew symmetric is a 0 matrix, right the only matrix that is both symmetric and skew symmetric is the 0 matrix. So W intersection W 2 is single term 0, okay so these are independent subspaces and any any vector in V has a unique representation.

So this is really, this condition for j equals 2 second condition that the intersection is single term 0 (okay finally) okay I think that is enough for examples let me just tell you what is a direct sum okay.

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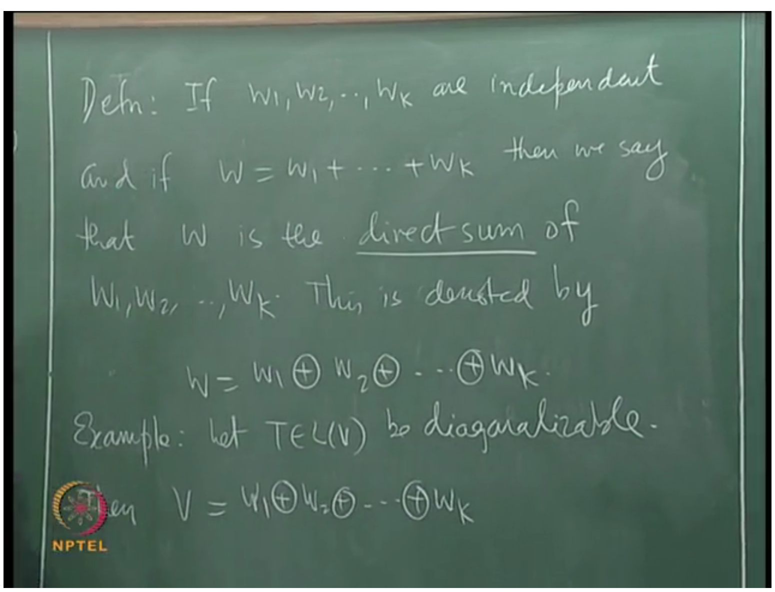

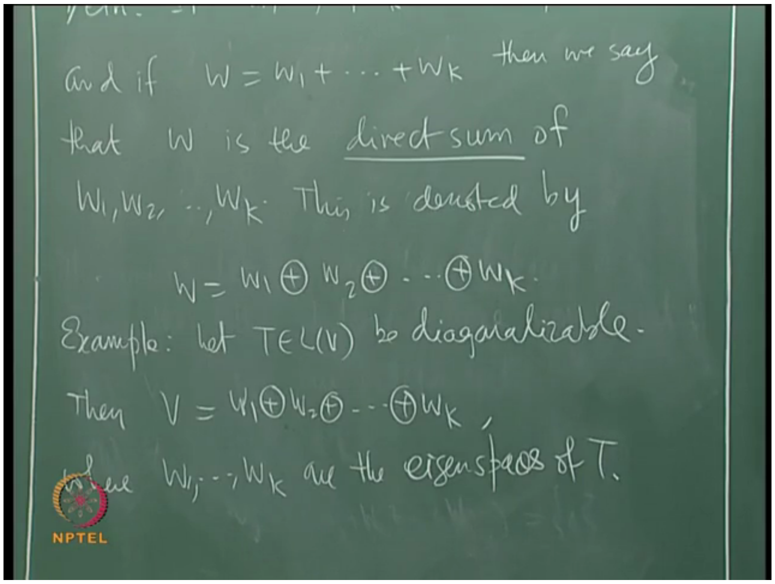

Let me give this definition if W 1, W 2, etc W k are independent subspaces, if these are independent subspaces if I denote W as the sum then we say that W is the direct sum of these subspaces we say that W is a direct sum of these subspaces sometimes it is called interior direct sum in group theory you must have studied such a notion interior direct sum, exterior direct sum, how these are isomorphic, etc okay.

So remember that (when I write) when I say that W is a direct sum of these subspaces what it means is that the subspaces must be independent. Now this is just the usual sum to differentiate the direct sum from this sum we will use this notation this is denoted by W equals W 1 plus with a circle you must have seen this in group theory, etc W k plus inside a circle. So when I write W in this manner it means that W 1, W 2, etc W k are independent subspaces this is the direct sum, okay.

Example of a direct sum several we have seen three examples before all the three the sum is a direct sum. I will give one final example for direct sum. Suppose T is diagonalizable, suppose T is a diagonalizable operator then V is the direct sum of the Eigenspaces, if T is diagonalizable then T can be written as a direct sum of these subspaces the subspaces being the Eigenspaces corresponding to the (Eigenvect) Eigenvalues of T, okay the proof follows from the fact that the Eigenspaces are independent and there is a basis of V each of whose vectors is an Eigenvector, okay.

So in this case you can write V as a direct sum of these subspaces, okay (so this will be) so you can already see that there is a connection from diagonalizability to direct sum of subspaces okay we will explore this relation a little further.

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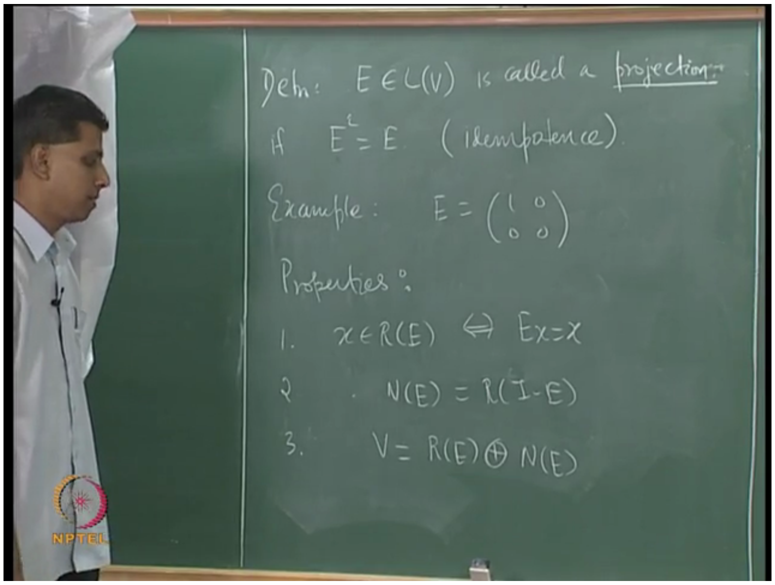

But before that let us look at the notion of a projection, what is a projection? An operator E on a vector space V is called a projection is called a projection (ion) is called a projection if E square equals E this is called idempotence, E is an idempotent operator the other name is projection because there is a geometric connotation, any idempotent operator will be called a projection, okay the geometry cannot be invoked immediately we will have to wait for orthogonal projections but let us look at some simple properties of projections.

Immediate example of a projection, we have seen these examples natural projections etc when we studied linear transformation so I will just give one example this is obviously a projection, okay E square is E, okay. I want to collect some properties of projections which will be useful later and then connect these two notions direct sum decompositions and projection operators. So some quick properties of projections the first property is that projection acts like the identity map on its subspace and 0 (on its null, on its range) projection acts like identity on its range space, 0 on its null space.

So this is like identity this behaves very close so E behaves somewhat like the identity operator in the following sense, first one if x belongs to range of E then Ex equals x and the converse is obviously true, a general operator will not be like this, if x belongs to range of T then x equals Ty that is all I know but if I know that T is also idempotent then Tx will be equal to x this is one of the properties.

Property 2 is that if x belongs to null space of E, okay I will write like this null space of E is range of i minus E this is the second property of projections null space of E is range of i minus E, so these are like complimentary kind of operators. Property 3 V can be written as range of E direct sum null space of E property 3 range and null space of a projection they form a direct sum of the vector space V in this case we say that they are complimentary subspaces if I have two subspaces W 1 and W 2 such that W 1 direct sum W 2 is E then W 1 and W 2 are called complimentary subspaces so if W 1 and W 2 are independent subspaces then they are complimentary subspaces again see that this will not hold for a general operator T range and null space need not be complimentary for a general operator but for a projection they are complimentary, okay.

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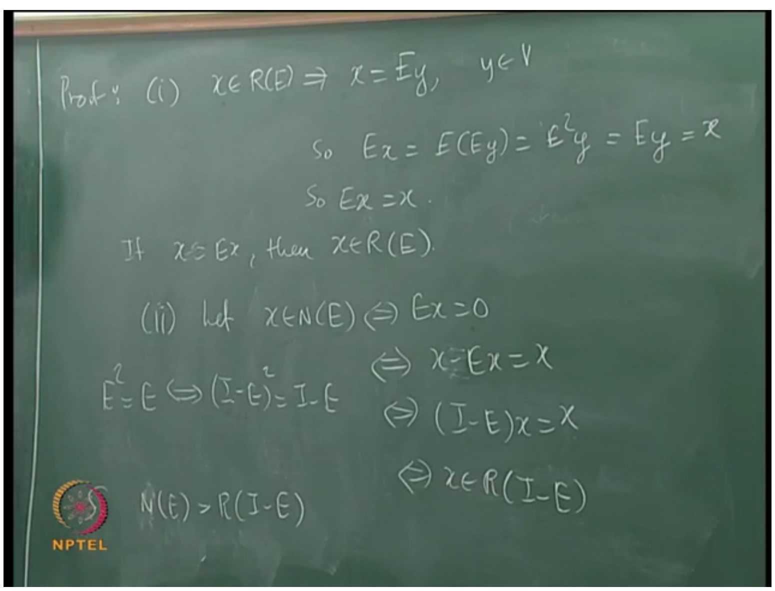

Quick proof of these for the first one x belongs to range of E implies that x can be written as let say Ey operate E on both sides so Ex is E of Ey that is E square y but E square is E so this is Ey, Ey is x again so Ex equals x if x belongs to range of E then Ex equals x, is that clear? Ex is E square y but E square is E because E is an idempotent operator, so this is Ey go back to this equation you get x, so if x belongs to range of E we have shown that Ex equals x converse is obvious if x is equal to Ex then x obviously belongs to the range of E okay.

If x equals Ex then x belongs to range of E that is trivial so you have both ways implication that is property 1, property 2 null space of E is range of I minus E, let x belongs to null space of E this implies Ex equal to 0 this means this means x minus Ex equals x, Ex is 0 so x minus Ex is x that is I minus E operating on x equals x.

So I have shown x belongs to range of I minus E, I have written x as I minus E of some y in particular x, so if it is a null space of E then it must be the range of I minus E and this whole process can be reversed that cannot be done here so let us do it again, this can be done if you observe that if E square equal to E then I minus E whole square is I minus E then this process can be repeated reversed.

Can you see that E square equal to E in fact if and only if I minus E the whole square is I minus E look at I minus E whole square it is I minus E minus E plus E square, minus E and plus E square get cancelled so we will get I minus E so is E a projection if and only if I minus E is a projection, use the first property suppose x belongs to range of I minus E, if x belongs to range of a projection then the projection acts like identity on that x.

So I minus Ex is equal to x that is what we proved in the first part, I minus Ex is equal to x gives this, this gives this, this gives this so this process can be reversed and so these subspaces coincide these subspaces coincide.

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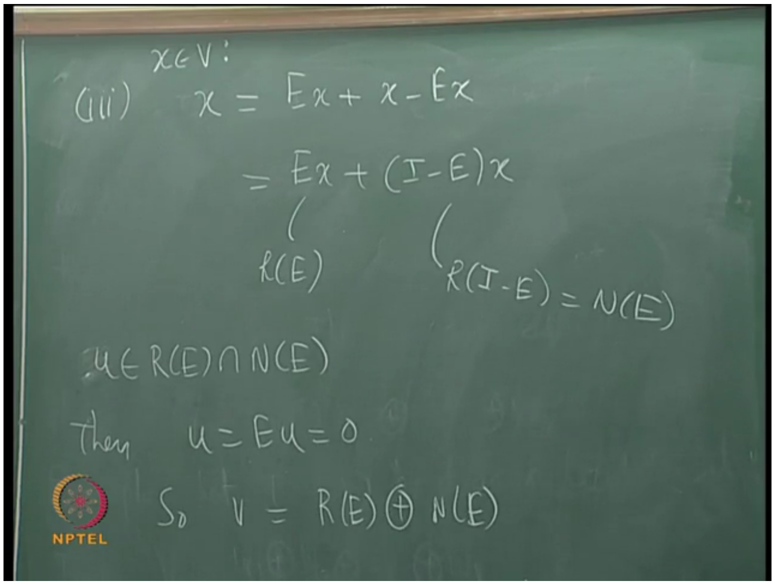

The last part that its it gives rise to a direct sum decomposition, maybe we must make the following observation that we have not used in any of these results the dimension of the vector space so this holds even for an infinite dimensional vector space, okay. You have a projection over infinite dimensional vector space all these properties hold.

Okay last part V is a direct sum just make this observation any x in V can be written as Ex plus x minus Ex, Ex gets cancelled this on the other hand can be written as Ex plus I minus Ex this belongs to range of E, this belongs to range of I minus E which I proved just now is null space of E and so any x has its representation okay obviously but I must prove that the intersection is single term 0 only then it will follow that this is a direct sum decomposition, okay.

So let us do that quickly but that is almost obvious from what we have proved till now, let us let us take u in range of E intersection null space of E, I want to show that u is 0, see I want to show that V equals range of E plus null space of E where the plus is a direct sum so I must show that the intersection of these two subspaces is trivial.

Let us take u and range of E intersection null space of E if u is in the range of a projection then Eu is u, anything in the range the operator acts like identity on that vector but u belongs to null space of E so Eu is 0 u also belongs to null space of E so Eu is 0 so u belongs to this intersection u must be 0 and so this is a direct sum. So V can be written as range of E plus null space of E where this plus is now the direct sum okay, so it has some nice property that there is if you have a projection then there is a direct sum decomposition, if you have a single projection. If you have several projections can we relate them to several subspace okay that is the question we will answer the answer is yes but before I prove that result there is another property which we must observe for a projection, every projection is diagonalizable.

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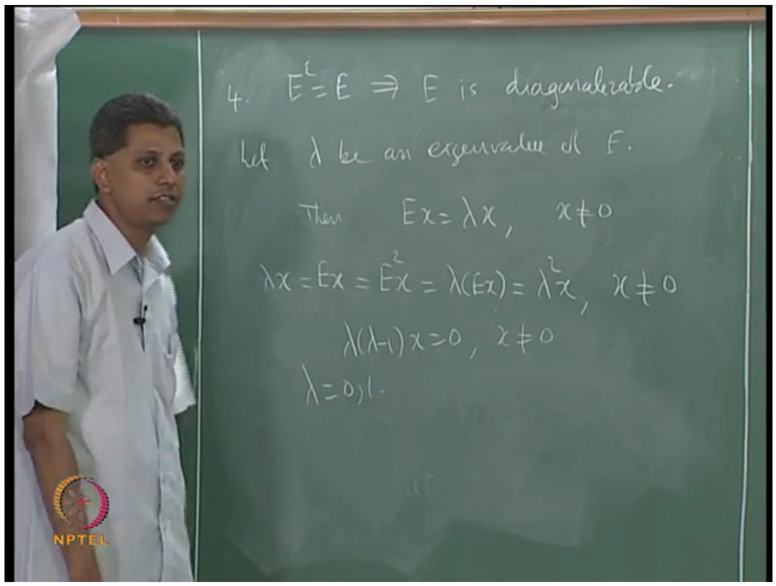

It is almost easy so I will state it as property 4 it is really property 4 if E is a projection then E is diagonalizable, to observe that E is it is almost trivial but to observe that E is diagonalizable you must tell me what are the possible Eigenvalues of E, 0 and 1 okay, let lambda be an Eigenvalue of E then Ex equals lambda x for sum x not equal to 0, look at E square x, E square x on the one hand is Ex which is lambda x on the other hand from this equation it is lambda Ex but Ex is lambda x again, lambda square x.

So if lambda is an Eigenvalue then lambda must satisfy this equation for sum vector x not equal to 0 lambda x equals lambda square x so lambda into lambda minus 1x equal to 0 with x not equal to 0 implies that lambda is 0 or 1, so these are the only possible Eigenvalues these are the only possible Eigenvalues of a projection if the Eigenvalue is only 1 then E is identity if the Eigenvalue is only 0 then E is a 0 operator all other projections in some sense lie between these two, okay.

Identity is trivially a projection, 0 is trivially a projection, I square is i, 0 square is 0, all other projections in some sense lie between these two in the sense that the Eigenvalues are either 0 or 1, okay I have got the Eigenvalues what about I the claim is that E is diagonalizable so can you give me a basis with a property that each vector in that basis is an Eigenvector yes, 0 and 1 why? That is okay a matrix having two distinct Eigenvalues does not mean its diagonalizable if it has two distinct Eigenvalues I know there are two independent Eigenvectors does it exhaust all the does it exhaust the number of basis vectors in any basis?

See (what) remember that E is an operator on V so if you look at the matrix of E it is n cross n, n cross n matrix has only two Eigenvalues obviously some of them will repeat 0 will repeat let say r times and 1 repeats n minus r times, when you have Eigenvalues repeating then there is no guarantee that it is diagonalizable but if it is a projection operator then there is a guarantee, how does that follow?

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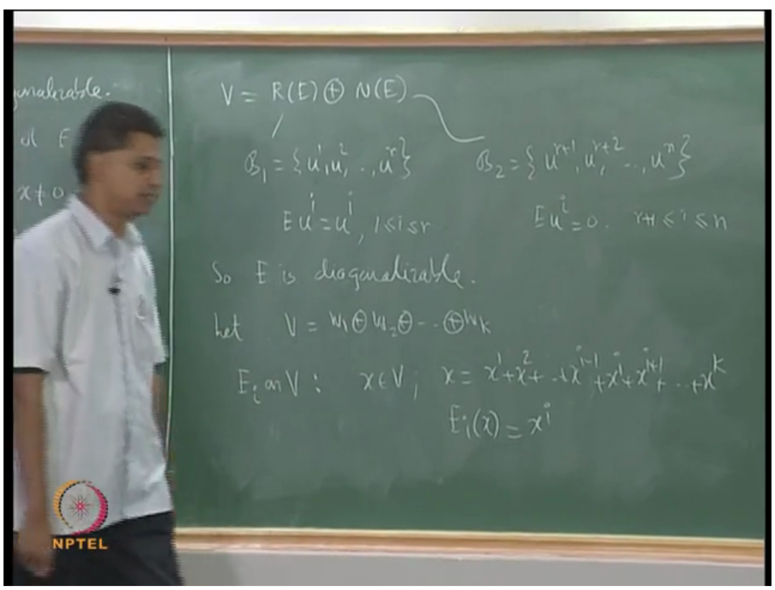

See it follows from the last property V is range of E direct sum null space of E, take a basis for range of E, I will call it B 1, u 1, u 2, etc u r and the basis for null space of E, I am saying that the union of these two, the union of these two is a basis of V is obvious because it is a direct sum decomposition but more importantly I must show that this each vector here, each vector here they are Eigenvectors that is a most important thing right, the fact that their union is a basis is a consequences of fact that W 1 and W 2 are independent subspace, these two are independent subspace we have seen this before but does it follow that this is each of the vectors here is an Eigenvector, what is the Eigenvalue?

Anything in the range of E will satisfy the conditions so u 1 is in range of E, so E u 1 is u 1 that is E u 1 is one times and u 1 so anything in the range of E is an Eigenvector corresponding to the Eigenvalue 1 anything in the null space of E is an Eigenvector corresponding to the Eigenvalue 0.

So this is a basis in fact any basis this is more than what one could ask for, you take any basis for the range space any basis for the null space take that union that will be a basis with the property that each vector in that is an Eigenvector, okay not just one basis each of whose vector is an Eigenvector, any basis for range of E union any basis for null space of E that union will form a basis each of whose vectors is an Eigenvector, is that clear?

So here E u 1 is u 1 so this holds for 1 less than or equal to i less than or equal to i less than or equal to r and E u i is 0 r plus 1 less than or equal to i less than or equal to n so u i u r 1 plus 1 etc u n these are Eigenvectors corresponding to the Eigenvalue 0, these are Eigenvectors corresponding to the Eigenvalue 1 their union is a basis and so E is diagonalizable, okay so any projection operator is diagonalizable. How are projections related to direct sum decomposition, okay I will maybe just state that theorem and prove it next time, one of the ideas that I want to use in that proof maybe I will discuss that today.

Let us say I have V let us say I have V as a direct sum decomposition in this manner W 1, W 2, etc W k are independent subspaces. Let me define the operator let us say E i on V the operator E i for i running from 1 to k for each subspace I will define for each subspace I will associate a projection as follows E i on V is defined as follows you take any x in V then this x has a unique representation let us call it x 1 plus x 2 plus etc x i minus comes somewhere x i after that x i plus 1 plus x k any vector x in the vector space V can be written as a unique sum in this manner, I will define E i of this x as ith term in this sum see when I write when I write this sum what is understood what is implicit is that I write it in this order first the term corresponding to W 1, second the term corresponding to W 2, etc implicitly the basis are ordered, vector addition is committed so this is same as x 2 plus x 1 plus etc okay but I am not looking at that representation, I am looking at that representation where the first term comes from W 1, second term comes from W 2, etc (given that decomposition) given that representation I define E i of x to be the ith term, ith term is x i E i of x is the ith term that is x i, I am claiming that this E i is a projection, I am also claiming that E i into E j is 0 when i is not equal to j.

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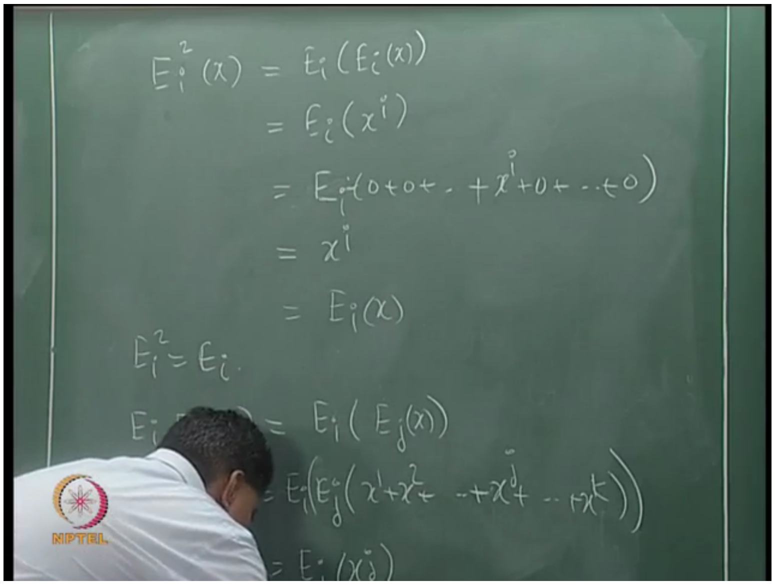

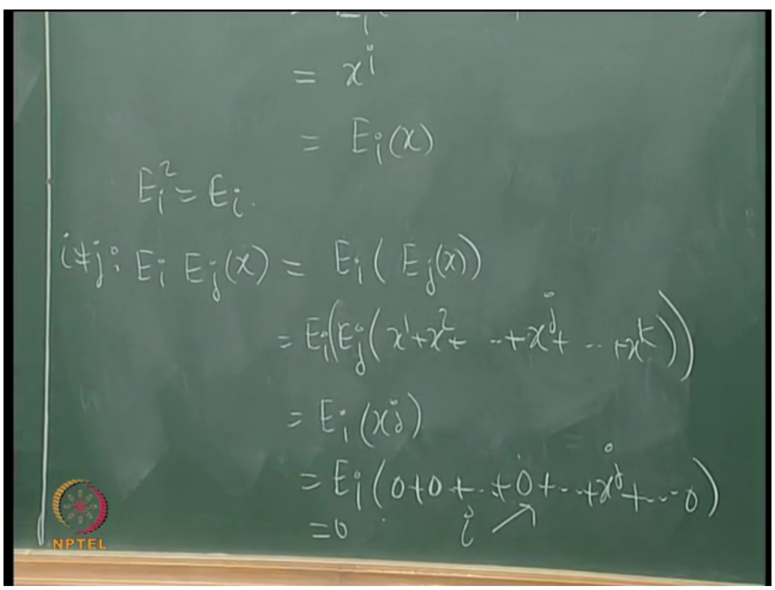

So maybe I will just prove E i is a projection that is easy to see what is E i square of x? E i square of x is E i operating on E i of x that is E i operating on E i of x, E i of x I have the definition it is ith term in this right hand side that is x i okay now I want E i of x i then I must write x i as a sum in such a way that I must write x i as a sum pick up the ith term okay but x i is

E i of 0 plus 0 etc plus xi plus 0 plus etc plus 0 where each of the zeros comes from W 1, W 2, W 3 etc not W i for W i I have x i all the others are 0.

So I now look at the ith term that is x i, E i of any vector is ith term where the terms are written in this order the first term is in W 1, second term is in W 2 etc, so I have shown E i square of x is xi. But what is x i, x i by definition is E i of x, x i is E i of x so what I have shown is that E i square is E i so this is an idempotent operator E i is an idempotent operator, it is a projection.

Second property E i into E j equals 0 that is also easy look at E i E j of x E i E j of x is E i of E j of x for for E j of x I must look at the representation of x, x 1 plus x 2 etc plus x j etc x k this E i E j of E j of any vector x is a jth term, E i of x j is a jth term so it is x j but E i of x j I must look at the ith term that is obviously 0.

So just to emphasize it is E i of 0 plus 0 etc ith term is 0, this is the ith term j comes somewhere here, I am assuming I less than j, j could be less than i there is no problem I comes somewhere here, j comes somewhere, all other terms are 0 I want ith term the ith term is 0 so this is 0 assuming i is not equal to j, i equal to j we have proved already E i square is E i if i is not equal to j then E i E j is the 0 operator okay we will prove another property the range of E i is W i and the fact that sum of these E i's is the identity operator, identity operator is the sum of E 1, E 2, etc E k okay and then also prove a converse that I will do in the next lecture.