Linear Algebra
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Module 8 Invariant Subspaces And Triangulability
Lecture 30
Invariant Subspaces

Okay, we are discussing the notion of minimal polynomial right, and see the objective as I stated before is to characterize diagonalizability, we already have one characterization, you compute the eigenvalues, compute the dimensions of the eigenspaces if the dimensions add up to the dimension of the vector space, then it is diagonalizable okay, dimensions of the eigenspaces add up to the dimension of the vector space then it is diagonalizable.

We will look at another characterization in terms of the minimal polynomial okay, this is another corner stone one of the corner stones in algebra that we have seen is a rank nullity dimensional theorem okay, now to prove this theorem you need certain notions, today we will discuss two of them one is the invariant subspace, invariant subspace of a linear transformation and then the notion of a T conductor okay.

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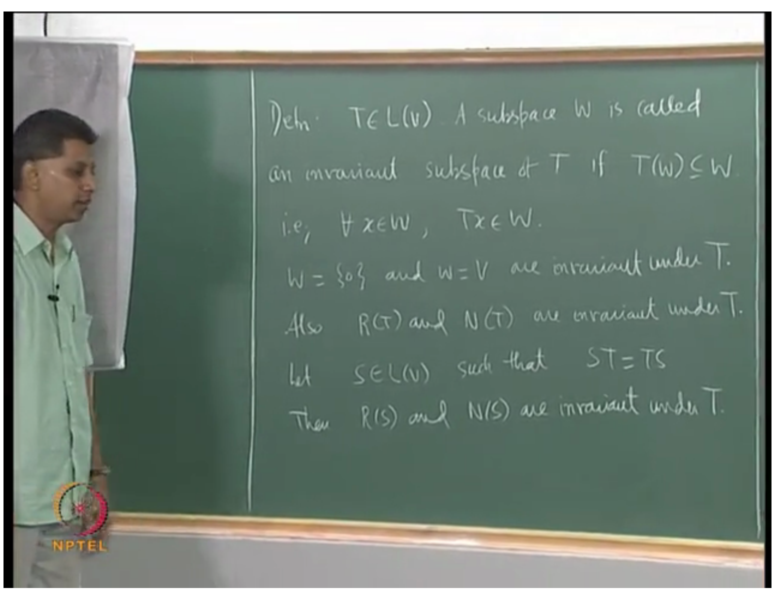

So let me make these notions precise, so I will first discuss the notion of an invariant subspace, T is a linear operator on V, a subspace W is called an invariant subspace of the transformation T of the operator T if this condition holds T W is contained in W that's an invariant subspace, okay what is the meaning that is for all x element of W it must follow the Tx belongs to W, okay let us dispose of the trivial examples of subspaces.

W equal to single term zero and W equals the whole space they are obviously invariant, so we will be interested if there are proper subspaces that are invariant under T, okay and you will see how this plays a role in characterizing diagonalizability. Let us look at some examples maybe before examples before numerical examples let me also tell you that the range space and the null space are invariant under T.

These are easy to verify so I will leave the proof for you, the range space and the null space are invariant under T, okay we also have another set of subspaces that are invariant under T, let S be a linear operator on V such that ST equals TS, S is an operator that commutes with T, suppose S is an operator that commutes with T then the range and the null space of S are also invariant under T, okay remember for us T is fixed for us T is fixed, S is an operator that commutes with T then range and null space are invariant under T, all these are little exercises for you okay to verify in particular what is relevant to the present context is the eigenspace corresponding to any eigenvalue these are invariant subspaces for any operator T because of this result.

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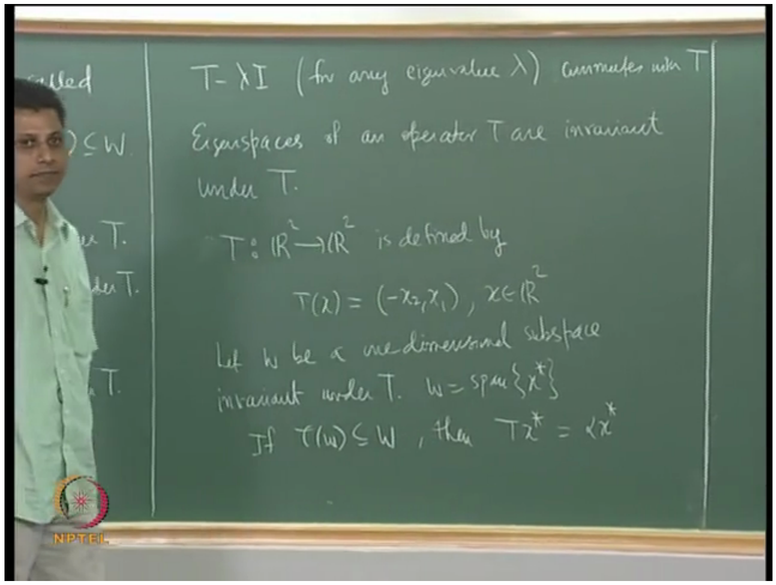

Look at T minus lambda I for any eigenvalue lambda this commutes with T, T minus lambda I commutes with T. So in place of S I have T minus lambda I where lambda is an eigenvalue, so by the previous theorem previous result which we did not prove null space of T minus lambda I is invariant under T but null space of T minus lambda I is precisely the eigenspace of the operator T corresponding to the eigenvalue lambda so eigenspaces are invariant under T, eigenspaces of an operator T are invariant under T, okay this is the most important consequence for us, okay.

Let us now look at one example where the operator does not have any invariant subspace if you want operators for which there are invariant subspaces just look at any operator which has an eigenvalue this theorem says the eigenspaces are invariant okay, so I want to give an example of an operator which does not have an invariant subspace, this goes back to the example that we have seen before, okay.

I want you to look at t from R 2 to R 2 whose matrix is okay I am really looking at the rotation operator, so the matrix of this okay I will straight away define T, T is defined by the rotation operator T x is minus x 2 x 1 this is the rotation by 90 degrees, rotating the vector x by 90 degrees this is the linear operator, I want to show that this operator does not have any non-trivial invariant subspace, okay 0 and R 2 are trivially invariant subspaces.

So if I take a subspace which is non-trivial which is invariant under T, the subspace must be of dimension 1. Let W be a one dimensional subspace invariant under T, I look for one dimensional subspace and see if it is invariant under T, so I am assuming there is one such, it is one dimensional so the basis consists of one element so let me take W to be span of a particular vector x star, x star is not 0, okay x star is not 0.

Suppose T W is contained in W, if T W is contained in W, T W is in particular for for x star I will have T x star also as an element in W, T x star should also belong to W, but W is span x star, so it is a multiple of x star, this must be alpha x star for sum scalar alpha W span by x star so anything is a multiple of x star but this means alpha is an Eigen x star is not zero this means alpha is an eigenvalue for T.

But we know that T does not have eigenvalues this is a contradiction, so this T does not have an invariant subspace.

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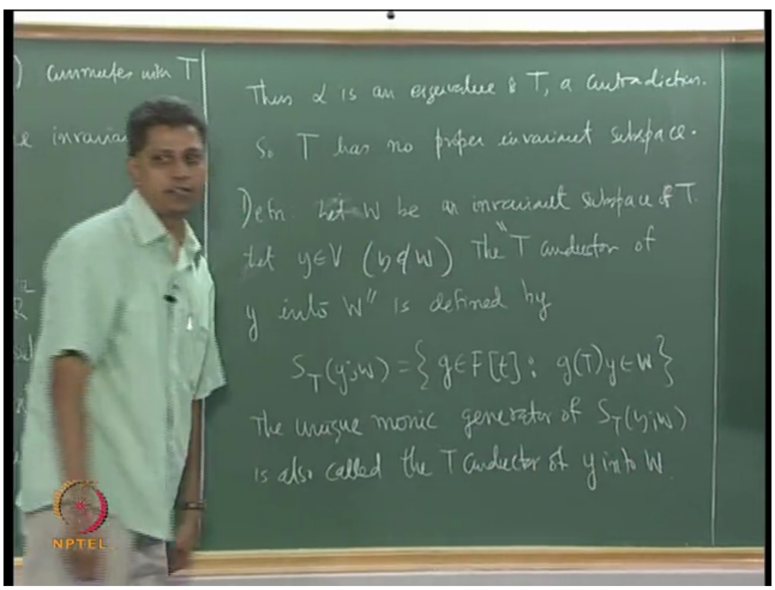

This is a contradiction to what we had seen earlier and so this T has no proper invariant subspace, okay so much so for invariant subspaces. Let us keep this aside and then look at the notion of the T conductor of a vector into a subspace, is this example clear? This operator does not have a proper invariant subspace.

We had looked at this example before and we have shown that this operator does not have an eigenvalue if lambda is an eigenvalue we had seen that 1 plus lambda square is 0 but this is an operator on R R 2 to R 2, so the eigenvalue must be real so there is no root for one plus lambda square equals zero so it does not have an eigenvalue, if it had an invariant subspace it would have had an eigenvalue okay.

Next is a notion which is rather similar to the minimal polynomial, okay but little more general than the minimal polynomial that is another notion we need for characterizing diagonalizability of operators, the notion is the following I am having a subspace let W be an invariant subspace of the operator T let W be an invariant subspace of an operator T see for us the frame work is T is an L of V, V is final dimensional we are trying to characterize diagonalizability, okay W is an invariant subspace of T.

Let y belong to V but not in W the T conductor the T conductor of y into W that is the name the T conductor of Y into W, I will define this to be the following subset of F t is defined by the notation is S T y, w the T conductor of y into W, W is an invariant subspace, y is an element that does not belong to W, T is the operator that we started with S T y, w this is the set of all polynomials g in F t set of all polynomials with coefficients coming from the underlined field, for us either real or complex over the single real variable T, we know it is a principal ideal domain. So collect all those polynomials that satisfy the property that g T y belongs to W, y does not belong to W but g T must belong to W g is a polynomial g of T is a linear operator on V, so g T y make sense I collect all those polynomials that satisfy this property.

For a fixed y fixed subspace W and T is of course fixed, collect all the polynomials that satisfy this property for a fixed y for a fixed invariant subspace W that will be called the T conductor of y into W, okay this is the (collect) set of all polynomials, I am going to leave it to you to prove that this is an ideal this is a subspace and has a property that if g belongs to this and f belongs to this to capital F then the product belongs to this.

This is an ideal so it has a generator that is each element, each polynomial in this is generated by a unique polynomial that polynomial will also be called the T conductor of y into W but before that is this non m T? First question, we have defined something similar the minimal polynomial is there any relationship between the minimal polynomial and the T conductor of y into W? What is the property of the minimal polynomial?

One important property it is an annihilating polynomial m T of any x is zero, so this y also m T of y is zero, what is a problem? The minimal polynomial is an annihilating polynomial so m of capital T of any vector must be zero W is a subspace so that must belong to W, so to begin with the minimal polynomial belongs to this, okay but there are more general elements that is a important point here there are more general polynomials but they cannot be annihilating polynomials remember that, because if they were annihilating polynomials of T different from m T there degrees will have to be greater they cannot be smaller, there are annihilating polynomials agreed but the degrees of those annihilating polynomials will be greater than the degree of the minimal polynomial, minimal polynomial definitely belongs to this.

So this is not m T okay this is non m T ideal of the principal domain f T. So this is generated by a single unique element that will also be called the T conductor of y into W okay.

So I will simply say that the unique we will be interested in the Monic generator, the unique Monic generator of S T y; w will also be called the T conductor of y into W is also called the T conductor of y into W, so depending on the context we will know whether we are talking about the subspace or the unique Monic polynomial, okay. See we have seen that the minimal polynomial belongs to this so can you immediately conclude that this T conductor divides the minimal polynomial? What is the meaning? How does it happen?

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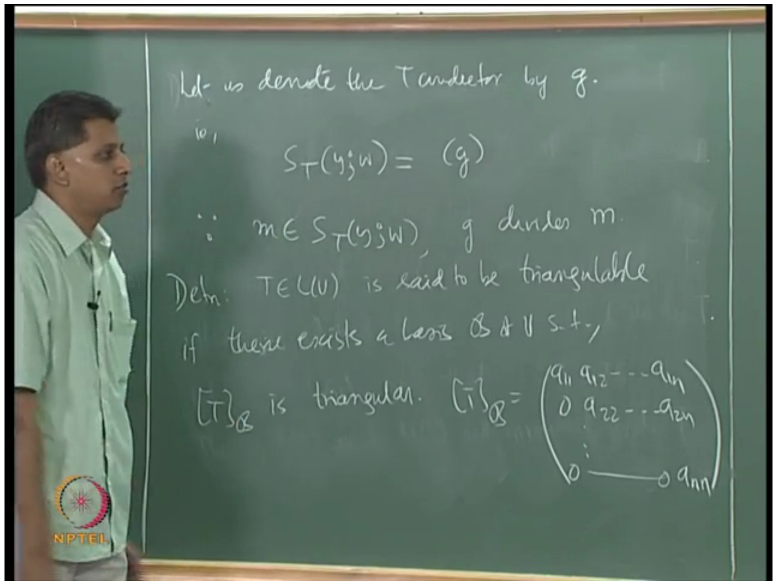

Let us denote g let us denote let us denote the T conductor by g that is we have S T of y; w this is generated by the single polynomial g that is what I mean g is the unique Monic polynomial which generates this ideal then what we know is that since m belongs to S T y; w it follows that g must divide m g must divide m g must divide m but this g cannot be here in an annihilator because if g divides them then the degree of g is less than or equal to the degree of m in general this cannot be the this cannot be an annihilating polynomial of the operator T, I will I will give you an example of this g okay but before that let us prove the following result so remember g divides m so this is more general than the minimal polynomial the T conductor of a vector into a subspace is more general than the minimal polynomial of the operator T, okay.

Then we have the following result, see our our problem is to characterize diagonalizability but what we would right now do is to settle with something less, is to settle with the triangulizability, so let me give the definition an operator T is said to be is said to be triangulable if there exists a basis script B of V such that the matrix such that the matrix of T relative to B diagonalizable it is a diagonal matrix, triangulable if it is a triangular matrix.

Such that this is triangular, triangular means either the lower of diagonal entries are 0 or the upper of diagonal entries are 0 we will stick to the lower being 0 so we will say that the T B it does not make a difference, we will say that the lower triangular part is 0 so it is upper triangular an operator T is set to be triangulable if the matrix of T relative to some basis of V is upper triangular.

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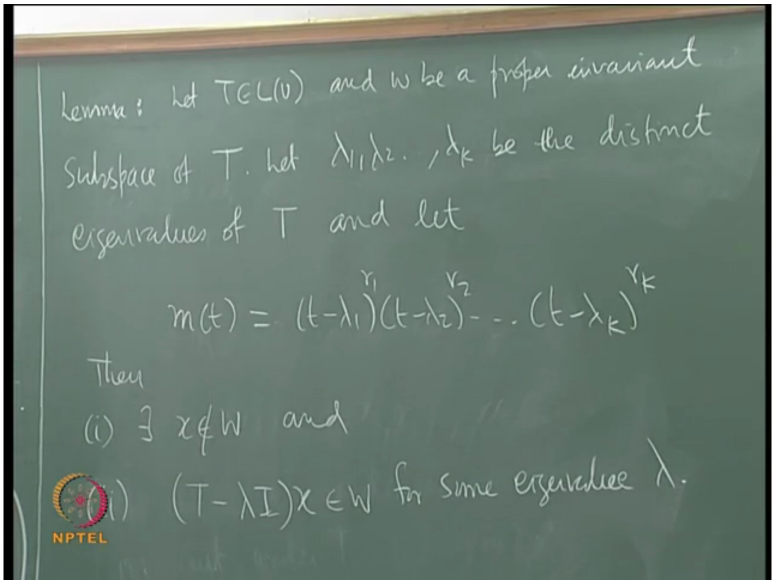

Let us first characterize upper triangular matrices we will need the following result it tells you something about the structure of g, I want to state this Lemma before before characterizing triangulability, this Lemma will be useful there. T is a linear operator on V and W be a proper invariant subspace of T, let lambda 1, lambda 2, etc lambda k be the distinct eigenvalues of T let suppose that the minimal polynomial m stands for the minimal polynomial, let suppose if a minimal polynomial can be written as a product of powers of linear polynomials that is t minus lambda 1 to the r 1, t minus lambda 2 to the r 2 etc, t minus lambda k to the r k the minimal polynomial is a product of powers of linear polynomials.

Then we have the following first there exists x which does not belong to W all I want to say is this, condition 2 is T minus let us say lambda I x, there exists in x that does not belong to the subspace W, T minus lambda I x belongs to W for some eigenvalue lambda so this what we will show, there exists an x which does not belong to the subspace W but which has the property that T minus lambda I of x belongs to W for some eigenvalue lambda, okay we will prove this Lemma and then use this to characterize triangulability and then characterize diagonalizability. See remember already the minimal polynomial appears here okay, so the minimal polynomial will play a crucial role in the diagonalizability with which what we will at least state today, okay.

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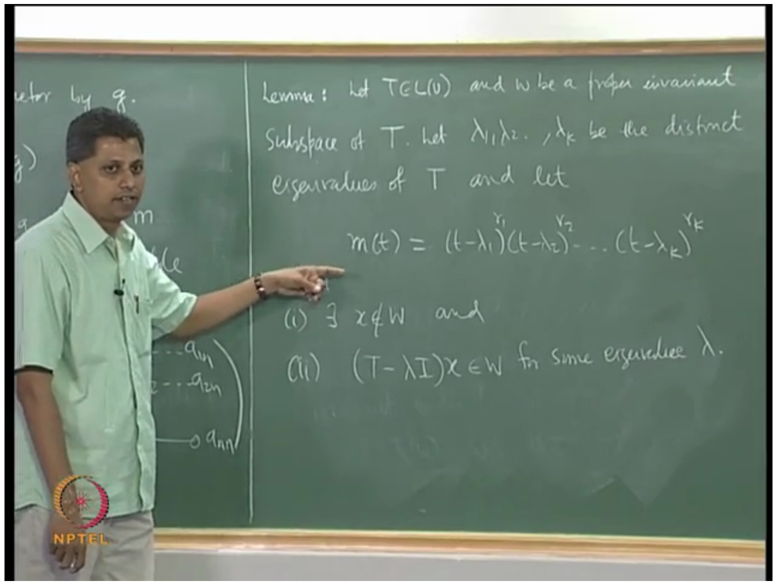

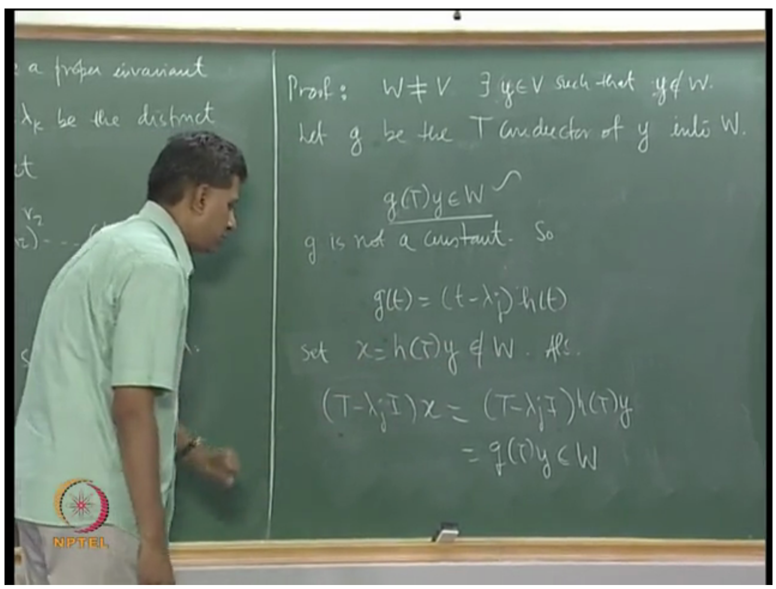

The proof is as follows see before that I told you I told you that we will have some information about the polynomial g, the information that is provided in this theorem is as follows under the very mild condition that the eigenvalues of the operator T exist, under the very mild assumption that the eigenvalues exist, if the eigenvalues exist then you can write the minimal polynomial in this form, okay. So except for operators like the rotation operator this condition is satisfied under this condition what this says is that this T conductor of the vector x into W is a linear polynomial, can you see that the T conductor, see we are interested in the T conductor of x into W from this can you see that x does not belong to W but T minus lambda I x belongs to W.

So all you have to do is to look at the polynomial little t minus lambda, this polynomial has the property that x does not belong to W means that g is not the constant, okay g is not the constant polynomial, from constant you go to linear polynomials this says that g must be a linear polynomial okay there exist an x which has the property that the T conductor of x into W is a linear polynomial that is second condition really okay that it is not constant is a first condition okay as I told you I will give a numerical example but let us first look at the proof of this result.

See W is not equal to V, W is proper subspace W is a proper invariant subspace so this is not the whole of V, so there exist x in V such that x does not belong to W, I will look at okay let me call it y there exist y such that y does not belong to W, I want to now look at let g be the T conductor of y into W, W is not the whole space V it is a proper invariant subspace. I pick one element y which does not belong to W and then construct in principal the T conductor of y into W, I am calling that T conductor as g, can you see that g cannot be constant g cannot be a constant why? Because if g were a constant then what is the property then g satisfies? g T y belongs to W okay this property is satisfied by g in fact among all those polynomials that satisfy this condition g is the one with the least degree coefficient of the highest degree is 1 etc coefficient of highest degree is 1, this is the unique Monic polynomial that satisfy this condition.

So if g were a constant then this would be a constant, constant times y belongs to W means y belongs to W but y is something that we started with does not belong to W. So g is not a constant g is not a constant polynomial, so it must be T minus lambda j into h T where the degree of h is strictly less than degree of g so h T y does not belong to W also if you look at T minus lambda j I x, it is T minus lambda j I h T y that is g T y which belongs to W coming from this okay g is not a constant so it must be at least a linear polynomial, so g is T minus lambda j into some h, h could be a constant.

But h definitely cannot be a constant because otherwise you will get a contradiction here, so the degree of h is less than the degree of g and so h T y cannot be in W, I am calling h T y as x. Now this x satisfies the second part, x does not belong to W alright but look at T minus lambda j x that is T minus lambda j I into h T y which is this operating on y that is g t y, g t y belongs to W so we are through okay. So this is the x that satisfies these two conditions so what follows is that g is a linear polynomial that is the consequence.

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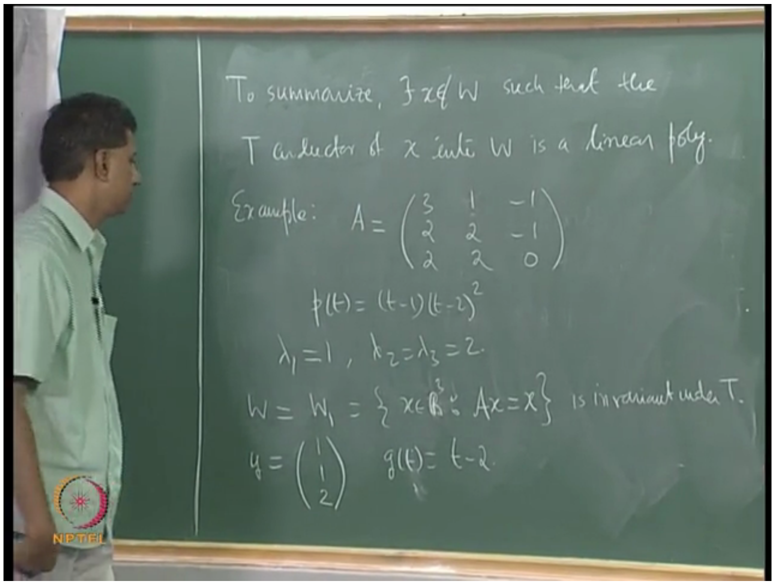

So to summarize there exist x which does not belong to W such that such that the T conductor of x into W is a linear polynomial, okay okay. I want to give a quick example you please verify the details ya yes x T x T cannot be a constant actually oh we are proving h T is a constant yes we are proving h T is a constant we are proving h T is a constant, h T is a constant is also consistent with this statement h (T) is a constant, this is a multiple of y, y does not belong to W so this does not belong to W, h (T) is a constant is what we are proving yes, okay okay.

The T conductor is a linear polynomial let us use this in characterizing triangulability okay, I told you I will give an example you please verify the details I just have the information, this example is of that of the second example of an operator which is not diagonalizable. I want to look at the operator over r3 whose matrix is this 3 1 minus 1 2 2 minus 1 2 2 0 the characteristic polynomial of this matrix is we are using p for that p of t is t minus 1 into t minus 2 the whole square that is what I remember t minus into t minus 2 whole square.

I will call lambda 1 as an eigenvalue 1, lambda 2 equals lambda 3 equals 2 this is an example of an operator which is not diagonalizable. Let me take let me take W as W 1 the eigenspace corresponding to the first eigenvalue set of all x in V such that r 3 such that Ax equals x, Ax equals lambda x lambda 1 x that is Ax equals x. So this is the eigenspace corresponding to the eigenvalue 1 the first eigenvalue. Remember that in this example we do not have eigenvectors for a second eigenvalue, the second eigenvalue has only one eigenvector span by 1 0 2, okay.

The second eigenvalue has only one independent eigenvector, the eigenspace corresponding to the second value is of dimension 1, I will take W 1 this is an invariant subspace any eigenspace is an invariant subspace of the operator T, so this W 1 is invariant, I want to give an example of the y that is constructed in the theorem. So let me give this y as 1 1 2 take this y, what is a T conductor of this y into W 1, verify it as t minus 2 for this y please verify it as t minus 2, okay.

Now remember that for in this example we have verified that the minimal polynomial is the same as the characteristic polynomial okay but the T conductor is a linear polynomial so this is something more general than the minimal polynomial, okay but it pertains to only a particular vector y, so please verify the details here this should sought of consolidate what we are doing there, okay.

I want to characterize triangulability, g T is just one factor no see minimal polynomial is t minus lambda 1 to the, t minus lambda 1 to the r 1 etc g is just one of those factors, what is the problem, m is a multiple of g, g divides m g divides m so m is a (multiple of t) multiple of g. So one factor t minus lambda j for some eigenvalue rest of them are here, no we are proving h is a constant, the constant is 1 that constant is 1 that constant is 1 h T is 1 pardon, we do not know presently we do not know in this proof we do not know what h T is, the only thing that I know at this stage see it is not a constant so it could be a linear polynomial, quadratic polynomial whatever okay.

But at this stage I can write this much there must be a linear factor, forget about h. But h now has a property that its degree is 1 less than g so h is h T y cannot be in W, that is the property we are exploiting, okay. Remember g divides m the degree is lesser or equal to the degree of m but g in general is not an annihilating polynomial, okay. So I need to ya use this Lemma to use this Lemma to characterize triangulability.

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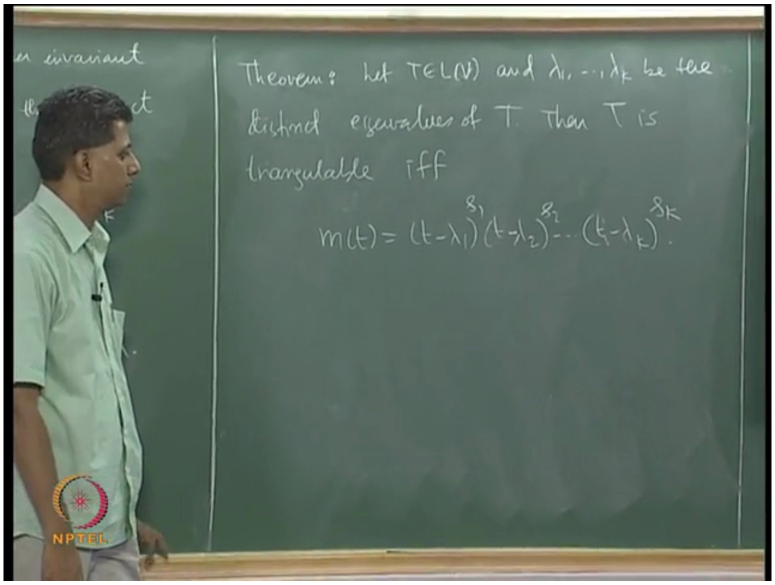

Let us see how far we can proceed, so I want to characterize triangulability this is the theorem. An operator T is triangulable if and only if the minimal polynomial is a product of linear factors an operator T is triangulable if and only if the minimal polynomial of the operator is a product of linear factors there are possible powers, okay but it is a product of linear factors, okay.

So I do not think I have enough time to prove it but I will at least make this observation that (if an operate) if the eigenvalues of an operator lie in the underlying field then the operator is triangulable, if an operator has all eigenvalues in the underlined field then it is triangulable, okay but not all operators are diagonalizable, okay we have already seen operators, examples of operator at least one which is not diagonalizable but all operators are triangulable provided you only meet this minimum condition, mild condition that the eigenvalues must belong to the underlined field, okay.