Linear Algebra
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Module 7 Eigenvalues and Eigenvectors
Lecture 29
The Cayley-Hamilton Theorem

Okay, see we are discussing the notion of minimal polynomials. We had looked at three numerical examples first so let me recollect this first example was that of a linear transformation which did not have eigenvalues, second example the matrix the linear transformation has eigenvalues but it does not have enough eigenvectors, third example it has eigenvalues in the field and it has enough eigenvectors that is it is diagonalizable, okay.

We are trying to calculate the minimal polynomial when we were trying to calculate the minimal polynomial we observed that for the first case the minimal polynomial coincided with the characteristic polynomial, second case again not diagonalizable the minimal polynomial was not the characteristic polynomial, third case I am sorry second example minimal polynomial was equal to the characteristic polynomial, third diagonalizable case minimal polynomial is just a product of distinct linear factors, okay and it appears as though the bug stops with the characteristic polynomial, okay it appears as though we could do with characteristic polynomials probably, okay.

The answer can we do is yes it is always we will prove that today this is called the Cayley-Hamilton Theorem, okay that is the characteristic polynomial of any linear operator over a finite dimensional vector space is an annihilating polynomial that is Cayley-Hamilton Theorem, okay once you prove we will prove the matrix case not the linear transformation case once you prove this what follows is that the minimal polynomial divides the characteristic polynomial follows that the minimal polynomial divides the characteristic polynomial.

Let us also remember this one important connection between characteristic polynomials and minimal polynomials the zeros of both these polynomials are the same the zeros of these polynomials are the same, okay okay. Before I prove this Cayley-Hamilton Theorem I also want to give a little implication which I do not mention when I was doing examples this could have been useful the implications are following.

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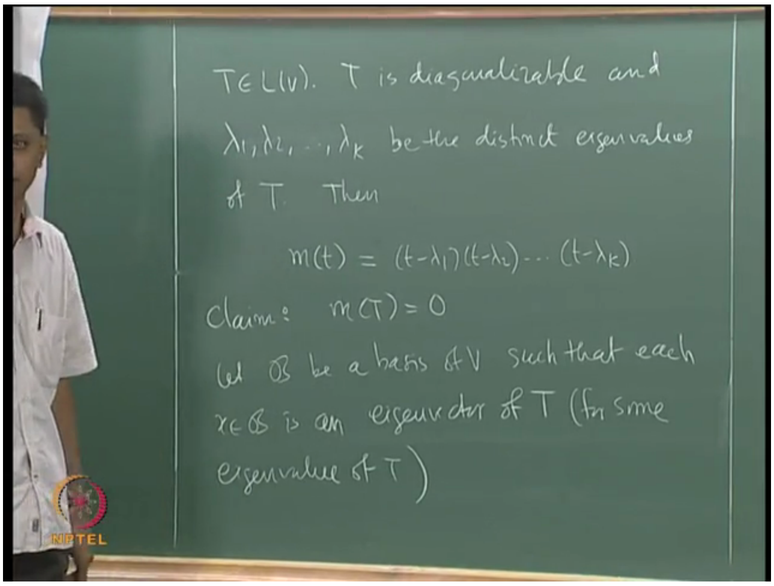

Suppose T is diagonalizable and let me say has before that lambda 1, lambda 2, etc lambda k be the distinct eigenvalues be the distinct eigenvalues of T. See remember I am not saying all the eigenvalues are distinct I am only saying that the distinct eigenvalues have been collected and I am calling them lambda 1, etc lambda k each one goes with a certain multiplicity lambda 1 goes with D 1, lambda 2 goes with D 2, etc lambda k goes with D k where D 1 plus D 2 plus D 3 etc upto D n is equal to the dimension of the space, okay okay.

If T is diagonalizable then what I want to prove it is a very simple proof the claim is that remember I am using small m for the minimal polynomial the claim is that the minimal polynomial is a product of distinct linear factors that is it is t minus lambda 1 into t minus lambda 2, etc t minus lambda k. If you go back to example 3 I simply ask you to verify that the minimal polynomial is a product of distinct linear factors the reason why I told you was this result in my mind.

So let us proof this first, the third example was that of a diagonalizable operator, okay. I want to show that okay for one thing the for one thing you cannot have the minimal polynomial of degree okay I want to prove the minimal polynomial is this just look at this polynomial the minimal polynomial cannot be of degree less than the degree of the right hand side polynomial do you agree with that?

That is because if it were less than the degree of the right hand side polynomial that is k then it would mean what is the meaning of that it would mean one of these factors have been missing, okay but if one of the factors is missing then I am in a situation where I have an eigenvalue it is not the zero of the minimal polynomial. So to begin with this is the possible candidate for the minimal polynomial, it cannot be less than this, the degree of the minimal polynomial cannot be less than the degree of this right hand side polynomial which is k then the only thing that I need to show is that this is annihilating polynomial I will show that this is annihilating polynomial then it follows that this is the minimal polynomial, this is of course monic the power of t the coefficient of t to the k is 1. So this is the monic polynomial I will only show that this is annihilating polynomial, okay.

What is if that we must show we must show that m of T is a 0 operator annihilating polynomial so m of T is a 0 operator, I want to show that an operator is 0 I will show that the action of this operator on a basis is 0 then it follows that the operator is 0 agreed, okay T is diagonalizable that means there exist a basis B with the property that each of the basis vector is an eigenvector of T, let B be a basis of V such that each x element of B is an eigenvector of T corresponding to some eigenvalue I am not too worried about that corresponding to some eigenvalue but I know that there is at least one such basis because T is diagonalizable.

I will show that m T of x is 0 it follows that m T must be 0, do you agree m T is a linear operator m of capital T that is the linear operator I want to show that that is the 0 operator I will show that m of T on the basis vectors if 0 so for every x in B I will show m T x is 0.

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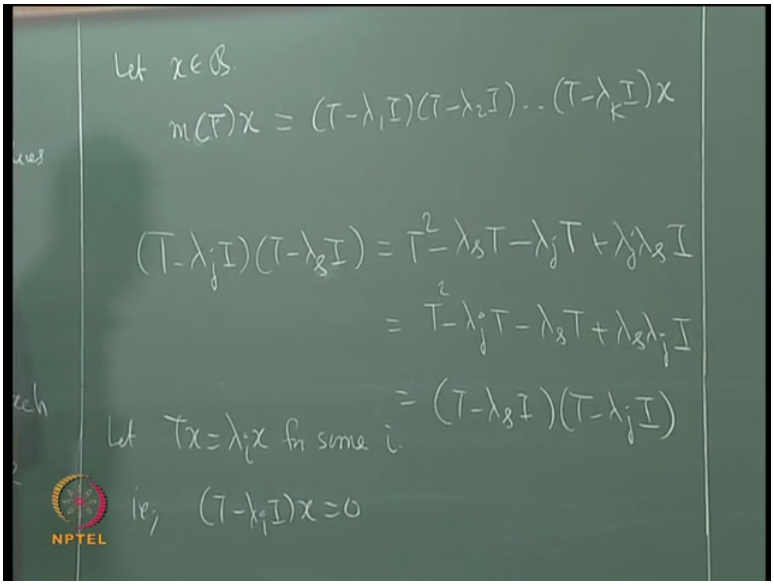

I will start with an x in B and consider consider m T x I must show that this is 0, okay but what is this m of T this is the expression it is T minus lambda 1 I T minus lambda 2 I etc T minus lambda k I operating on x. I want to show that this is 0, this is 0 vector but observe that the property that these factors have is that any two of them commute any two of them commute.

T minus lambda j I into T minus lambda s I this is equal to T square minus lambda s T minus lambda j T plus lambda j lambda s I this is (T minus) T square minus lambda j T minus lambda s T plus lambda s lambda j I and this is nothing but T minus lambda s I into T minus lambda j I. Remember that in general if s and T are operators s T is not equal to T s, okay that is matrix multiplication is not commuted but this is about special matrices special linear operators, operators of the form T minus lambda I these commute, what is the advantage?

You go back here this x is x belongs to B and each vector in B is an eigenvector so this x is an eigenvector corresponding to some eigenvalue lambda I push that to the end operate on it you get 0. So let me say that I know that x is in B and each vector is an eigenvector so there is Tx equal to lambda i x for some for some eigenvalue lambda i. So I now that this equation holds for x that is T minus lambda I of x equals 0 now I use commutativity of this factors.

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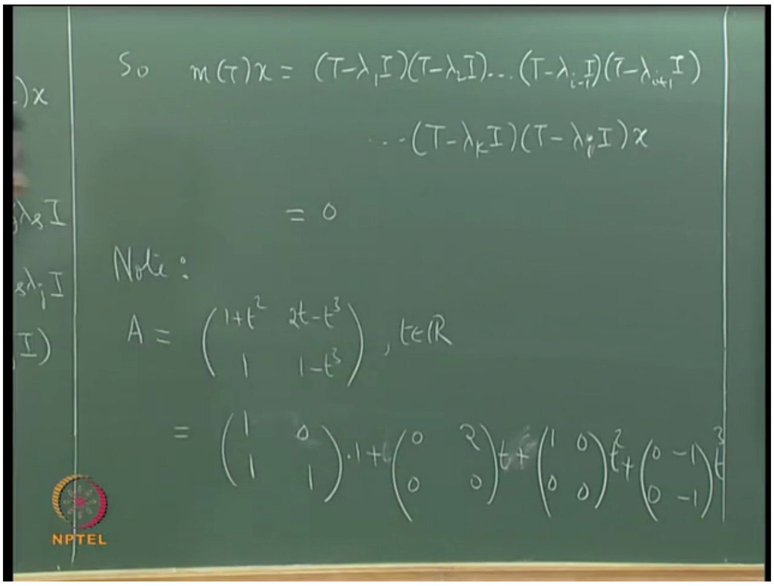

To write m T of x as T minus lambda 1 I lambda 2 I etc the Ith factor I can take T minus lambda i minus 1 T minus lambda i plus 1 T minus lambda k is the last one but I can push T minus lambda i to be the last one because of commutativity. Now I see that T minus lambda x is 0 make use of that the rest of the factors are linear so again 0, okay this is really easy. If T is diagonalizable then the minimal polynomial is a product of distinct linear factors at the end of next lecture we will be able to show that the converse is also true, if the minimal polynomial is a product of distinct linear factors then T is diagonalizable, okay which will then give you another characterization of diagonalizability, okay.

So the converse will be proved a little later we need the notion of invariant subspaces, annihilating polynomials I will do that later but this is one result which could be used in those examples if you know that it is diagonalizable then calculating minimal polynomial is easy if you know the eigenvalues, okay. As I told you we will discuss Cayley-Hamilton Theorem and let us proof for matrices, okay.

I just need one little observation before proving the Cayley-Hamilton Theorem if I have a matrix with polynomial entries then the matrix can be written as a polynomial whose coefficients are matrices. I will just give you an example if I have matrix with polynomial entries let us say 1 plus t square (t) 2t minus t cube 1 and let us say 1 minus t cube I have a matrix with polynomial entries then I am claiming that this matrix can be written as a polynomial whose coefficients are matrices as a polynomial whose coefficients are matrices, what is the meaning?

Simply collect the coefficients of like powers look at the constant for example constant is 1 here, 0, 1, 1. So I can first write it as 1 0 1 1 times 1 plus corresponding to t there is only one term 0 2 0 0 corresponding to t square, okay let me write this into t this is the constant term of the polynomial for A the polynomial is in terms of 1 t t square t cube the coefficients are matrices this is one coefficient, this is second coefficient into t plus for t square I have only one term 1 0 0 0 t square plus finally for t cube I have 0 minus 1 0 minus 1 t cube is it okay please calculate verify this right hand side is correct I have 1 plus t square is the first term, second is 2 minus t 2t minus t cube, the third goes with 1 in this entry, the fourth one is 1 minus t cube, okay. So I have written a matrix whose entries are polynomials this matrix has been written as a polynomial in in t the same variable t but with matrix coefficients let us remember this.

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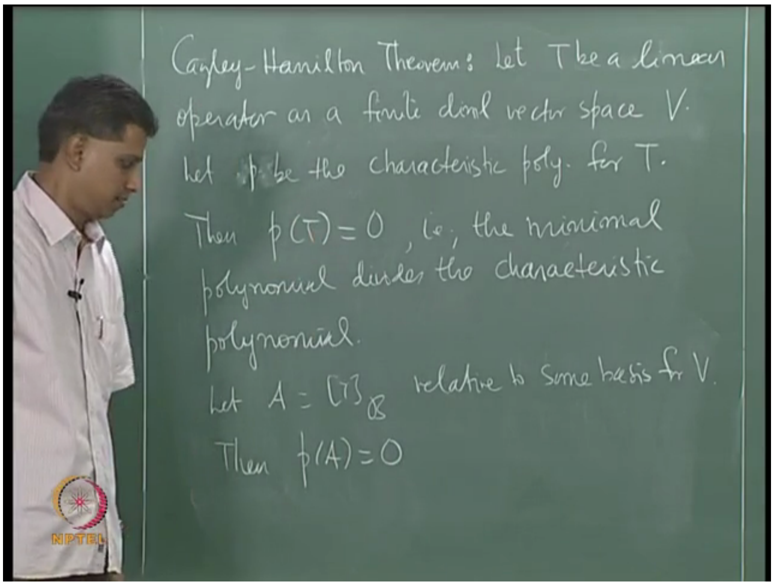

Now next to get Cayley-Hamilton Cayley-Hamilton Theorem I will give the statement for an operator and also write down the statement for the matrix and prove the matrix case T is a linear operator on a finite dimensional vector space V. Let okay I am using word p let p of (T) let p be the characteristic polynomial for T. Then Cayley-Hamilton Theorem states that p of T is capital T that is the 0 operator that is okay what this says is that the characteristic polynomial is an annihilating polynomial but we know that among all the annihilating polynomials the minimal polynomial is a one with the least degree so the characteristic polynomial must be a product of the minimal polynomial and another polynomial, okay. In other words the minimal polynomial divides the characteristic polynomial.

This is an important information and we must compare this with the result that we proved right in the beginning when we looked at the definition of the minimal polynomial. So I remember having made the statement that to define the minimal polynomial first of all you must know that there is an annihilating polynomial show that there is an annihilating polynomial remember we took the operators identity t, t square etc and went upto t to the n square and show that this polynomial must be in a and showed that these vectors these operators are linear independent that gives rise to annihilating polynomial, okay.

Now look at that estimate for that annihilating polynomial it is polynomial of degree n square, okay what this theorem says is that you could do with much less you do not have to go beyond n that the degree of the characteristic polynomial is n this theorem says you do not have to go beyond n at most the characteristic polynomial, is that clear? Okay proof as I told you I will prove the matrix case let me just write down this statement for matrices.

Let A be the matrix of T relative to some basis then we would like to show that p of A is the 0 matrix this is what we want to show.

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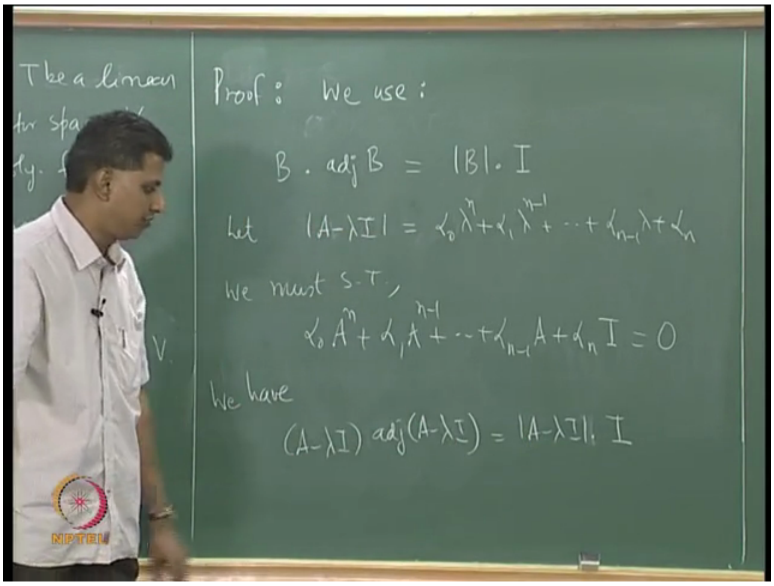

See for proving this I will make use of of course this observation and also another non trivial result on determinants which I will not prove this non trivial result on determinants for the proof we need the following this non trivial result is what I will write down next we use the following for any square matrix B for any square matrix B if you look at the matrix B into the adjoint of B for any square matrix B the matrix B into the adjoint of B, what is this? Yes that is what I want this is determinant of B times I this is not A inverse you remember that see the determinant of B could be 0, okay that is quite possible.

So there is no inverse coming here if you know the determinant of B is non-zero then you can remember this is an equation involving matrices first B is a matrix adjoint of B is a matrix, on the right hand side determinant B is a number times identity matrix. So this is an equation involving matrices and then if determinant of B is not 0 we could divide by that number and then write this as 1 by determinant of B into adjoint of B, call that as some T, okay T is a linear transformation let us call it C then BC equals I, you can also show CB equals I. So it follows that this is a inverse, okay. But this is more general than this determinant B equal to 0 non-zero case.

So I will make use of this result, this is again a non-trivial result using the property of determinants this is not easy but it is not difficult again. So I will make use of this result and then proof Cayley-Hamilton Theorem I will apply this identity for the matrix A minus lambda

I in place of B, okay. Before that let me call let me call determinant A minus lambda I, I know that this is a polynomial of degree n.

So let me call this as alpha not lambda to the n plus alpha 1 lambda to n minus 1, etc alpha n minus 1 lambda plus alpha n this is the polynomial of degree n I am writing it like this. Now this alpha not could be 1 or minus 1 depending on the order but let us keep it as it is what is that we need to show we need to show that p of A is equal to 0. So we must show that when I replace lambda by A when I replace lambda by A I get a polynomial I A I must show that this polynomial is the 0 polynomial I am sorry I get a matrix I must show that this matrix is a 0 matrix, okay. So we must show that alpha not A to the n plus alpha 1 A to the n minus 1 etc alpha n minus 1 A plus alpha n identity we must show that this is 0 matrix, okay okay.

Let us now make use of A minus lambda I in place of B and then see what we get so I am using this formula and applying A minus lambda I for B, okay in this I am applying A minus lambda I. So I have this A minus lambda I into adjoint of A minus lambda I is equal to determinant of A minus lambda I into identity matrix I will substitute determinant of A minus lambda I the expression that I have written down just now here and keep this right hand side and then look at what I have from the left hand side.

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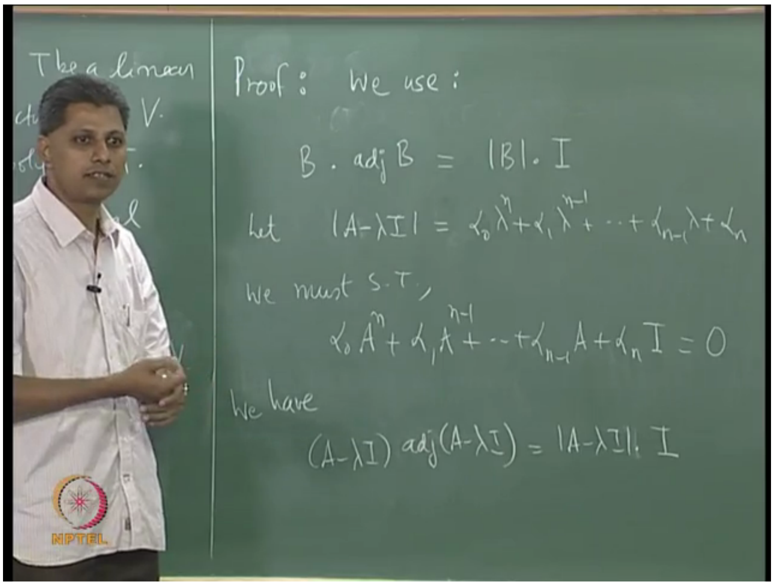

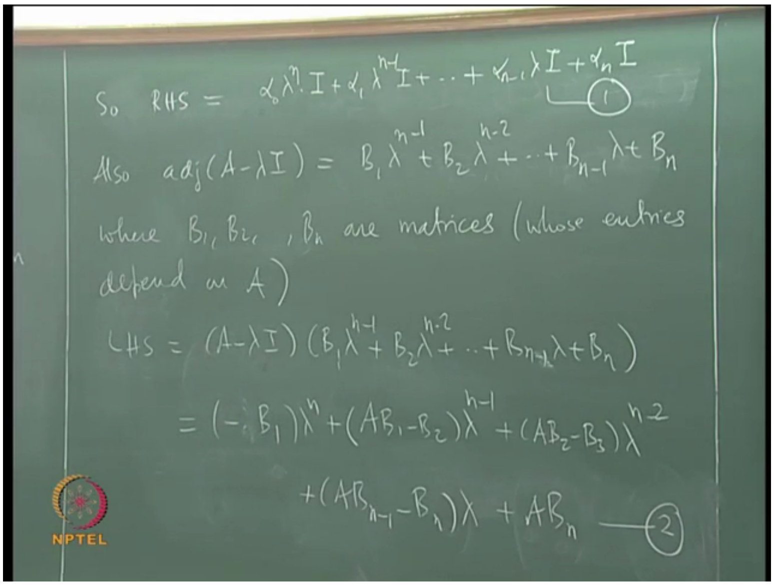

So let me write the right hand side the right hand side is alpha not lambda n I is that correct ya times identity plus alpha 1 lambda to the n minus 1 identity etc alpha n minus 1 lambda identity plus alpha n identity, okay just to remind myself the right hand side is a matrix determinant of A minus lambda I into I that is a matrix, okay. Now look at the left hand side look at the second factor adjoint of A minus lambda I how do you compute the adjoint of a matrix?

The adjoint of a matrix B let us go back to this is computed by using minors, cofactors taking transpose first compute the minors, what is the minor of an entry? The minor of an entry is the determinant of okay if you have an n cross n matrix the minor of an entry Aij is a determinant of the n minus 1 cross n minus 1 sub matrix obtained by deleting the ith row jth column, okay okay.

We are calculating minor of A minus lambda I, okay A minus lambda I along the diagonal I have A 11 minus lambda, A 22 minus lambda, etc. Let us take any arbitrary element calculate its minor I will not calculate its minor I will only say what this minor the form must be. Can you see that okay I am deleting I am deleting a row a row and a column and then I am looking at an n minus 1 cross n minus 1 sub matrix I calculate the determinant of that.

Now that is the determinant will be a polynomial of degree n minus 1 that is what I want, that is the polynomial of degree n minus 1 also it is a determinant of a matrix whose entries are polynomials so I can write this polynomial like I will make use of this observation I can write this polynomial in terms of in the variable lambda but the coefficients are matrices. So this is the point that is probably the most crucial in this apart from this formula.

Adjoint of A minus lambda I, I can write this as it is a polynomial of degree n minus 1 but the entries are the coefficients are matrices this is the point. So I will use a similar notation let us say B 1 lambda to the n minus 1 plus B 2 lambda to the n minus 2 plus etc plus B n minus 1 lambda plus B n but I must observe that where B 1, B 2, etc B n are matrix coefficients. So they are matrices adjoint of A minus lambda I adjoint of A minus lambda I is a matrix whose entries are polynomials in lambda adjoint of A minus lambda I is a matrix whose entries are polynomials in lambda but the maximum degree of each polynomial in the adjoint of A minus lambda I is lambda to the n minus 1 because you are calculating the determinant of a n minus 1 cross n minus 1 sub matrix.

So these are matrices whose entries of course depend of A but that is not important it is mentioned only for completeness, these the entries of these matrices depend on A. Now substitute this into the left hand side equate coefficients of like terms you get term you do a one more operation and then you get p of A equals 0, okay.

Let us substitute to calculate the left hand side left hand side is A minus lambda I into adjoint of A minus lambda I the expression is available here, I have B 1 lambda to the n minus 1 plus B 2 lambda to the n minus 2 etc B n minus 1 lambda plus B n this is the left hand side. Let me collect the coefficients and write down the terms accordingly there is only one term corresponding to lambda to the n that term is minus lambda B 1 sorry minus B 1 there is only one term corresponding to lambda to the n that is this.

There are two terms corresponding to lambda to the n minus 1 one coming from this minus B 1 the other one I am sorry one coming from this and the other coming from this AB 1 minus B 2 lambda to the n minus 1. Let me may be write down one more term for lambda to the n minus 2 you can verify AB 2 minus B 3, AB 2 there is a B 3 to the lambda n minus 3 into that will give lambda n minus 2 so minus B 3. So it follows this order plus coefficient of lambda coefficient of lambda will come from contributions from second term in the last term, last but one and the first term AB n minus 1 minus B n lambda and constant term is just AB n, okay these are the terms corresponding to the degrees lambda to the n lambda to the n minus 1 etc. So I must compare these two.

So maybe I will call this equation 1 right hand side formula 1, left hand side formula 2, okay look at right hand side I have formula 1, left hand side I have this expression these two must coincide.

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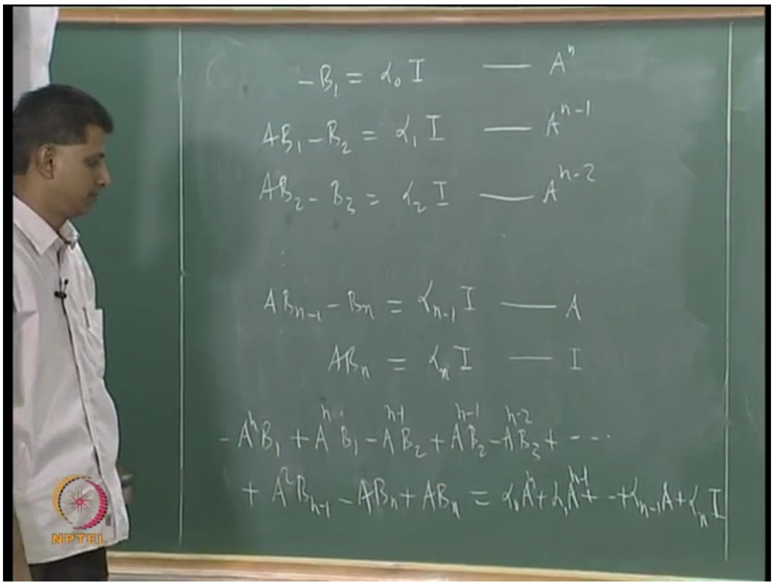

So I get the following equation coefficient of lambda to the n on the right hand side is alpha not I minus B 1 is alpha not I, AB 1 minus B 2 is alpha 1 I, AB 2 minus B 3 is alpha 2 I, etc AB n minus 1 minus B n that corresponds to what lambda that is alpha n minus 1 times identity, the last equation is AB n equals alpha not sorry alpha n identity. So I have these equations n plus 1 equations polynomial of degree n so there are n plus 1 terms these are the n plus 1 equations, multiply the first one by A to the n minus 1, this by the n minus 2, this by this by A, this by A square and then add you get term, okay.

So let us say I multiply this by A to the n, this by A to the n minus 1, n minus 2, etc this will be multiplied by A square and I will just leave it right so no this will be multiplied by A, this is multiplied by identity so that term ya I must go from A to the 0 to A to the n multiply and add A to the n maybe I will just do that quickly minus A to the n B 1 pre multiply, matrix multiplication is not committed pre multiply minus A to the n B 1 plus A to the n minus 1 B 1 minus there is already n here minus A to the n minus 1 B 2 plus A to the n minus 1 B 2 minus A to the n minus 2 B 3 plus etc plus this one is multiplied by A A square B n minus 1 minus AB n plus AB n on the one side, on the other side alpha not A to the n plus alpha 1 A to the n minus 1 etc plus alpha n minus 1 A plus alpha n identity.

You can see that the left hand side is 0 these two get cancel, these two get cancel etc. You take any term the first term each term has two terms you take any term the first term gets cancel with the second term or the previous term etc. So the left hand side is 0, the right hand side is what you have that is Cayley-Hamilton Theorem this left hand side is 0 matrix, right hand side is p of A, okay.

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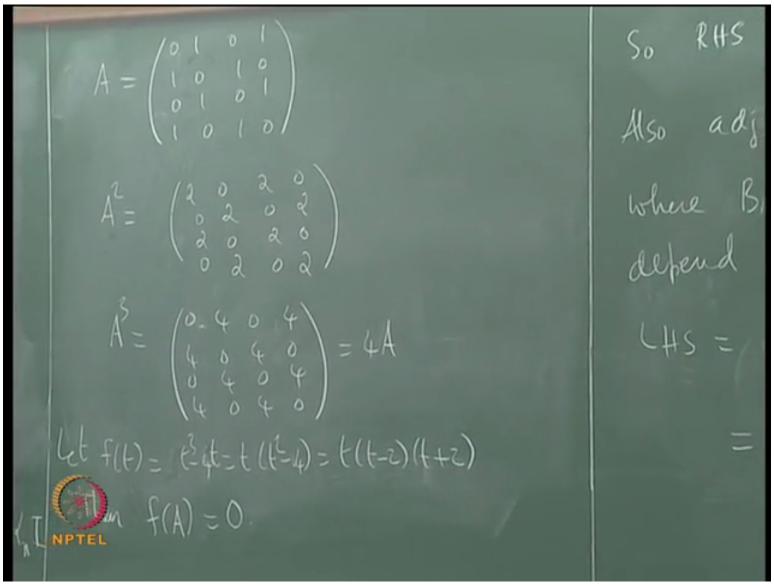

Let us do one final problem in this section which is purely for fun may be I want to look at this 4 by 4 matrix. See remember calculating the characteristic polynomial is not difficult alright but calculating the eigenvalues is difficult but in certain problems it will be very clear. So I want you to look at this problem let us look at the linear operator T on R 4 whose matrix with respect to the standard basis is this 0 1 0 1 1 0 1 0 0 1 0 1 1 0 1 0, I would like to calculate the characteristic polynomial of this as well as the minimal polynomial this matrix has some nice properties which we will make use of (())(33:03), okay.

Look at A square see I am not trying to advocate this procedure for all matrices, for this matrix it is nice to see how the minimal polynomial, characteristic polynomial can be calculated very quickly, okay. What is A square? A square is 0 1 0 1 into 0 1 0 1 it will be 2 0 2 0 0 2 0 2 2 0 2 0 0 2 0 2 this is A square, okay still there is no pattern. Look at A cube A cube can be shown to be see it is a product of this into this that is 4, so you will have 0 4 0 4 4 0 4 0, mistake no problem this is correct, what is your question? I am calculating A square here is that correct, is something wrong here? 2 0 2 0 0 2 0 2, okay this is fine this into this, okay and A cube instead of this I will go back to this pattern.

See before coming to the class I work it out and come that is call preparation so this is correct please check this this is equal to 4 times A that is what happens in this example, okay this is 4 times A which means I have caught hold of a annihilating polynomial, right A cube minus 4A is 0. So I will look at I look at the polynomial I call it f this time f of t is t cube minus t cube minus t that is t into t square minus 1 sorry t cube minus 4t that is the t square minus 4 that is t into t plus 2 t minus 2 whichever be if I call f as this polynomial then f of A is 0 that much I know I know that f of A is a 0 matrix, okay.

I know that minimal polynomial is a polynomial of least degree among the annihilating polynomials, okay. So the minimal polynomial is see the degree of this is 3, the degree of this is 3, can the minimal polynomial be a linear polynomial? Can the minimal polynomial be a linear polynomial? Linear polynomial is a polynomial of the form t minus lambda linear polynomial is t minus lambda, okay.

So can this be the minimal polynomial for this matrix? Can it be linear? If the minimal polynomial for this matrix is linear then I must have something like A equals lambda times I, A equal to alpha times I for some scalar alpha. Obviously not so A is not alpha times identity A is not which means can you see that all these three individual factors do not constitute minimal polynomial that is minimal polynomial is not a linear factor do you first agree. It is not linear it is all that I want to say is that this is the minimal polynomial, okay. I want to roll out the possibility that it is not quadratic nor is it linear, linearity is immediate.

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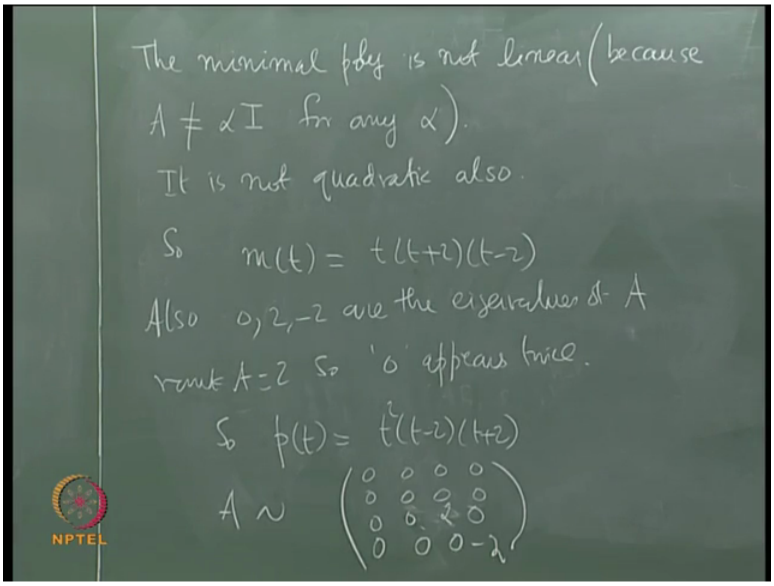

The minimal polynomial is not linear what is the reason? If it were linear then A would be a multiple of identity and so that is rolled out. If the minimal polynomial is linear then the minimal polynomial is of the form t minus alpha for some alpha but if that were the case a minimal polynomial must be annihilating polynomial. So I replace t by A I must get A minus alpha I is 0 that is not the case. So it is not linear, can it be quadratic? If it is quadratic what are the choices? If it is quadratic remember minimal polynomial must divide this polynomial so this is like a bench mark.

If it is a quadratic then it is either of the form t into t (square) t minus 2 or t into t plus 2 or t minus 2 into t plus 2, okay. If it is of the form t into t minus 2 then t square minus 2t must be an annihilating polynomial that is A square equals 2A that is not the case. If it is of the form t into t plus 2 then I must have A square plus 2A equals 0 that is not of the form, the last possibility is t square minus 4 t square minus 4 is a possibility for the minimal polynomial in which case A square equal to 4I that is again not the case t square minus 4 is the third quadratic polynomial that is not a minimal polynomial because that is not the minimal polynomial because A square minus 4I is not 0, A square is not equal to 4I and so we have already seen that this is a annihilating polynomial so this must be the minimal polynomial, okay. So please give the reasons for the quadratic case.

It is not quadratic also and so the minimal polynomial for this matrix is t into t plus 2 into t minus 2. Now the minimal polynomial has the property that the zeros of the minimal polynomial are the characteristic values. So the characteristic values of the matrix that we started with are 0 plus 2 minus 2. Also 0, 2, minus 2 are the eigenvalues of the matrix A there is a relationship between the characteristic polynomial and the minimal polynomial they have the same set of zeros.

The dimension is 4 I have three eigenvalues so one of them comes twice which one? One of the eigenvalues must come once more the algebraic multiplicity the algebraic multiplicity of one of this eigenvalues must be 2 which one is that? If you give the answer you must also give the reason, okay. Indians invented 0 so let us take 0 first I want to know the dimension of the set of solutions of Ax equals 0x that is Ax equals to 0, okay.

What is the dimension of the set of solutions of Ax equal to 0, A is given to you it should be 2 is okay but I want you to give a reason, why is it 2? Wait wait wait rank of A is 2, this row is same as this row, this row is same as this row these two are obviously independent rank of A is 2, nullity of A must be 2 so dimension of set of all solutions of Ax equal to 0 is 2, so 0 comes twice. Rank of A is 2 and so the eigenvalue 0 appears twice as an eigenvalue of this matrix and so what is the characteristic polynomial?

So I will simply say appears twice so p is the notation we have for the characteristic polynomial. So p of t is t square into t minus 2 into t plus 2 the characteristic polynomial should be of this form t square into t plus 2 into t minus 2, is the matrix diagonalizable? See the answer is yes there are atleast two reasons the answer is yes the matrix is diagonalizable there are atleast two reasons one reason if you assume my result that a matrix is diagonalizable if and only if the minimal polynomial is a product of distinct linear factors that has not been proved but you could have taken the statement.

The other thing is distinct eigenvalues have linear independent eigenvectors, I am assured of three independent eigenvectors 0, minus 2, plus 2 three independent eigenvectors come alright, do I have one more eigenvector which is independent with these three coming from Ax equal to 0, okay the problem lies the question is is 0 deficient that is do I have two independent eigenvectors for the eigenvalue 0 that is easy to see in this problem.

Ax equal to 0 there are two independent solutions, rank is 2, nullity is 2 that is what we determine just now the matrix is diagonalizable, okay the matrix is diagonalizable and it is similar even if you view this as a matrix over rational field the field of rational numbers the matrix A is similar I will use this the matrix A is similar to this 4 by 4 matrix let say I follow this order then it is this matrix, okay this matrix I will come back this matrix will that was here doubt I think 0 0 2 0 I am taking it in this order 0 comes twice 2 and then minus 2.

So A is similar to this that means p inverse A p equals this matrix, how do you get p, p is a matrix whose first two columns are the independent solutions of Ax equal to 0, third column is the only solution of Ax equals 2x, fourth column is the only solution of Ax equals minus 2x only non-zero solution that is a matrix p that is invertible and p inverse A p equals a diagonal matrix, okay okay I think I will stop here.

I want to discuss the notion of invariant subspaces, annihilating polynomials and how it helps these two notions help in proving the converse statement that if the minimal polynomial is a product of distinct linear factors then the matrix is then the operator is diagonalizable. I will proof this in the next class, okay let me stop here which argument see you want to show that the matrix is diagonalizable then you must show that it has 4 independent eigenvectors, 3 independent eigenvectors are already there because of 0 minus 2 plus 2 they are distinct so I look at a non-zero solution of those eigenvalue equations, I want one more that one should one more should correspond to the eigenvalue 0 because the characteristic polynomial is this that should correspond to 0.

So do I have two independent solutions of Ax equal to 0 is what the question is but the rank of A is 2 we observed just now the rank of A is 2 so nullity is 2, so there are two independent solutions take those two independent solutions put them in the matrix p you get p inverse A p as this diagonal matrix, okay so this is see this is just to illustrate how certain problems can be (())(46:47) do not apply this for example in the exam you will waste time, okay we will meet on Monday.