Linear Algebra
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Module 7 Eigenvalues and Eigenvectors
Lecture 28
The Minimal Polynomial

Okay, we are discussing the minimal polynomial, okay.

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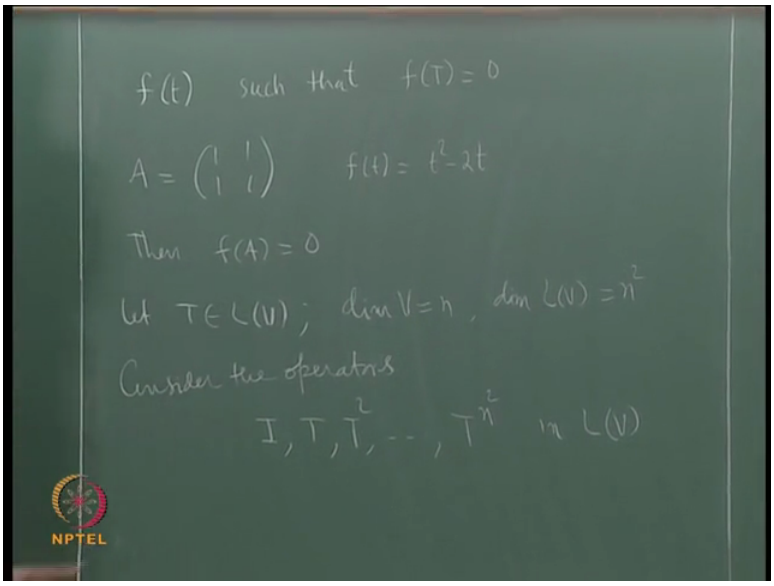

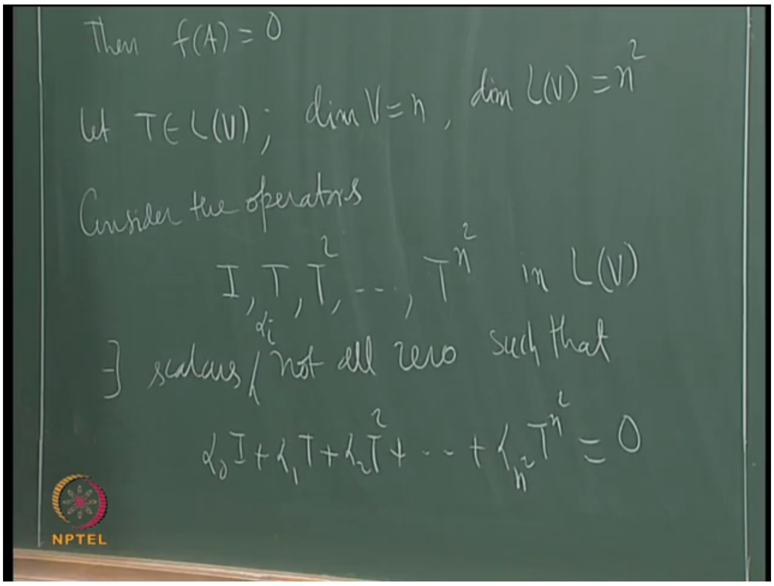

Minimal Polynomial is a polynomial which has the property is a polynomial f of t which has the property minimal polynomial is a polynomial f such that f of capital T equal to 0, okay I will give the precise definition but the question is whether there exist such a polynomial, okay I had an example yesterday example of a matrix 2 by 2 matrix and the polynomial.

Let me recall that I had given this matrix I am looking at the linear transformation whose matrix relative to the standard basis is this linear transformation on R 2 and I defined f of t to be t square minus 2t, okay then we notice yesterday that f of A is a 0 matrix, okay. So atleast for this matrix we have shown that there is a polynomial f such that f of A equal to 0, in the general case let me give an argument and then give the precise definition. We are we are seeking a polynomial f which has the property that f of T is 0 given a transformation given an operator T, okay.

In the general case let us look at the following T is a linear operator on a finite dimensional vector space, what is the dimension of V? As usual it is n, what is the dimension of L V? n square dimension of L V is n square. So if you look at and see L V is the space of all linear operators on V so consider these operators consider the following operators starting from I, T, T square, etc operators on V these are elements of L V, etc I go upto T to the n square consider these operators these are in L V.

So these are elements, these are vectors in this vector space these are n square vectors in this vector space n square plus 1 rather here n square plus 1 vectors in the vector space L of V the dimension of L of V is n square so these must be linearly dependent. So there exist scalars not all 0 atleast one of them is non-zero not all 0 there exist scalars not all 0 such that scalars I mean alpha i not all 0 such that alpha not i plus alpha 1 T alpha 2 T square etc alpha n square T n square equals a 0 operator these vectors in this space L of V are linearly dependent so there are scalars atleast one of which is not 0 all you have to do is pick up the polynomial from this equation.

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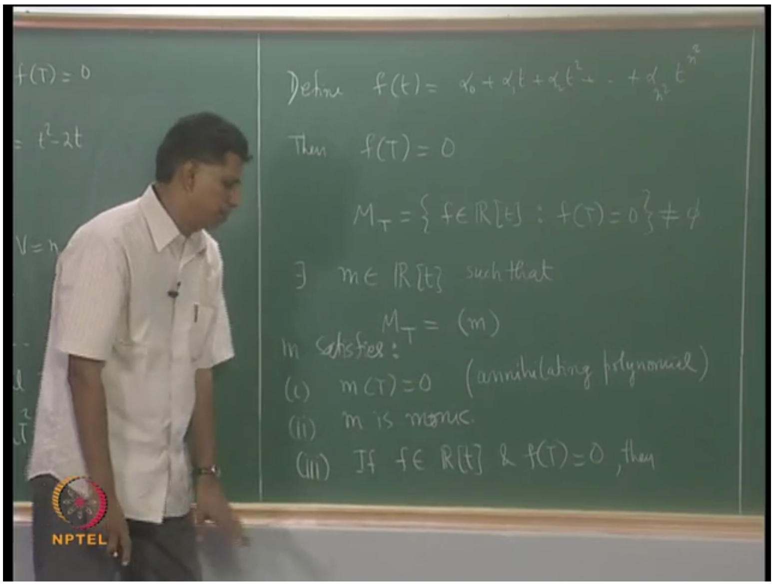

Define let us say f of t by alpha not plus alpha 1 t alpha 2 t square etc plus alpha n square t to the n square. The polynomial coming from this left hand side this polynomial has a property that f of T equals the 0 matrix, okay. So we have proved in the general case also for any linear operator on a finite dimensional vector space there is at least one polynomial in fact there are infinitely many there is at least one polynomial f that satisfies f of T equals a 0 operator.

What we do is look at see as I described yesterday we look at the collection M T this is the set of all polynomials f so I am using f in R t for instance set of all polynomials f with real coefficients over the real variable t such that f of T equals a 0 operator, what we have just now shown is that this is not empty for any operator this set is not empty, this is a subset of R t this is a sub bring this also has the property that it is an ideal and this is an ideal in R t R t is an euclidean domain it is also called a principle ideal domain f x over the variable x is a principle ideal domain. Principle ideal domain means that every ideal is generated by a unique element.

So there exists m in R t such that such that this capital M sub T is generated by this m is generated by this m that is every every polynomial in M T is a multiple of this little m every polynomial in M T is a multiple of little m, what are the properties of this little m? Little m has a property that it is a monic polynomial it is a monic polynomial the coefficient of the highest degree of m is 1, what is the other property? m has a property that m of T is 0, right it comes from this, what is the other property that m has? Degree of m is less than or equal to degree of f whenever f is such that f of T is 0, okay.

So let me just write down these properties m satisfies the following the first condition is that it must be such that m of T is 0 such a polynomial is called an annihilating polynomial so I will just write down on this side annihilating polynomial it annihilates T it destroys T that is it takes T to 0 this is called an annihilating polynomial so m of T any polynomial f that satisfies f of T equal to 0 is called an annihilating polynomial in particular m of m must be an annihilating polynomial.

Second property m is monic monic means the coefficient of the highest degree is 1 remember if the coefficient of the highest degree is not 1 we can always divide by that number the coefficient and you will get it. So m is monic that is another property. Property 3 we are calling it a minimal polynomial minimal in what sense minimal in the sense of degree. If f belongs to R t and f of T equals a 0 polynomial if f belongs to R t and f of T equal to 0 0 operator then so I told you that M T is generated by little m which means I also told you that anything in M T is a multiple of this little m, okay.

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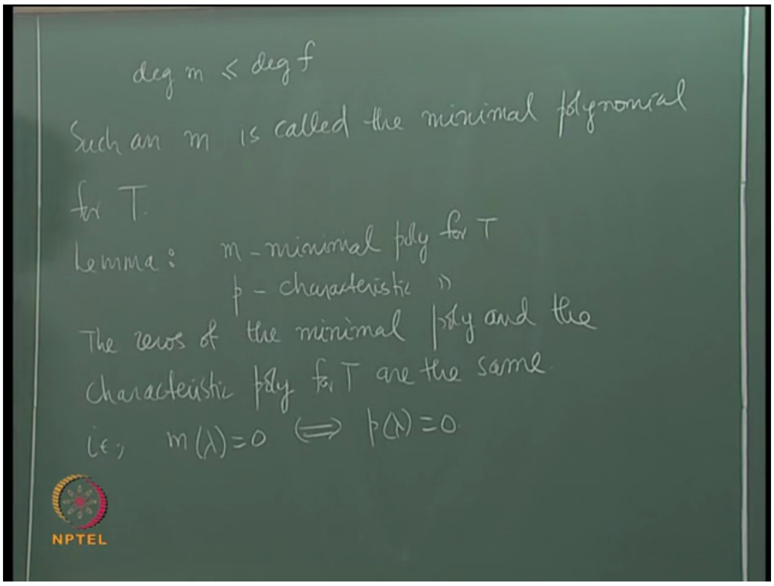

So I have this condition degree of m is less than or equal to degree of I missed have this condition satisfied by m. For example f could be a multiple of m in which case the degree of f and degree m are the same, okay but it cannot be less than the degree of m any annihilating polynomial must have degree cannot have degree strictly less than the degree of the minimal polynomial, okay so this m is called the minimal polynomial.

Such an m is called the minimal polynomial the minimal polynomial for T for the operator T it depends on T different operators will have different minimal polynomials, okay okay. In the worst case when you do not know what an algebra is, what a sub bring or what an ideal is you can take it for granted that this definition is sensible there is there is a minimal polynomial for each operator, okay okay.

Let us look at some examples but may be before that I need to tell you the relationship between the minimal (polyno) so we are discussing eigenvalues, eigenvectors what is the relationship what is the relationship between the minimal polynomial and the eigenvalues. Let us first settle this so let me just proof the following following result T is in L V and V is finite dimensional and so I will always reserve this little m for the minimal polynomial for an operator and the characteristic polynomial I will reserve p, okay.

So m for me will always be the minimal polynomial for T and p will be the characteristic polynomial p will be the characteristic polynomial any other annihilating polynomial I will use f. The zeros of the minimal polynomial and the characteristic polynomial for an operator T are the same, okay this is the connection the connection between the minimal polynomial and the characteristic polynomial is that they have the same zeros same zeros means that is m of lambda equals 0 if and only if p of lambda equals 0 if lambda is a root of the equation m of T equals 0 than lambda is also a root of the equation p of T equals 0, okay okay how do you prove it?

Can you see that there is a connection now between p is the characteristic polynomial so any root of the equation p T equal to 0 lambda is an eigenvalue so the zeros of the minimal polynomial are the eigenvalues of the operator.

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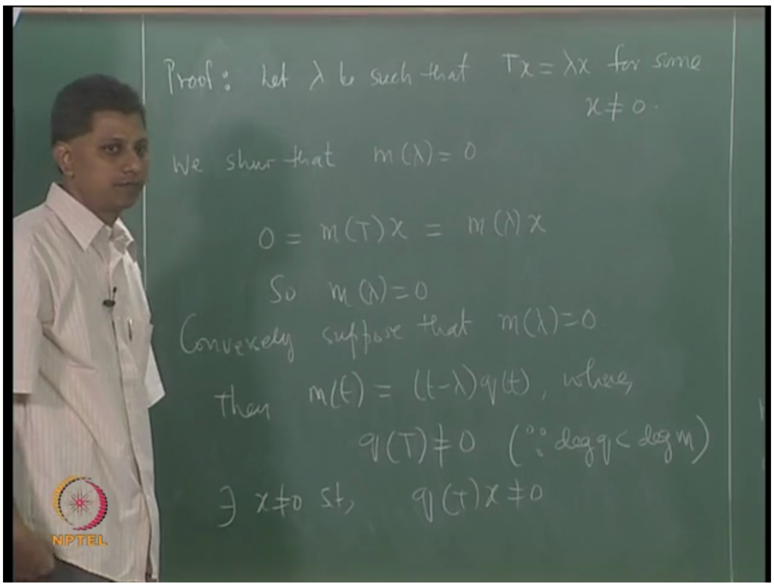

Proof we need to show that let lambda be such that Tx equals lambda x for some x not equal to 0 that is lambdas and eigenvalue of T, x is the corresponding eigenvector, okay we are actually proving the sufficiency power first ya if lambdas and eigenvalue then I am showing that lambda is 0 of the minimal polynomial, okay is this okay m T x equals 0 do you agree with this first m is a minimal polynomial so it is annihilating polynomial.

So for any x m T x is 0 in particular for this eigenvector, 0 is m T x but you remember this result that we proved m T x if lambda is an eigenvalue this can be written as m lambda x Tx equals lambda x you saw this result in the last lecture this is m lambda into x m lambda is a number, x is a vector x is an eigenvector so it is not 0, m lambda must be 0. So one part is very simple if lambda satisfies p lambda equal to 0 then we have shown m lambda is 0, okay we have only used the factor that m T x is m lambda x.

Conversely let us suppose that m lambda is 0 we must produce a vector we must show that there exist a vector let us say y such that Ty equals lambda y we must show that there exist a vector y not equal to 0 such that Ty equals lambda y it would then mean that lambda is an eigenvalue and so p of lambda is 0, okay m lambda equal to 0 means that if I use T as the variable for the polynomial m do you agree that I can write m of t as t minus lambda into q of t, if lambda is 0 of m of t equal to 0 then t minus lambda is a factor of m of t, t minus lambda is a factor of m of t this is this is by definition.

What you know about q of t? The degree of q is at least 1 is precisely 1 less than the degree of m. So q cannot be annihilating polynomial, where, q cannot be annihilating polynomial that is q of capital T cannot be the 0 polynomial the reason since degree of q is strictly less than degree of m and any polynomial of degree less than the minimal degree of the minimal polynomial cannot be annihilating polynomial.

So q of T is not the 0 operator, q of T not the 0 operator means that there is one non-zero at least one non-zero vector x such that q of T x is not 0 there exist x not equal to 0 such that q of T x is not 0 that is the meaning of saying that an operator is not 0 this is this will be my eigenvector.

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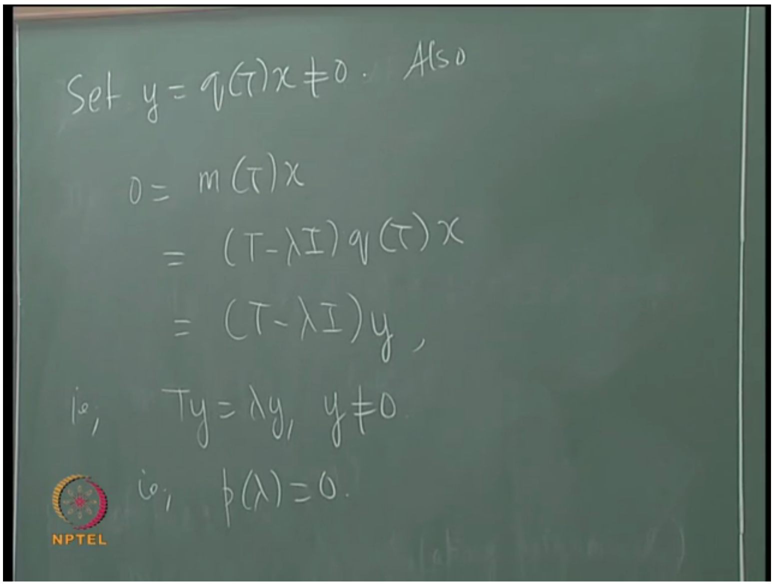

So I will call this y call y as q T x then for one thing y is not 0. Also I will start with this 0 m T x that must be 0 because m T is minimal polynomial 0 equal to m T x this is written as T minus lambda I times the polynomial q into x this is T minus lambda I q T x is y into y. So what I have is what I have is Ty equals lambda y with y not equal to 0. So I have shown that y is an eigenvector corresponding to the eigenvalue lambda. In terms of p this means p of lambda is 0, okay.

Now this is an important connection between the minimal polynomial and the characteristic polynomial, they have the same roots except for multiplicities, okay remember that that we have not shown we have not shown that the multiplicities are the same, okay but they have the same roots this is an important point.

Maybe we should now look at the examples, okay some numerical examples my first example will be one that exemplifies this diagonalizability I realized I have not given an example. So for diagonalizability let me first do an example and also consider the examples that we have discussed earlier we have discussed at least two examples. The first example of an operator which does not have an eigenvalue so it is not diagonalizable, second example of an operator which has enough values but not enough eigenvectors again it is not diagonalizable, third example that I am going to discuss now will be diagonalizable this will also tell you what diagonalizability means for matrices if it is not already clear to you.

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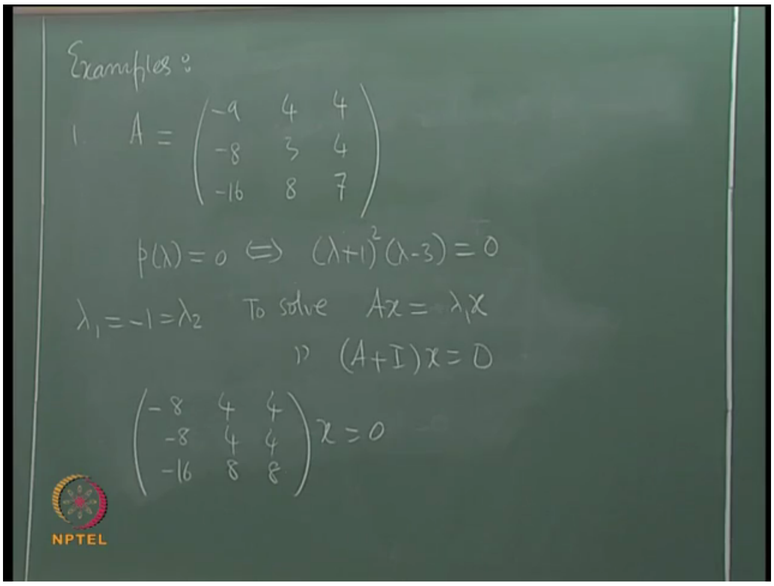

So I am going to discuss examples and also by the side discussed the minimal polynomials, okay okay. The first one let me consider the operator T whose matrix relative to the standard basis is given by this 3 by 3 matrix I will take this matrix this is the matrix of a linear operator relative to the standard basis let us say. I will leave it to you for verifying that the characteristic polynomial of this matrix we are using p the characteristic polynomial of this matrix please check that it is I will go with a minus sign so I will say characteristic equation for this problem the characteristic equation will be please verify that it is lambda plus 1 the whole square into lambda minus 3 this is the characteristic equation of this matrix, okay.

Expand this 3 by 3 determinant A minus lambda I equal to 0. The eigenvalue minus 1 comes twice eigenvalue 1 comes once, what I know is the eigenvectors corresponding to one eigenvector corresponding to minus 1 and the eigenvector corresponding to 3 are independent I must verify if minus 1 the eigenspace corresponding to minus 1 has dimension 2 I must verify if the eigenspace corresponding to the eigenvalue 1 has dimension 2, okay okay what is lambda equals to minus 1 so I need to go back to this.

Look at look at look at the case lambda 1 is minus 1 I am looking at the number of independent solutions of this equation Ax equals minus lambda 1 x that is A plus lambda 1 I A plus I x and I solve this is that correct minus 1 I meant lambda 1 x I meant lambda 1 x so I have Ax so A plus lambda 1 lambda minus 1 what is going on ya minus lambda 1 so I need to solve this A plus I x equal to 0 so what are those equation?

These equations are minus 8 4 4 minus 8 I must add 1 4 4 minus 16 8 8 into x the row reduced echelon form of this matrix will have only 1 nonzero row, 2 zero rows the first row is minus 2 1 1 or if you want you can say it is 1 minus 1 by 2 minus 1 by 2. In any case the number of solutions of this equation is 2 the rank of this matrix is 1 the row rank of this matrix is 1 because second and third rows are just multiples of the first row.

So the row rank of this matrix is 1 which means the null space will have nullity is 2, so null space has dimension 2. So there are two independent solutions of this equation, okay.

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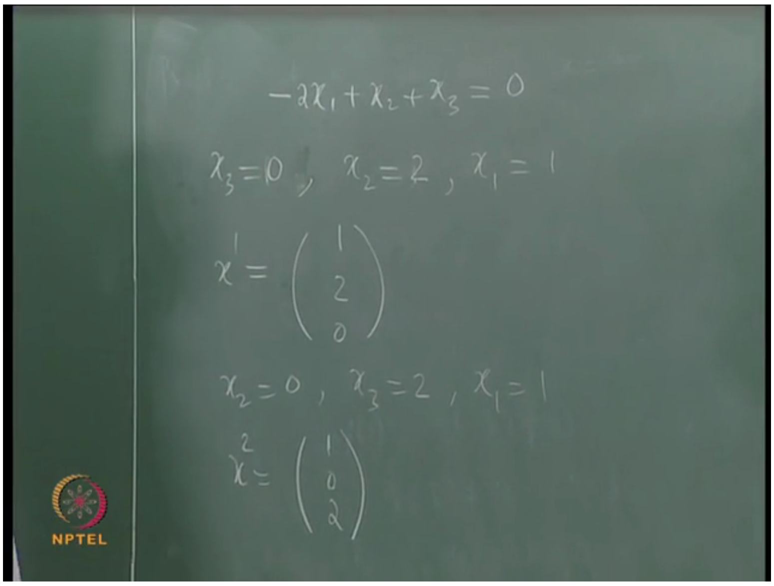

Let us compute for completeness I have there is only one equation there is only one equation that I must solve I must write down two independent solutions this is one equation and three unknowns I can fix two of them. So I will take x 3 to be 1 for the first case and x 2 equal to 1 so I have this let us say x 2 equals 2, so x 1 is 1 is that okay minus 2 plus 2 plus 1 so that is not 0 so I am taking x 3 I can take x 3 to be 0, x 2 was 2, x 1 is 1 so that gives me one vector.

Let me write x1 the super script 1 that is one vector for me 1 2 0 I will write down other vector for the same equation. Let us say that comes from I will write that here, another choice is (x 3 x 1) sorry x 2 is 0, x 3 is 2, then x 1 is 1 that gives me another independent vector which is I think I have not written correctly here 1 2 0 x 2 ya 1 0 2, right is that correct 1 0 2 x 2 and x 3 behave similarly.

So I can interchange the rows that is what I have done. So this is another this is one, this is another obviously they are independent so I have two independent eigenvectors for the eigenvalue minus 1 which comes twice as an eigenvalue, okay.

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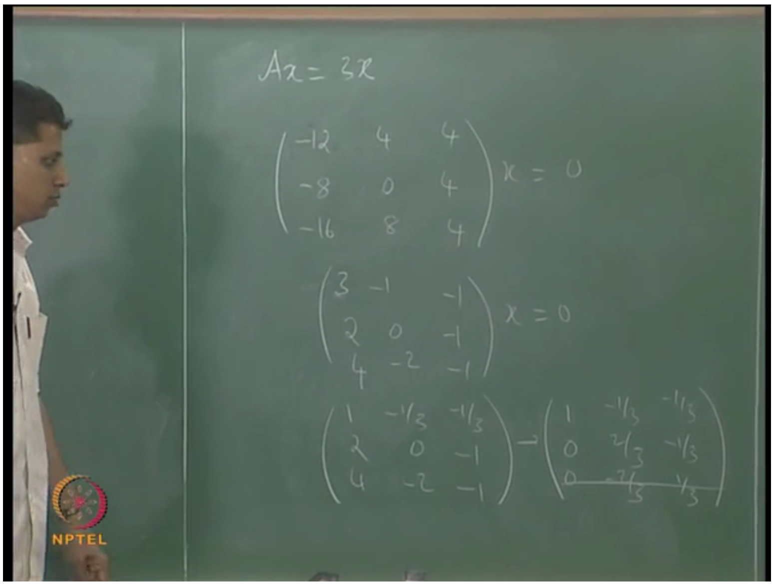

Let us also calculate an eigenvector for the eigenvalue 3, I need to solve Ax equals 3x A minus 3x so I must solve minus 12 4 4 minus 8 A minus 3 0 is a correct 4 minus 16 8 7 minus 3 4 I am looking at A minus 3 minus 12 0 4 all the other entries are the same. So this into x, I need to do row reduced okay may be just by observation let us let us do this quickly minus 2 1 1 let us say 2 minus 1 1 that is the first equation, second equation is 2 0 1 I divide by 4 4 minus 2 minus 1 I divide by minus 4 minus 2 minus 1 that is correct, second row I am dividing by 4 minus 4 2 0 minus 1 I am dividing by minus 4 3 I am dividing by minus 4 3 minus 1 minus 1.

Okay, 1 minus 1 by 3 minus 1 by 3 2 0 minus 1 4 minus 2 minus 1 and then minus 2 times this plus this 1 minus 1 by 3 minus 1 by 3 minus 2 times this that is 2 by 3 2 by 3 minus 1 minus 1 by 3 minus 4 times this plus this, okay. So I have this to be 0 4 by 3 minus 2 that is minus 2 by 3 minus 4 times this 4 by 3 minus 1 minus 1 by 3 1 by 3, okay so please check I can remove the last row which is same as the second row without reducing it to the row reduced echelon form etc okay. Now I see that I must fix x 1 x 2 x 3 can be determine uniquely, okay.

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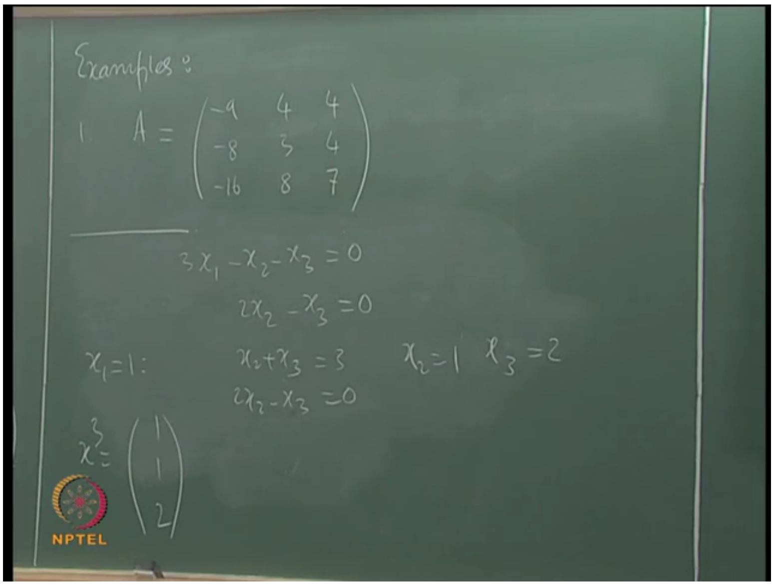

Same example I can fix x 1 let me multiply it by minus 3 3x 1 minus x 2 minus x 3 multiply this by 3 2x 2 minus x 3 let me take x 1 to be 3 may be x 1 to be 1 then I must solve for these two x 2 plus x 3 is 3 2x 2 minus x 3 is 0 3x 3 3x 2 is 3 x 2 is 1 what is x 1 x 2 is 1 sorry x 3 x 1 has been fixed x 3 is 2 I will call this the third vector x 3 first coordinate is 1, second coordinate is 1, third coordinate is 2 so please check that this is another eigenvector 1 1 2 minus 12 plus 4 plus 8 0 minus 8 plus 8 minus 16 plus 8 plus 8, okay so this is an eigenvector for the eigenvalue 3.

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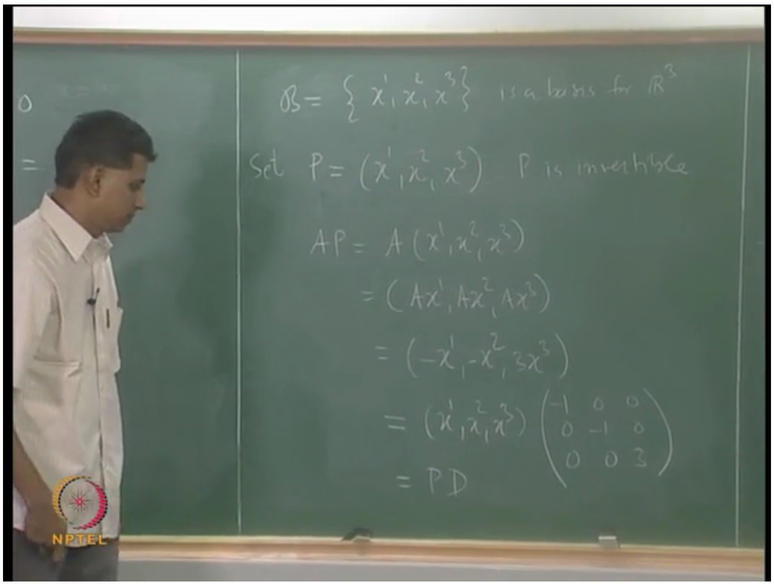

Now look at all these three vectors I will use this portion now script B equals x 1, x 2, x 3 these two vectors x 1 and x 2 obviously are independent and x 3 must be independent with x

1 x 2 because x 1 x 2 correspond to the eigenvalue minus 1 x 3 correspond to 3 so these three are linearly independent vectors. So this forms a basis for R 3 such that each vector is an eigenvector so the operator T that we started with must be diagonalizable in particular in particular this matrix A is diagonalizable. What is the meaning of A being diagonalizable?

Let us call P as a matrix whose entries are the columns x 1, x 2, x 3 I define a new matrix P whose columns are the eigenvectors taken in this order. First correspond to minus 1, second corresponds to 3 is this T invertible? T is invertible because the columns are independent so the homogeneous equation P x equal to 0 has x equals to 0 as the only solution this P is invertible and P is invertible and locate AP we have done this calculation before AP is A into x 1, x 2, x 3 this A can be brought inside Ax 1, Ax 2, Ax 3 but x 1, x 2, x 3 are the eigenvalues.

So this is minus 1 right minus x 1 minus x 1 3x 3 minus 1 minus 1 are the eigenvalues for the first two vectors. So the third one 3 is eigenvector, right. Can I write this as can I write this as x 1, x 2, x 3 into minus 1 0 0 0 minus 1 0 0 0 3 I can please check this I can write it in this manner. In our notation this matrix is P let me call the other matrix as D, D stands for the diagonal matrix sorry yes this is P into D, okay.

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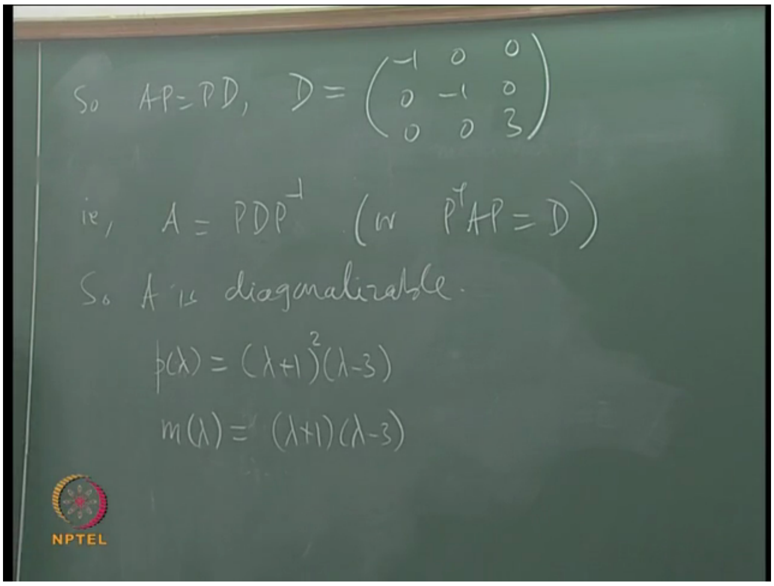

Where D is that is A equals PDP inverse we can also look at P inverse AP as D this is what diagonalizability for a matrix means this is what diagonalizability for a matrix means I have also mentioned this when we wrote down the matrix of a linear transformation A is similar to the diagonal matrix D, A is similar to the diagonal matrix D and so A is diagonalizable, T is diagonalizable if and only if A is diagonalizable, A is diagonalizable means that A must be similar to a diagonal matrix, okay so this is the first example of a matrix that has been shown to be diagonalizable.

What is the minimal polynomial for this matrix? We have written on the characteristic polynomial. Let me recall the characteristic polynomial I am using the notation p for this matrix is lambda plus 1 to the 2 into lambda minus 3 probably it goes with the minus sign but does not matter I am looking at p of lambda equals 0, okay. So the characteristic polynomial is lambda plus 1 square lambda minus 3, what is the minimal polynomial for this, where do we start with, what is the first choice for the minimal polynomial? Lambda plus 1 into lambda minus 3, okay.

One choice is lambda plus 1 into lambda minus 3, what is the other choice it must it must also have another possibility is lambda plus 1 into lambda minus 3 whole square, right the number of the zeros must coincide, okay but for this matrix I wanted to verify that the minimal polynomial is just lambda plus 1 into lambda minus 3 please verify for this matrix that the minimal polynomial is lambda plus 1 into lambda minus 3 the reason why this must be true I will explain a little later but this can be verified by just two matrix multiplication you need to verify that m of A equals 0 that is A plus I into A minus 3 I is 0, okay.

So please verify in this example where the minimal polynomial is this and observe that the minimal polynomial is a product of distinct linear factors this example the minimal polynomial is a product of distinct linear factors that is it does not have lambda plus 1 whole square or lambda minus 3 whole square it is just a product of these two distinct linear factors, okay that is the minimal polynomial for this example.

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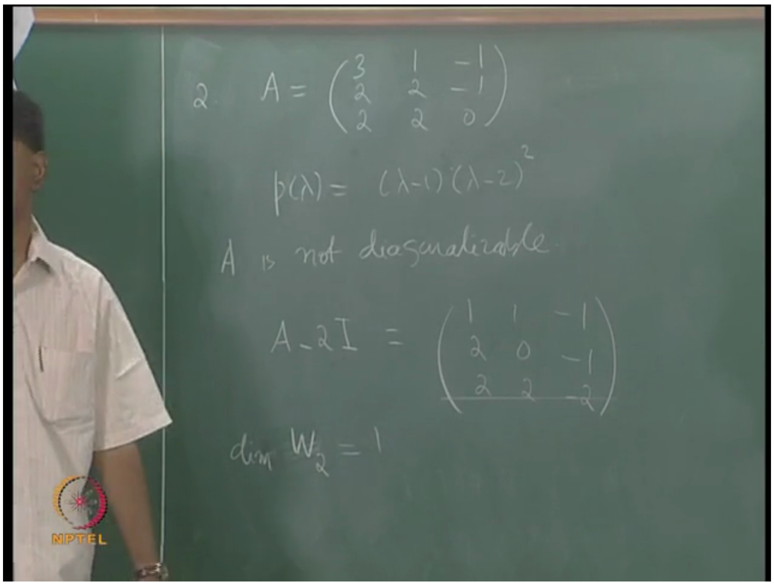

Let us go back to the second example the second example let me discussed earlier tell me if the entries are correct 3 1 minus 1 2 2 minus 1 2 2 0 can you just check and tell me if this is the second example, this is an example of a matrix which is not diagonalizable in spite of the fact that the eigenvalues exist are the entries okay okay.

What is the characteristic polynomial for this matrix we have verified what lambda minus 1 the whole square into lambda minus 2, okay lambda minus 1 into lambda minus 2 whole square this matrix is not diagonalizable. Simply because it is defective with regard to the number of eigenvectors there is no problem with the eigenvalues, eigenvalues are there 1 comes with multiplicity 1, 2 comes with multiplicity 2 but it does not have enough eigenvectors it has only two eigenvectors that is if you look at the matrix lambda minus if you look at the matrix A minus 2I we have discussed the example before I just want to consolidate.

If you look at the matrix A minus 2I that is 1 1 minus 1 2 0 minus 1 2 2 minus 2, okay you observe that the third row is a same I mean is a multiple of the second. So I can delete it their rank is 2 the rank of this matrix is 2, the nullity of this matrix is 1 that is nullity of the linear transformation whose matrix is this nullity is 1 which means the dimension of the null space of A minus 2I is 1 but dimension of null space of A minus 2I is precisely the eigenspace the eigenspace is of dimension 1, okay.

So that is only 1 eigenvector corresponding to the eigenvalue 2, there is 1 eigenvector corresponding to 1 in any case I do not have 3 independent eigenvectors for this matrix so this is not diagonalizable, okay. So please check that nullity check that the eigenspace corresponding to the eigenvalue 2 that is 1 dimension of the eigenspace corresponding to the second eigenvalue is 1.

And so there are only 2 independent eigenvectors for this matrix A so A is not diagonalizable, what are the choices for the minimal polynomial? The choices are lambda minus 1 into lambda minus 2, lambda minus 1 whole square into lambda minus 2, or lambda minus 1 into lambda minus 2 whole square, okay these are the possible choices. I wanted to verify in this example that this is the same as the minimal polynomial.

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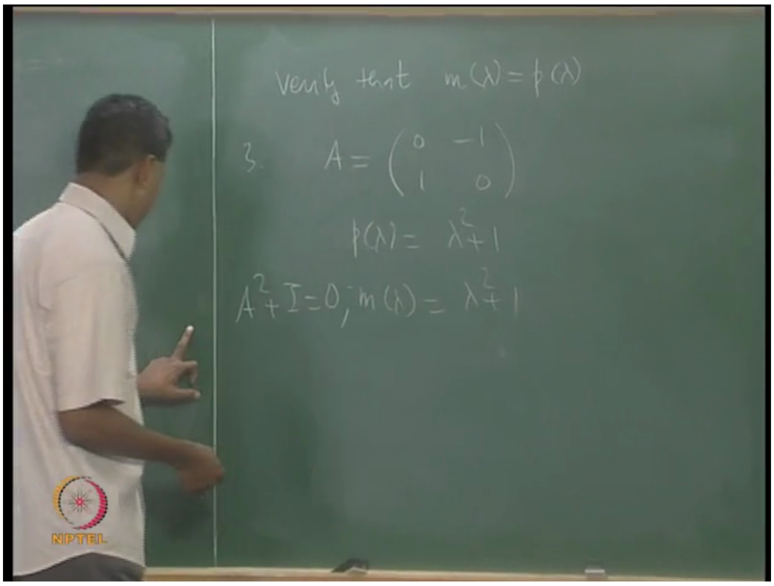

Please verify that the minimal polynomial in this example is same as the characteristic polynomial I told you there are three choices for the minimal polynomial one is lambda minus 1 into lambda minus 2 you can verify A minus I into A minus 2I is not 0, the other choice is lambda minus 1 whole square into lambda minus 2 that is A minus I the whole square into A minus 2I verify that that is also not 0, finally this is the other choice verify that m lambda is p lambda, okay in this example.

Finally if you look at the third example which is the first example the matrix A is 0 minus 1 1 0, this simply does not have eigenvalues, what is the characteristic polynomial for this matrix? Characteristic polynomial is lambda square plus 1 if you look at this as a complex matrix this can be factorized as lambda plus I into lambda minus I so as a complex matrix this has I and minus I as the eigenvalues, okay.

The minimal polynomial is a product of if you look at it has a complex matrix then the minimal polynomial will have to have both these roots, you please also verify that A square plus I equals 0 and so the minimal polynomial in this example is equal to the characteristic polynomial, okay the minimal polynomial is the characteristic polynomial for the reason that A square plus I is 0 and that both both the factors both the roots minus I and plus I must appear in the minimal polynomial, okay okay probably I will stop here.