Linear Algebra
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Module 7 Eigenvalues and Eigenvectors
Lecture 26
Eigenvalues and Eigenvectors of Linear Operators

In the next few lectures we will discuss the notion of eigenvalues, eigenvectors, etc but the motivation comes from the following problem.

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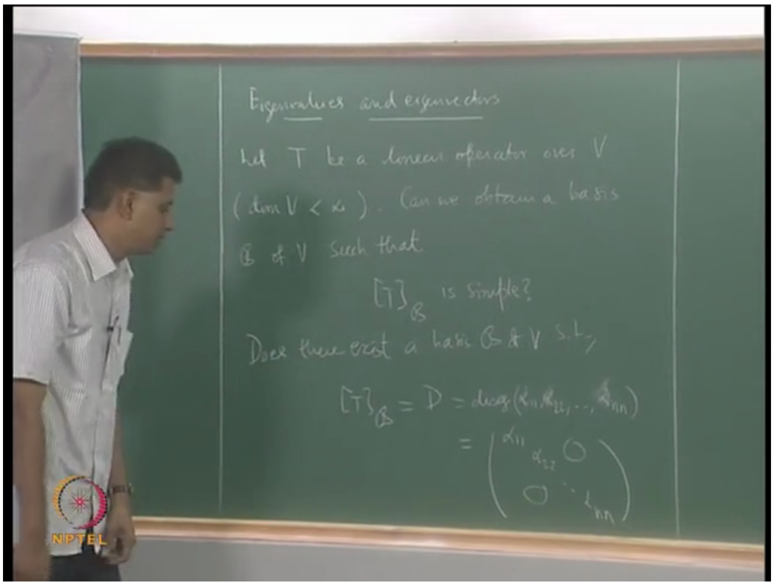

Okay, let me write down the topic first eigenvalues and eigenvectors for linear transformations, okay. Let us say we have an operator T over V a finite dimensional vector space V, okay then we know how to write down the matrix of the linear operator corresponding to a basis. The question is can we obtain a basis script B such that such that the matrix of T relative to this basis B B but we use the short notation such that this is simple given a basis is there a basis? The question is that is there a basis such that the matrix of T relative to that basis is simple what we need to interpret this word simple make it precise.

The simplest linear the simplest matrices apart from the scalar matrices that is the matrix is called the scalar matrix if it is k times identity that is the simplest. The next simplest is the class of diagonal matrix. So we will ask this question to begin with can we find a basis B of V such that the matrix of T relative to that basis is a diagonal matrix, okay. So this is one of the interpretations of what a simple operators.

So let us precisely ask this question does there exist a basis B of V such that the matrix of T relative to that basis is a diagonal matrix let me use the letter capital D for that what is this this is diagonal of let us say d 11, d 22, etc d nn that is this matrix has this form okay I will choose alpha, alpha 11, alpha 22, etc alpha nn all other entries are 0 all other entries are 0 the diagonal entries are these numbers alpha ii. Now it is quite possible that some of this alpha ii's are also 0, okay but that does not matter, off diagonal entries are 0 that is what I am imposing on the matrix of a linear transformation relative to a fixed basis B.

What is the advantage of this? First of all what is the meaning of this, what does it say about the transformation T and what is advantage of this? If a matrix T if linear operator T has this property then we will say that T is diagonalizable, okay.

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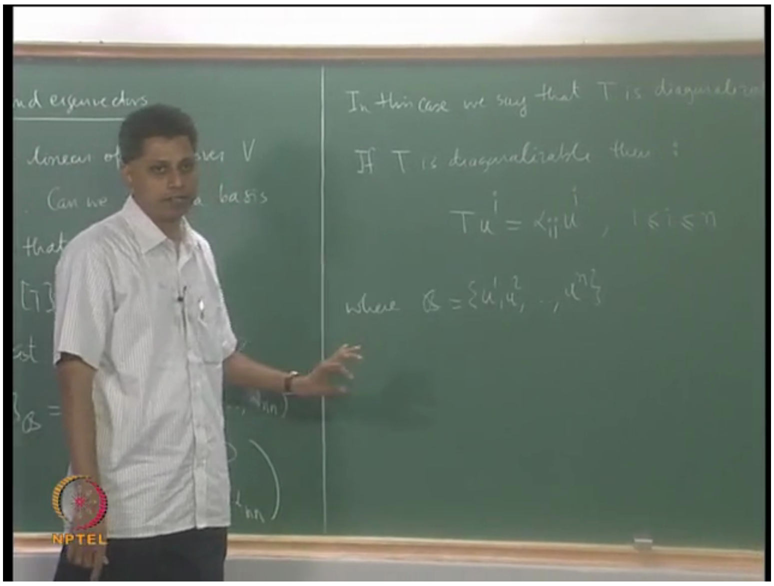

Let us first give this definition we will simply say that in this case T is said to be diagonalizable in this case we say that T is diagonalizable we say that T is diagonalizable. The question is are all operators on finite dimension vector space is diagonalizable? Before answering this question let us see what it means so what is advantage of suppose we know that an operator is diagonalizable then what are the advantages?

If T is diagonalizable then we have the following I will use the previous notation that we have developed here, T is diagonalizable what this means is that T u 1 for instance can you see it is alpha 11 u 1 where I am using where I am using the notation u 1, u 2 etc for the basis (()) (5:52) if I denote the basis B by u 1, u 2, etc u n, n is the dimension of the space B then T u 1 is alpha 11, u 1 T u 2 was alpha 22 u 2 etc I have these equations T u i is alpha ii u i, okay T u i is alpha ii u i this describes we know that the action of T on a basis describes T completely this of course describes T completely.

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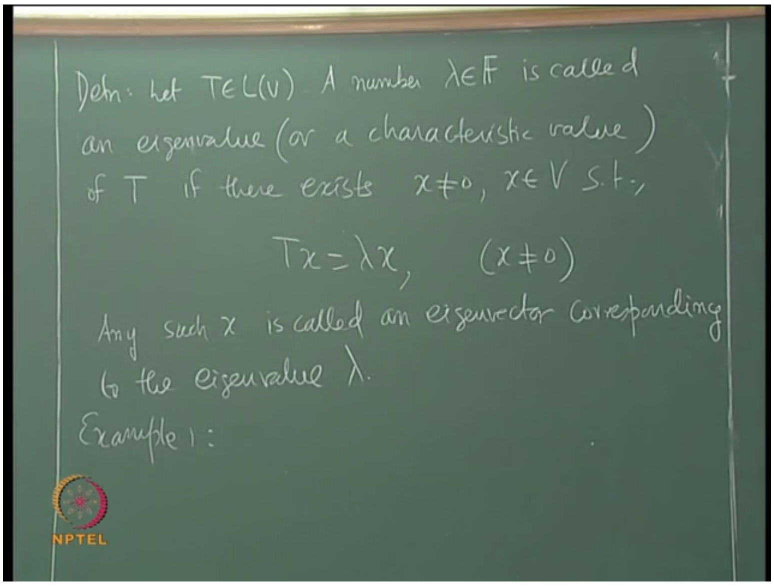

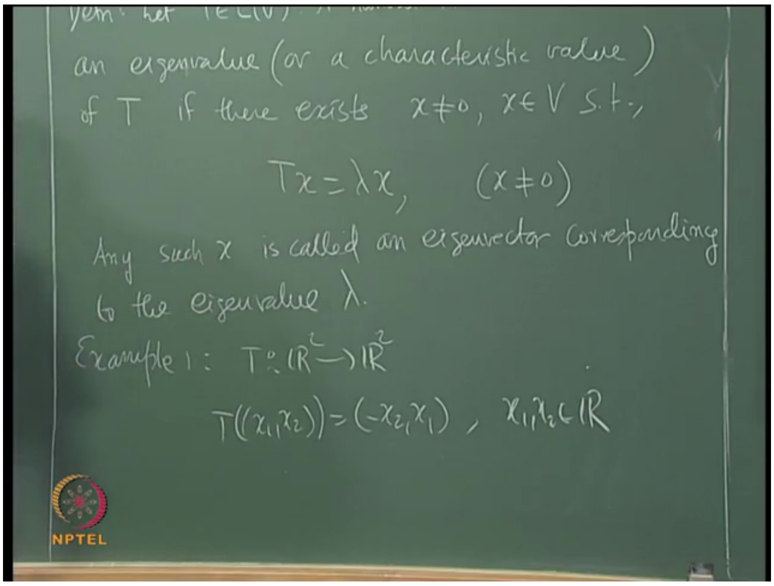

So let me give the definition of an eigenvalue let T belong to L V a number lambda in F F is underlying field either R or C ya no please sit this is correct a number lambda in F is called an eigenvalue or a characteristic value there are other names like latent value, etc a number lambda in F F is underlying field for me V is defined over F a number lambda in F is called an eigenvalue of T eigenvalue of T if there exist x not equal to 0 in V such that Tx equals lambda x.

Now let me emphasize that this x must not be 0 so I will write that here this should go along with the equation Tx equals lambda x, does this equation have a solution? For some x nonzero, for some number lambda this lambda comes from the underlying field, okay if it is a real vector space I demand that the number must be a real number if it is over the field of quotients the number lambda must be a fraction, okay.

So we demand that this lambda belongs to F then this lambda is called an eigenvalue any vector x that satisfies this equation is called an eigenvector. I will simply say any such x satisfying the above is called an eigenvector there is a correspondence is called an eigenvector corresponding to the eigenvalue corresponding to eigenvalue lambda any such vector x that satisfies this equation is called an eigenvector or a characteristic vector or a latent vector corresponding to the eigenvalue lambda, okay.

Question do all linear transformations have eigenvalues let us first look at a simple example maybe before we look at that example can you see that this is really solving a homogeneous equation it is like T minus lambda i of x is equal to 0 then I will look at the matrix of T relative to some basis then T minus lambda i x is equal to 0 is equivalent to a minus lambda i x is equal to 0, okay okay.

Let us look at first an example let us take the rotation transformation let us take the rotation transformation rotation transformation remember this equation Tx equals lambda x the operator T acts on x and then the resultant vector must be along the direction of x that is what this means the resultant vector must be along the direction of x multiple of x so rotation vector rotation matrix rotation operator will that have an eigenvalue? Geometrically.

Provided okay let us say what is the rotation matrix it corresponds to what cos theta minus sin theta sin theta cos theta let me say that the angle theta lies between 0 and 180 strictly does it have a eigenvector? So I have this in particular define T from R 2 to R 2 real space by T of x 1, x 2 is minus x 2, comma x 1. So this is what I said corresponds to 90 degrees really, right minus x 2, x 1 does this have an eigenvalue?

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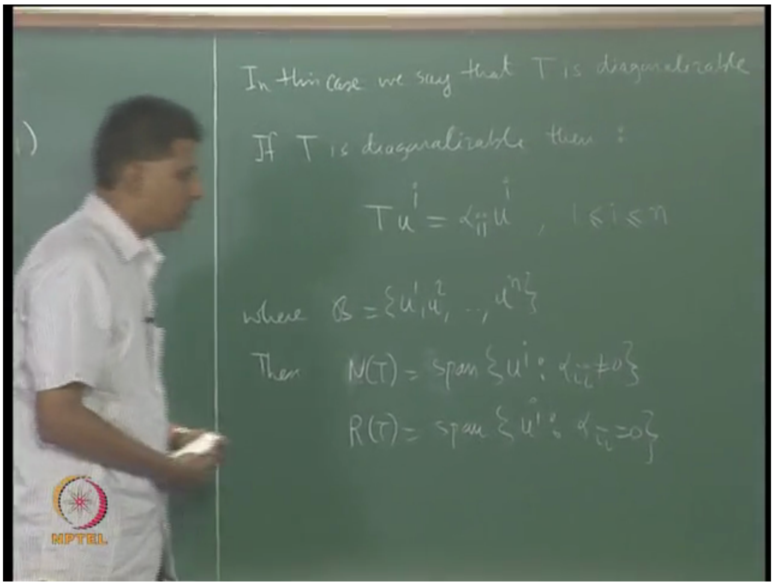

So for this problem let us do it by looking at the first definition I have look at (Tx) lambda x equals Tx equals minus x 2, comma x 1 but lambda x on the other hand is lambda x 1, lambda x 2, okay. Let us remove the case lambda as 0 if lambda is 0 can you see that x is 0 that is because this so I need I am looking for an eigenvector so this cannot be 0 so lambda cannot be 0, lambda 0 then x is 0 I am looking at an eigenvector I am seeking an eigenvector so lambda cannot be 0.

Now look at these two equations from this second one x 1 is lambda x 2, x 2 was minus lambda x 1 so this is lambda into minus lambda x 1 that is minus lambda square x 1 that is 1 plus lambda square x 1 is 0 lambda is a real number we are seeking lambda a real number. So from this it follows that x 1 is 0 go back to this equation x 1 is lambda x 2 lambda is not 0 x 1 is 0 so x 2 is 0 so x is 0. So in either case there is no nonzero vector x that satisfies this equation Tx equals lambda x, Tx equals lambda x is not satisfied by any nonzero vector x.

So this linear operator does not have an eigenvalue this linear operator T does not have an eigenvalue now this is the problem with the underlying field the underlying field is R R is not algebraically closed from group theory a field is said to be algebraically closed if any polynomial of degree n whose coefficients come from the underlying field has preciously n 0's.

What is the polynomial which does not have 0 here the polynomial T square plus 1 the polynomial p of T equal to T square plus 1 does not have a 0 over R, R is not algebraically closed. So the problem in this case the operator T does not have an eigenvalue has come from the deficiency of the field the deficiency is really from the field okay this is one possibility I will look at the other possibilities but before that for the case of 2 by 2 for an operator T on R 2 all this can be done or if I have an operator on R 3 then this is this can get quite complicated. So we need to translate this into the language of matrices. So let us translate this problem into one for matrices and then look at matrix.

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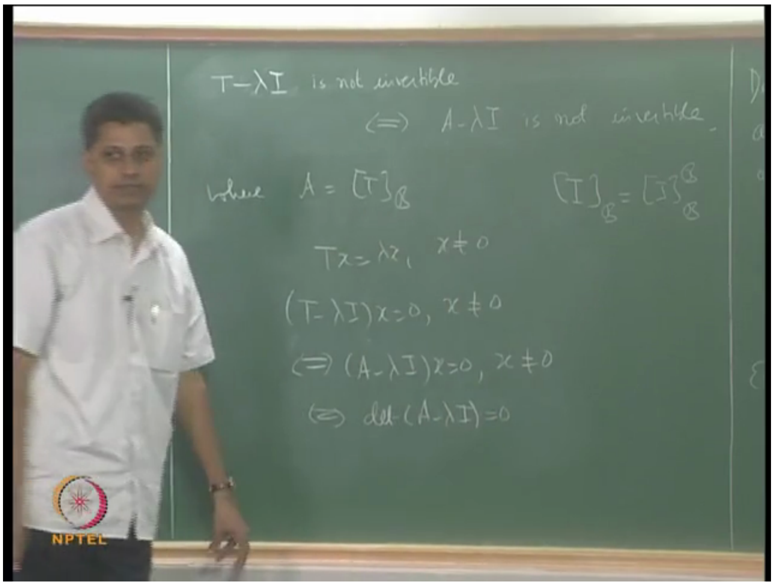

Let us remember that T is okay okay tell me this is okay T minus lambda I is not invertible if and only if A minus lambda I is not invertible we have proved this before where where A is a matrix of T relative to a particular basis B. Do you remember that we have proved this as a if T is a linear transformation A is a matrix of T and if T is invertible then A is invertible in fact the matrix of T inverse is the inverse of the matrix A, okay the matrix of T is the inverse of the matrix A.

So remember that here I is identity transformation I am applying T minus I am looking at the matrix of T minus lambda I relative to B I relative to B the matrix of identity transformation relative to B is identity matrix here here remember that we are okay I hope you remember that this is what we have relative to two basis we write a linear transformation if it is an operator then we work with only one basis, okay.

If you have two basis then the identity transformation relative to two basis need not be the identity matrix, okay but with respect to a single basis this is identity matrix. So let us remember this is the identity transformation, this is identity matrix. So this is not invertible if and only if this is not invertible, okay then the question boils down to. So I am seeking Tx equals lambda x with x 0 equal to 0 this is is the same as t minus lambda I x equal to 0 null space of T minus lambda I that is null space of A minus lambda I by this identification. So I have A minus lambda I x equal to 0 x not equal to 0.

Now the question is over homogeneous equations I have a matrix let us say B x equal to 0 I want x not equal to 0, I know that this has a solution if and only if the row reduced echelon form of this coefficient matrix has atleast one 0 the last row atleast the last row is 0 there may be more 0's there may be more 0 rows, okay.

In other words this matrix cannot be invertible if A minus lambda is invertible I can pre multiply by A minus lambda I to conclude that x is 0. So if I am seeking x to be nonzero then A minus lambda I cannot be invertible in terms of determinants this means determinant of A minus lambda I is 0.

So the question is does this equation have a solution now what kind of an equation is this? T having lambda as an eigenvalue reduces to lambda satisfying this equation what equation is this determinant ya it is a polynomial equation determinant expansion along let us say first row or the first column this is a polynomial of degree n it is a monic polynomial the coefficient of lambda to the n is 1 or minus 1 does not matter you have 0 here.

So this is a polynomial equation where the polynomial is of degree n it is what is called as a monic polynomial the coefficient of the highest degree is 1 so we need to solve this polynomial equation, okay.

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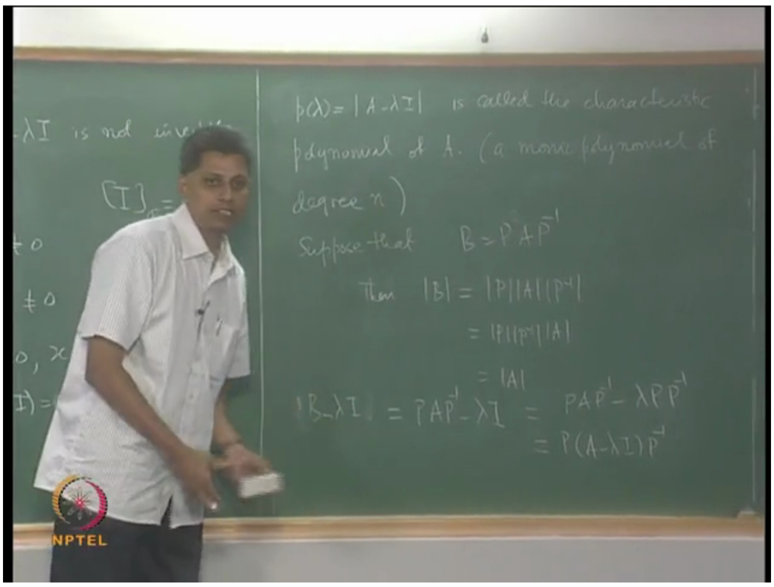

That polynomial equation is called the characteristic I will call (p of T) as okay p of lambda as determinant of A minus lambda I, okay this is this notation is for the determinant this is called the characteristic polynomial of A is called the characteristic polynomial of A we observe that it is a polynomial of degree this is the monic polynomial of degree n the monic polynomial of degree n.

Now this is the difficult problem in numerical linear algebra but we are going to do problems of the size 3 by 3 so it should not be difficult, okay before we proceed to other examples let us also make the following observation I am going to use some of the properties of determinants that I am sure you are aware of for instance determinant of a product is a product of the determinants, okay determinant of A B is equal to determinant A into determinant B, I am assuming that A and B are both square.

What about matrices that are similar to each other? Let us say P A P inverse I have two matrices A and B related by this equation B is similar to A for instance if I write down the matrix of a linear transformation relative to two basis let us say the matrix A with respect to one basis, matrix B with respect to some other basis then A and B are related by this equation, okay.

Suppose that B equals P A P inverse then can you do you remember that determinant of B is equal to determinant of A, determinant of B is determinant of P into determinant of A into determinant of P inverse that is determinant is multiplicate determinant A B is determinant A into determinant B. Now these are numbers this is determinant P into determinant P inverse into determinant A but determinant P determinant P inverse that is 1 so this is determinant of A, okay.

Similar matrices have the determinant this property we will need where do we need? We need in the following look at B minus lambda I, I have B equal to P A P inverse look at B minus lambda I, B minus lambda I let me write like this this is also equal to instead of B I have P A P inverse minus lambda I, I can write this as P A P inverse minus lambda P P inverse identity I have written like this.

Now I take P to the left P inverse to the right and write this as P into A minus lambda I P inverse. So B minus lambda I is P times A minus lambda I times P inverse again use the same formula apply determinants on both sides determinant of B minus lambda I is determinant of A minus lambda I that is if B is similar to A then determinant of B minus lambda is determinant of A minus lambda I, so is it clear from this that A and B have the same eigenvalues.

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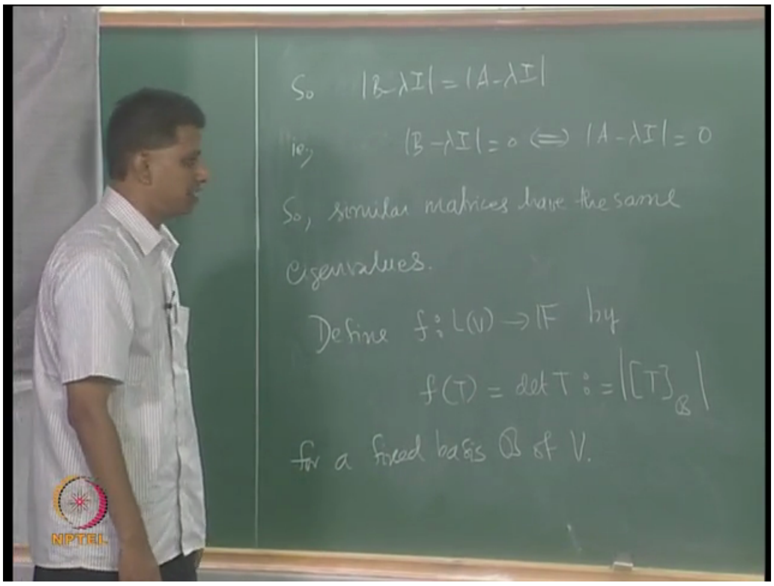

Similar matrices have the same eigenvalues similar matrices have the same eigenvalues one of the advantages of this result is that the following function is well defined I am calling the determinant of T, I am defining the determinant of a linear transformation, okay I define a function from the set of all linear transformations the underlying field this function I am calling as a determinant function.

Determinant of T is a matrix of T relative to B, B is any fixed basis B is any fixed basis is this well-defined? In other words if I change the basis will I am sorry I meant the determinant of that I meant the determinant of this matrix, okay is this well-defined? Determinant of the matrix of T relative to some basis relative to some fixed basis is this well-defined? If I change the basis will I get a different value for that determinant I will not because when I change the basis then the matrices are related by similarity transformation but we just now saw that similar matrices have the same determinant, okay.

So this is well defined this allows us to go from determinant of a matrix to determinant of a linear transformation, okay but let us get back to matrices that is what we want when we want to look at example.

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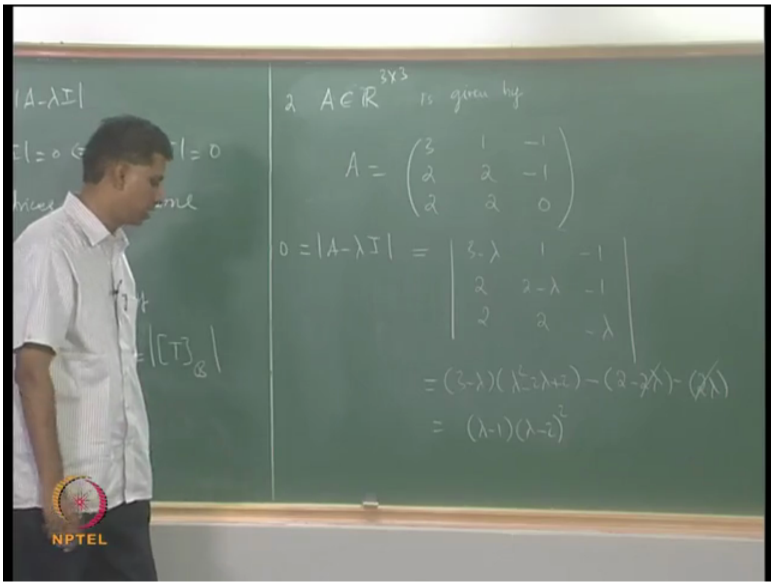

So let us look at the eigenvalues and eigenvectors of matrices first, okay I am really looking at second example suppose I have A defined as follows let us look at this matrix let us calculate the eigenvalues possibly the eigenvectors. Remember the first example of a linear transformation for which we try to compute the eigenvalues. In the first example we have seen that it has no eigenvalue that efficiency coming from the field what happens in this example?

I need to calculate the eigenvalues, eigenvectors I must look at the polynomial equation determinant of A minus lambda I equal to 0, okay so let us do this quickly this is A minus lambda I along the main diagonal I must delete lambda and then calculate the determinant. So I want the determinant of this matrix B minus lambda 1 minus 1 2 2 minus lambda minus 1 2 2 minus lambda.

Let us expand it along the first column, okay first row 3 minus lambda into lambda square minus 2 lambda plus 2 minus minus 2 lambda plus 2 2 minus 2 lambda please check the calculations minus just 2 lambda this and this get cancelled goes the (())(28:39) minus 2 please check that the rest of simplification follows lambda minus 1 into lambda 2 whole square please verify this expression simplifies to this lambda minus 1 into lambda minus 2 whole square this is the characteristic polynomial of A this matrix.

So we have no problem with regard to eigenvalues lambda equal to 1 is an eigenvalue with multiplicity 1, lambda equal to 2 is an eigenvalue with multiplicity 2, okay. So apparently no problem with regard to eigenvalues are concerned, what about eigenvectors?

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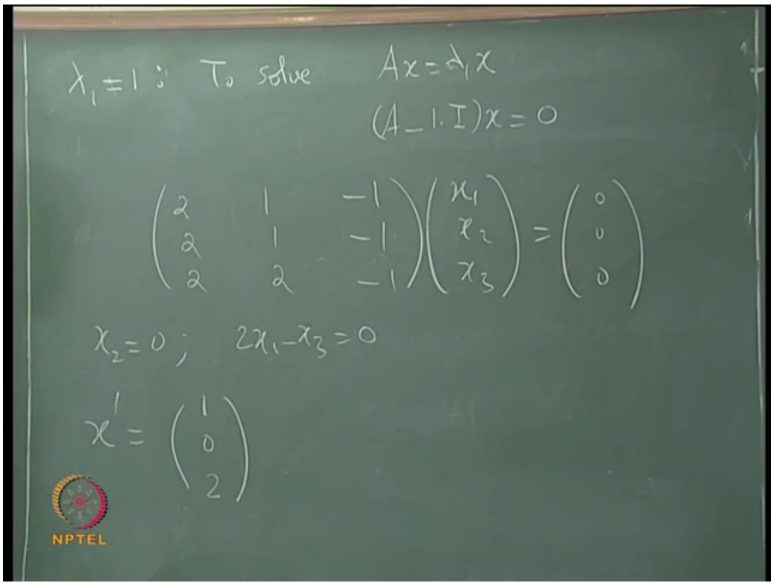

So let us take the case lambda 1 equals 1 we must solve this equation that is A minus 1 identity x is 0. So I must solve 2 1 minus 1 2 1 minus 1 2 2 minus 1 I am seeking nonzero x that satisfy this equation any to the elementary raw operations etc okay but I can observe quickly that this row can be deleted, I can push it to the last row then that will become the 0 row and then from the first equation and the third equation what follows is that the second coordinate is 0 we do not have to do elementary operations, second row can be removed it is same as the first row that in affect gives the 0 row at the bottom, I have only two rows first row is 2 1 minus 1 second row 2 2 minus 1 it is a homogeneous equation 2 x 1 plus x 2 minus x 3 2 x 1 plus 2 x 2 minus x 3 cancel one from the other I get x 2 to be 0.

So x 2 is 0 the other equation single equation is 2 x 1 minus x 3 equals 0 any vector x that satisfies these two conditions is an eigenvector corresponding to the eigenvalue 1 there are infinitely many I agree but there is precisely one independent vector. I will call it x 1 let us say first coordinate is 1, third coordinate will be 2, second coordinate I know is 0 there is precisely one independent vector that satisfies these two equations any other vector is a multiple of this it is a homogeneous equation any other solutions is a multiple of this that is because this is two see this is actually two equations in three unknowns we must fix two one of them is already fixed either x 3 or x 1 I must fix for convenience I fix x 3 then I can determine x 1 in terms of x 3, three equations in two unknowns two must be fixed x 2 is fixed to be 0 fix x 3. So there is only one solution the dimension of the solution space is 1 so there is only one independent solution for this.

So this is an eigenvector corresponding to the eigenvalue 1 any other vector that satisfies a x equals lambda x will be a multiple of this because it is a homogeneous equation any multiple is also a solution. So this is one eigenvector, what about the eigenvalue 2?

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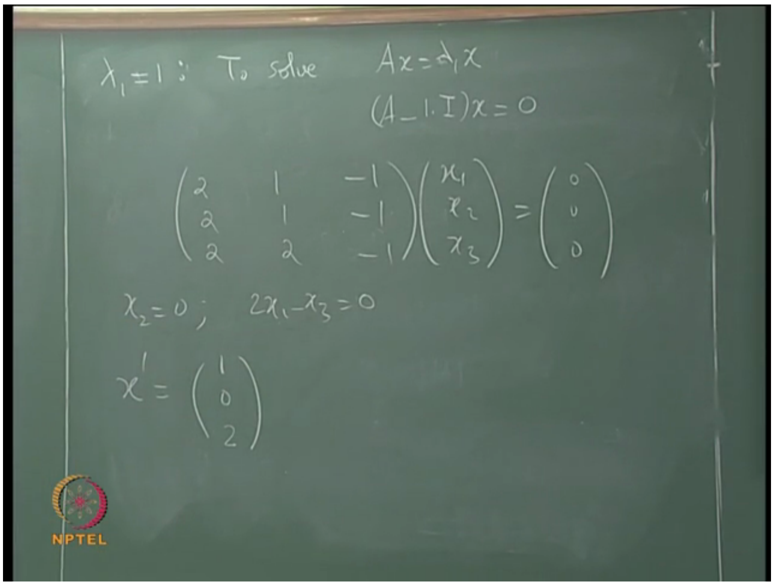

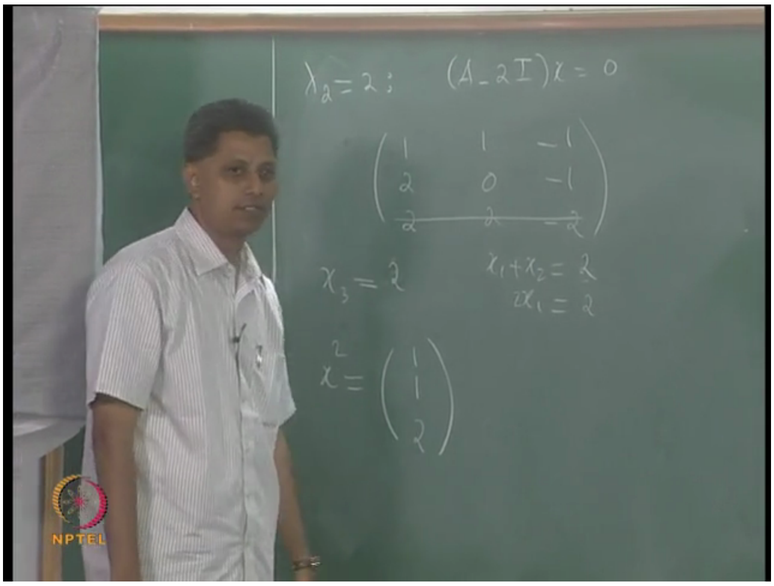

I need to solve A minus 2 times I x is 0 with x not 0 so I must delete 2 1 1 minus 1 I must delete 2 this into x is 0 this into x is 0 I observe that this is the multiple of the first equation so this gives me 0 row. Now I have two equations in three unknowns I must fix fix only one of them the other two can be determine in terms of this is it okay you can do elementary operations that is obvious the rank is 2 the rank is 2 so nullity is 1.

So let us say for convenience I fix x 3, I fix x 3 to be 1 by the way if you go back to this go back to these equations I have taken x 3 to be x 3 to be 2, can I take x 3 to be 0? If I take x 3 to be 0 it follows that x 1 is 0 so I will get 0 0 0 but I want a nontrivial solution, okay that is the reason why I have taken x 3 to be 2 we are looking at nontrivial solutions. I do a similar thing here see if I take x 3 to be 0 if I take x 3 to be 0 then this is gone I must look for x 1 x 2 such that this into x 1 x 2 is 0 this is an invertible matrix I will again get 0 so I cannot take x 3 to be 0.

So one of the nonzero choices simplest x 3 equals 1 then I need to solve for x 1 plus x 2 is 1 2x 1 is 1 so I will change it to 2 only for convenience 2x 1 is 2 is that okay these two equations. So x 1 is 1 I will call this x 2 the new vector that we obtain by solving this equation x 1 is 1, x 3 is 2 (oh this one is wrong) is it okay now 1 1 2 2 minus 2 1 plus 1 2 2 minus 2 so this is the this is the only linearly independent solution of this equation any other solutions are multiple of this, okay.

So this time even though we have obtained three eigenvalues without counting multiplicity if I count multiplicity there are only 2 eigenvalues 1 come once 2 comes twice but let us say it has three eigenvalues I do not have three eigenvectors I do not have three eigenvectors this is not the deficiency of the field this is the deficiency of the operator this is the deficiency of the operator or deficiency of the matrix that we started with, okay.

Now even at this point you can verify that this operator is not diagonalizable because it has only two independent eigenvectors, okay you can take this as an exercise this operator is not diagonalizable because it has only two linearly independent eigenvectors.

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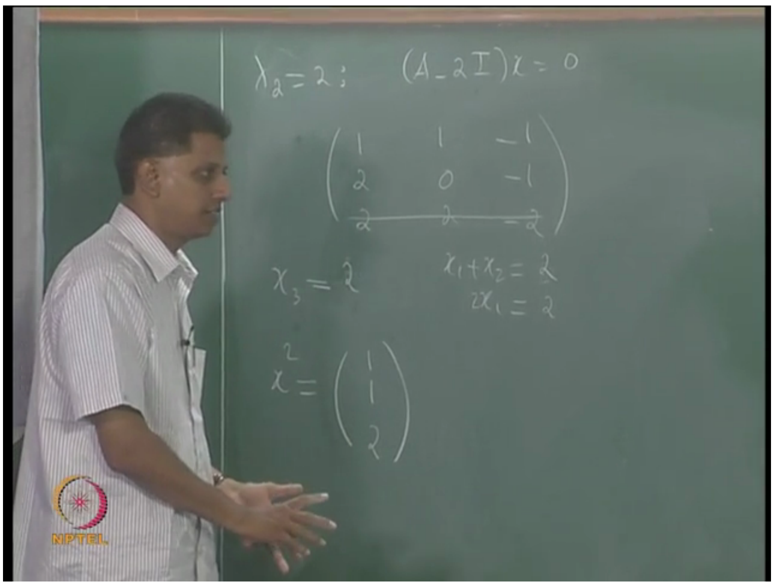

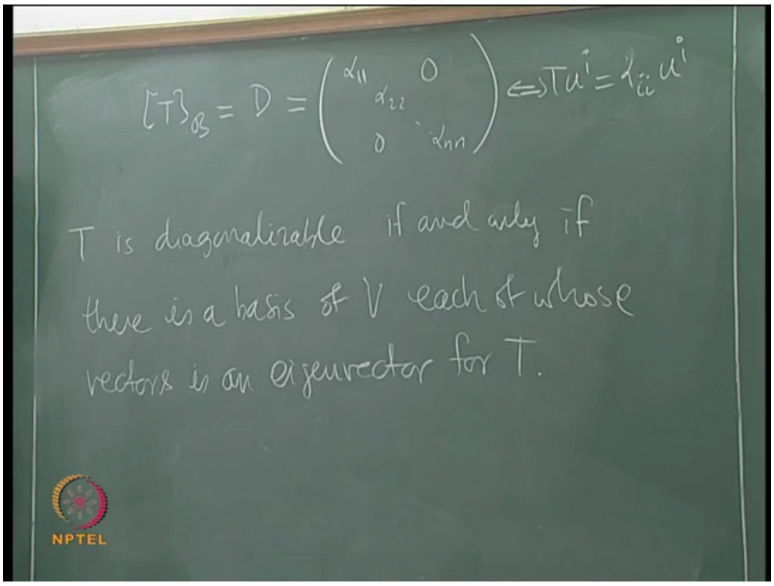

So let us see the connection see the connection between eigenvalues, eigenvectors and diagonalizability. So remember we said that T is diagonalizable if the matrix of T relative to B is a diagonal matrix, can you see that from this it follows that T is diagonalizable if and only if there is a basis of V each of whose vectors is an eigenvector for T, T is diagonalizable if and only if there is a basis B of V with the property that each basis vector is an eigenvector.

The reason why this is true is that if you write down this then this goes along with the equation Tu i equals alpha ii u i that is the reason why this statement is true is this clear? What this means is that if T is diagonalizable then each of the basis vectors that I started with is an eigenvector u 1, u 2, etc u n are basis vector so they are not 0 so they are eigenvectors corresponding to the eigenvalues alpha ii.

Now what are alpha ii? These are the diagonal entries of this diagonal matrix, okay. So if T is diagonalizable then each of the basis vector that I started with is an eigenvector conversely if I have a basis u 1, u 2, etc u n such that each each vector there is an eigenvector then I must have some such equation satisfy I must have Tu i equal some gamma ii u i. If I get this then I write down the matrix of T relative to that basis u 1, u 2, etc u n I must get the diagonal matrix so these statements are equivalent, okay.

Now go back to the previous example the previous example we have only two independent eigenvectors, so T is not diagonalizable the matrix A is not diagonalizable that language of the transformation the linear transformation T induced by A is not diagonalizable is this clear? Non diagonalizability can come from two factors one from the underlying field, the other one the inherent nature of the transformation T, okay.

In the second example the problem is the transformation T, okay. In the next lectures let us look at necessary sufficient conditions for diagonalizability apart from this apart from this necessary sufficient conditions for diagonalizability and then properties, examples, okay let me stop here today.