Linear Algebra

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Module 6 The Dual Space
Lecture 25

The Transpose of a Linear Transformation. Matrices of a Linear Transformation and its Transpose

So let us move to the next topic as I told you yesterday the last lecture we will discuss the transpose of a linear transformation prove the natural proof the fact that the transpose of a linear the matrix of the transpose of a linear transformation respective to the dual basis is the transpose of the matrix of the original transformation with respect to given basis this is one of the results that we will proof we will also proof this row rank equals column rank in a much simpler manner using the notion of transpose, okay.

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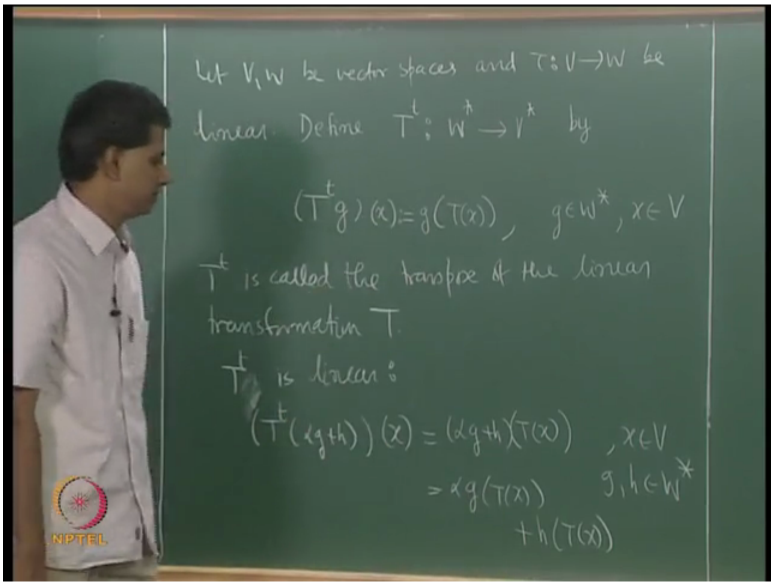

So I need to define the transpose of a linear transpose I have V, W vector spaces over the same field over the same field and T is linear be a linear transformation. I will define T transpose T transpose is from the dual space of (W) W star to the dual space of (V) V star define T transpose by T transpose g, T transpose domain is W star co domain is V star. So T transpose g must be a linear functional on V first thing T transpose of something belongs to V star so T transpose of that must be a linear functional on V then this g belongs to W star.

So let me say that g belongs to W star and T transpose g g is in W star so I will look at a mapping is defined if its action on an arbitrary element is defined T transpose g of x T transpose g of x this is a function this is a function this is this must be a functional on W. So T transpose g of x what is the definition? It is defined as g of T x, okay tell me if this is clear I am defining T transpose g at x as g of T of x. Remember this x belongs to V, T is the linear transformation from V to W. So T of X is in W g is the functional linear functional on W so g of this this is in W that we have seen just now.

So this right hand side is clear that is the definition of T transpose g. This T transpose is called the transpose transpose of the linear transformation is called the transpose of the linear transformation. What we have not shown is as yet is that T transpose g is linear only if I show that this is linear I can write I can write this, it is a linear function on V, I must show that I have not yet established so let us do that quickly.

T transpose g is (linear) sorry T transpose is linear I am going to proof that T transpose is linear T transpose is linear T transpose of alpha g plus h equals alpha T transpose g plus T transpose h. So consider T transpose alpha g plus h this function is known by its action an arbitrary vector. So this of x for x and V this by definition T transpose of something some function f of x is f of T of x alpha g plus h T of x I have use one more bracket but remember that (alpha) g and h are from W star and W star is a vector space we need to look at the operation in that vector space it is precisely alpha g of T x plus h of T x but go back to this definition.

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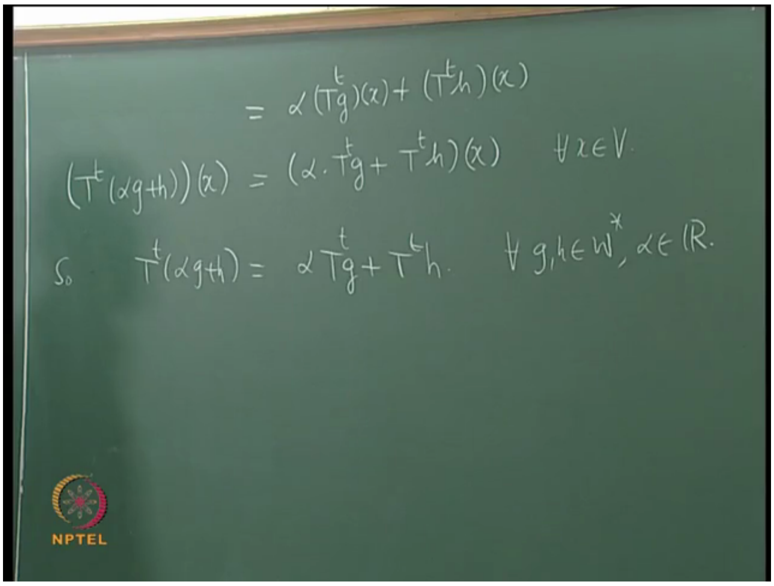

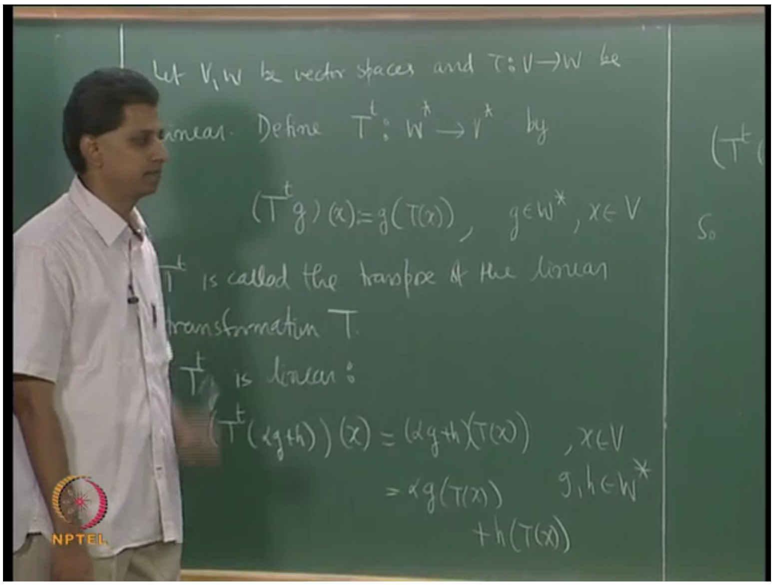

This is the same as alpha T transpose g of x plus T transpose h of x, I can write this as this is like addition of two functions f plus g of x is f x plus g x point wise. So this is alpha into T transpose g plus T transpose h acting on x so what did I start with I started with T transpose of alpha g plus h of x, I have shown that this is what I have on the right hand side this is true for all x in V for one thing.

So what it means is that f x equals g x for all x then f is equal to g. So T transpose alpha g plus h is equal to alpha T transpose g plus T transpose h now this equation I know holds for all g, h in W star and alpha from the underlying real field, okay. So it is an easy exercise really provided you understand the definition the definition of the transpose is given by this formula, okay.

Let us so what is it that we have shown we have shown that T transpose is linear and so T transpose g is a functional on V so T transpose is a linear functional on V, I am sorry ya T transpose g is a linear function on V so it belongs to V star, okay.

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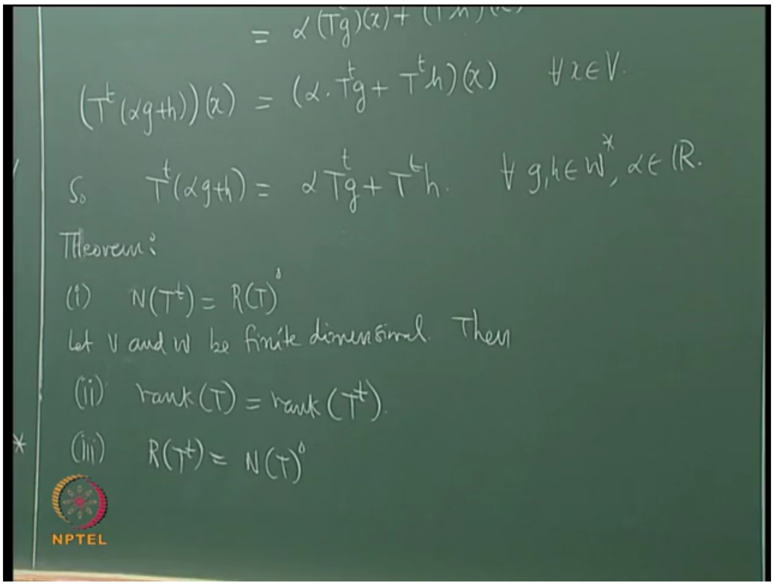

Let us look at some properties of this transpose really one theorem where I list all the properties. I remember that I do not take I do not demand that V and W are finite dimensional this definition goes through for a arbitrary vector space, okay. First property is the following one can show that null space of T transpose is the annihilator of range of T this is the first property for properties 2 and 3 to hold we need V and W to be finite dimensional.

Let V and W be finite dimensional then condition 2 property 2 that transpose operation transpose map satisfies is that rank of T is equal to rank of T transpose rank of T is equal to rank of T transpose. In order to talk about rank rank is a dimension of the range of T for instance in order to talk about the dimension of the range of T we must know that W is finite dimensional.

In order to talk about rank rank of T transpose you observe that T transpose is a functional on V so I need to know that V is also finite dimensional. So V and W have been assumed to be finite dimensional this is this is the formula that will really lead to this row rank equals column rank. And property 3 property 3 is range of T transpose equals the annihilator of null space of T range of T transpose equals null space of annihilator of T, okay we will proof this and then look at one or two consequences really.

Before proving let us understand this equation see on the right hand side I have annihilator of range of T, T is a linear transformation from V to W. So range of T is a subspace of W and annihilator of that means it is a functional on W. So anything in range of T-annihilator belongs to W star, look at what we have on this side null space of T transpose does that also belongs to W star that is what is the case T transpose null space of T transpose will be contained in W star so it is first meaningful to write down such a such an equation, okay let us proof this.

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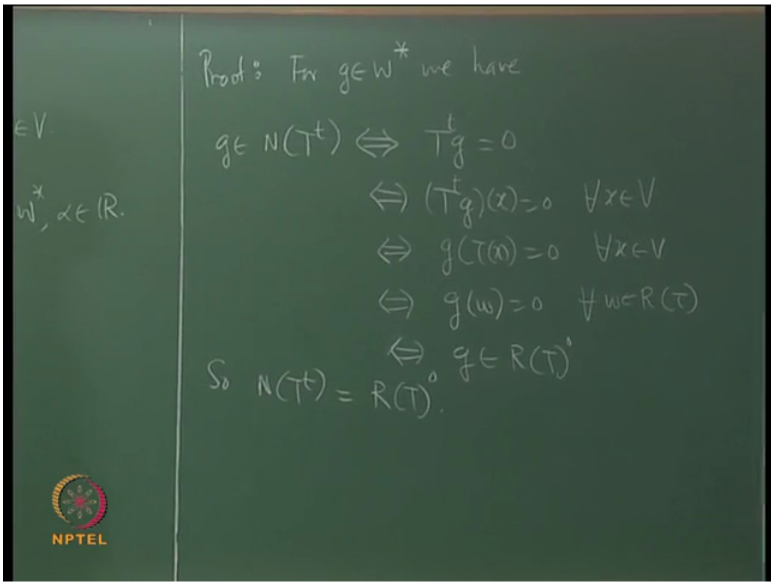

We want to show that two sets are equal so that each is contained in the other and I observe that these are subspaces of so let us take g in W star for g in W star we have the following g belongs to null space of T transpose implies T transpose g equals 0 g belongs null space of T transpose, T transpose g remember is in V star so it is functional it is a functional it is a function a function is 0 means its action on any element is 0.

So this implies what we also know that it is a functional on V so T transpose g of x this is 0 for all x in V the function f is identically 0 if f of x equal to 0 for all x in the domain. T transpose g x let us use the definition of T transpose this means g of T of x is 0 for all x in V definition of T transpose g of T of x is 0 for all x in V. Now when x varies in V this T x is in W and this T x is in the range of T.

So from this can I write g of W equal to 0 for W in range of T, I want to show g of W equal to 0 for all W in range of T, let us take W in range of T then W can be written as T x, question is g of T of x is that 0? Yes, g of T of x is 0 for all x, g of T of x is 0 for all x the thing you need to observe is that T x belongs to range of T. So g on any element in the range of T must be 0 this is preciously what we mean by saying g belongs to range of T-annihilator, g belongs to range of T-annihilator range of T is a subspace if what is the notation I am using W let us say if Z is a subspace of a vector space V then Z 0 is the set of all functionals f that satisfy the condition that f of Z is equal to 0 for all Z in Z that is precisely what we have done.

So g belongs to annihilator of range of T can we retrace the steps what have we proved? Null space of T transpose is contained in range of T-annihilator, arbitrary g in null space of T transpose we have shown g must belong to range of T-annihilator to show the we also need to show that range of T-annihilator is contained in null space of T transpose but can we retrace the steps here?

Suppose g belongs to range of T-annihilator can I conclude g of W equal to 0 for all W in range of T that is the definition, so from here to here I am allowed to go, can I go from here to here? g of W is 0 for all W in range of T, can I conclude g of T of x is 0 yes that is the definition, g of W equal to 0 in place g of T of x is 0 from here it is the definition of the transpose.

So from here to here I have gone just now, from here to here definition of transpose if I have a function f such that f of x equal to 0 for all x then f is 0 function. So from here to here I can go and if T g is 0 that is same as saying g belongs to null space of T transpose so these two sets are the same, is that clear. So this proofs the first formula so null space of T transpose is the annihilator of range of T.

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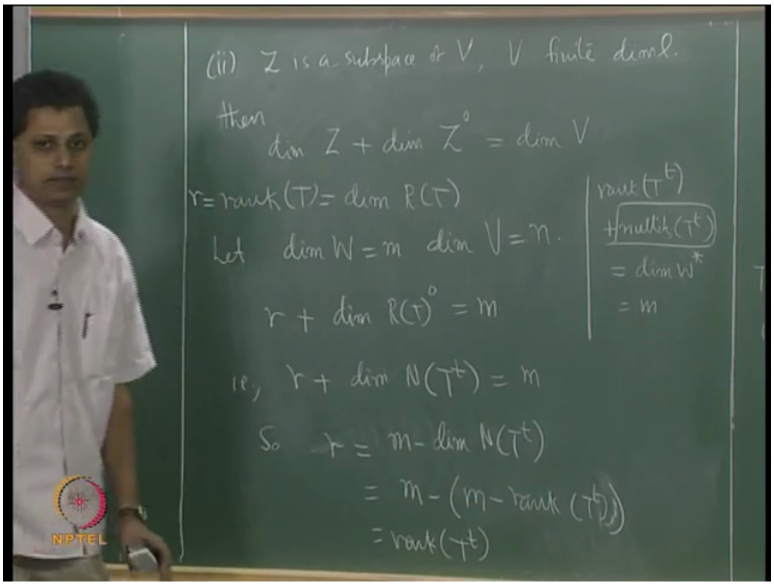

Let us proof the second formula when the spaces are finite dimensional, proof of the second formula recall that if Z is a subspace of V, V is finite dimensional then this was shown in the last lecture or in the previous class then dimension of Z plus dimension Z 0 this is equal to dimension of V, so I will make use of this formula. Let me start with range of T, rank of T is the dimension of the range of T, I will use r for that r is a rank of T that is the dimension of the range of T, I will apply this formula for the subspace range of T that happens in W.

Let me now assume that dimension W is m, dimension V is n. Now look at rank plus dimension of range of T-annihilator this is subspace of W so this is equal to m but we have just now shown that range of T-annihilator is null space of T transpose, okay r equals m minus dimension null space of T transpose. Now while apply rank nullity dimension theorem for the operator T transpose, rank nullity dimension theorem for the operator T transpose I must know that the domain of T transpose is finite dimensional domain of T transpose is W star, W is finite dimensional so W star is finite dimensional.

So I can write this as m minus rank plus nullity is a dimension, rank of T transpose plus nullity of T transpose equals dimension of what dimension of W star that is m, rank plus nullity of T transpose, I want nullity of T transpose, nullity of T transpose is m minus r I am sorry not r see I am using this equation.

May be I will write it on this side somewhere I am using this equation rank of T transpose plus nullity of T transpose equals dimension of the domain of T transpose that is W star that is m, from this I will take this number. Nullity of T transpose is this number dimension of null space of T transpose nullity of T transpose nullity of T transpose is m minus rank of T, sorry rank of T transpose nullity of T transpose is m minus rank of T transpose, cancel m rank of T transpose.

So I have shown that r is equal to rank of T transpose but r is by the definition of rank of T, okay. So rank nullity dimension theorem plus this formula connecting the dimension of a subspace and the dimension of its annihilator, is this okay?

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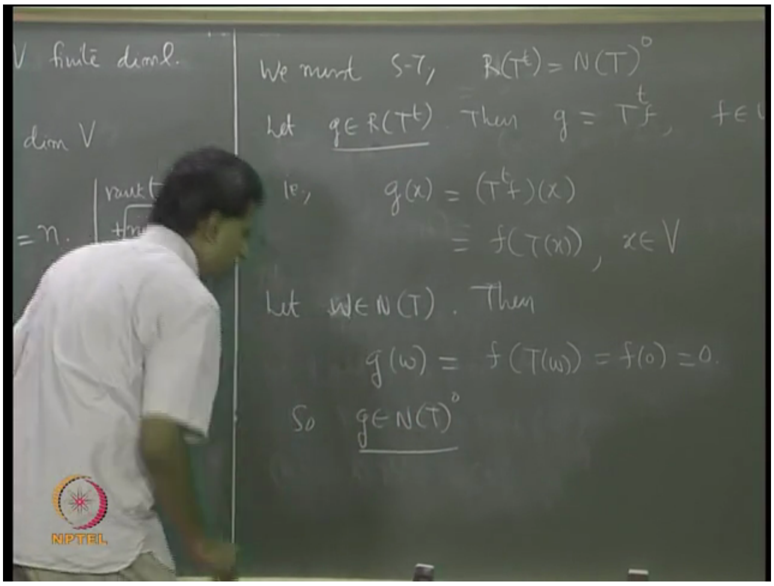

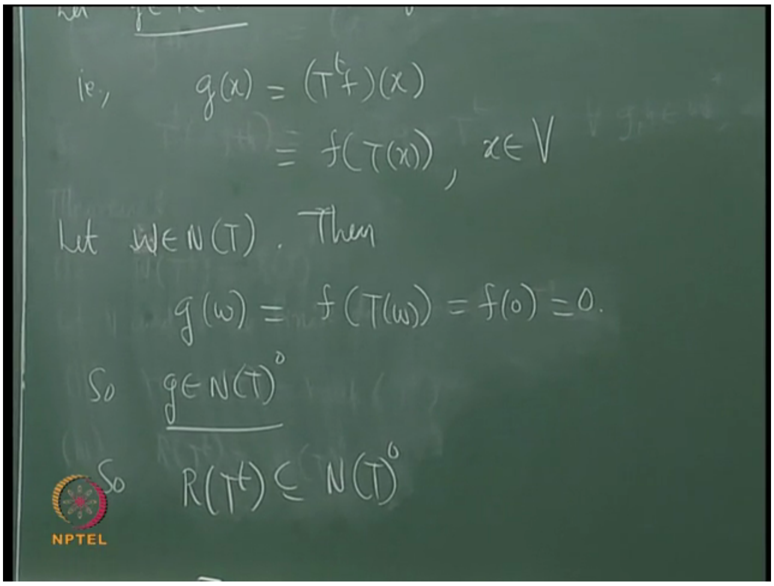

Finally the last formula let me write down we are seeking to proof that range of T transpose equals null space of T-annihilator. Let me first proof that this left hand side subspace is contained on the right hand side subspace and proof that their dimensions are the same then it will follow that they are the same, okay we have seen this kind of an argument before.

So let us take g in range of T transpose I am seeking to show that g belongs to null space of T-annihilator that is I want to show that g of W is 0 whenever W belongs to null space of T, okay. Let g belong to range of T transpose an element is in the range of a transformation. So this g can be written as g can be written as T transpose f for some f in W star ya that is correct T transpose is a function from W star to V star so g is T transpose f that is what is the meaning of this g of x is equal to T transpose f of x which by definition of the transpose is f of T of x, x belongs to V, g must g ya g is a functional on V so g belongs to V star so this x belongs to V, okay.

Now I look at now I look at let us say W suppose W belongs to null space of T, I am seeking to show that g of W is 0. Suppose I show g of W is 0 then it means g belongs to null space of T-annihilator but that is straight forward. Look at g of w, g of w is equal to anything in W I have this formula f of T of w but W belongs to null space of T, so T w is 0 so this is f of 0 f is linear is 0 so 0 is 0.

So if W belongs to null space of T then we have shown g w is 0 so g is a functional that must belong to the annihilator of null space of T. So g belongs to annihilator of null space of T, I started with g in range of T, I have shown this so it follows that range of T transpose is a subset of subspace of null space of T-annihilator I will next show that dimensions coincide so it follows that these two subspaces are the same.

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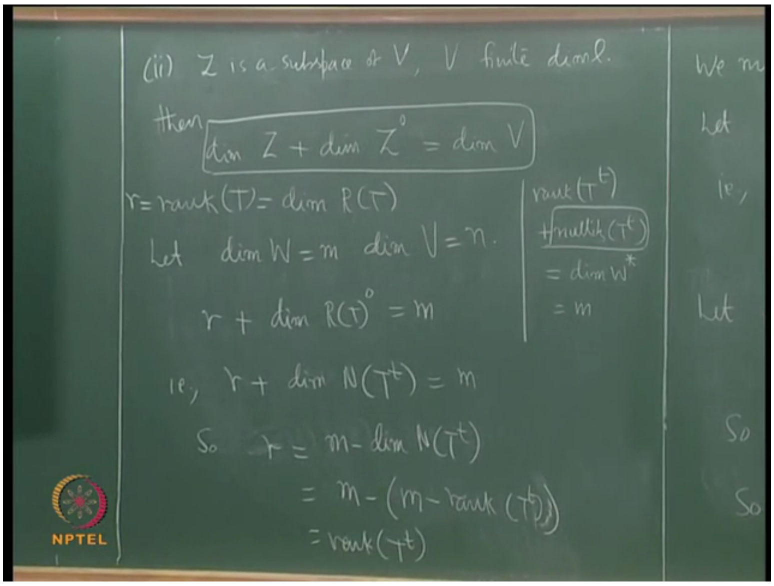

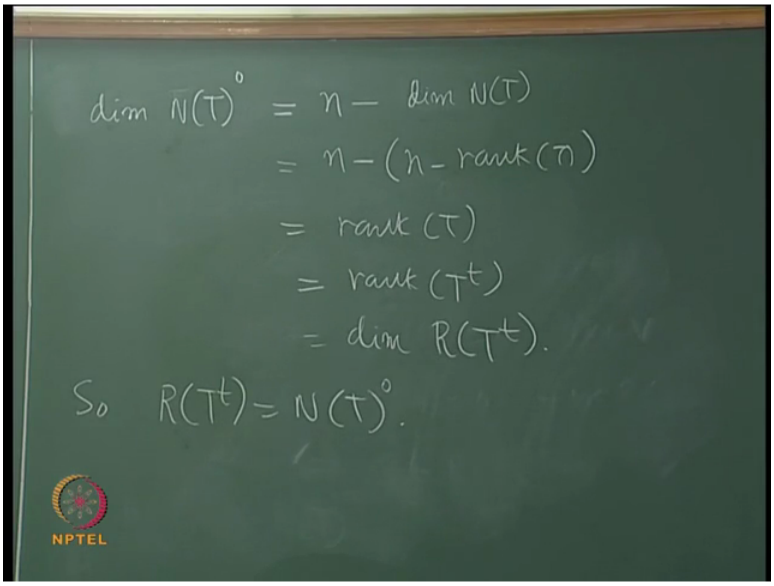

Look at dimension of the annihilator of the null space of T I am appealing to this formula dimension of the annihilator of null space of T this happens in V there is the dimension of the vector space V minus dimension of null space of T, I will apply rank nullity dimension theorem n minus dimension of null space of T, I will go back to this formula rank of T plus nullity of T equals dimension of V that is n.

What I want is dimension of null space of T nullity of T that is n minus rank of T, n minus n minus rank of T that is rank of T but just now we have shown that rank of T is rank of T transpose rank of a linear transformation is the dimension of the range space of the transformation which is what we wanted to show, you agree we have shown that range of T transpose is contained in null space of T-annihilator, we have also shown that these two subspaces have the same dimension so these two subspaces coincide, okay that proofs the theorem.

So again you see that the essential formulas are rank nullity dimension theorem and the fact dimension Z plus dimension Z 0 is a dimension of the space where Z is a subspace, okay.

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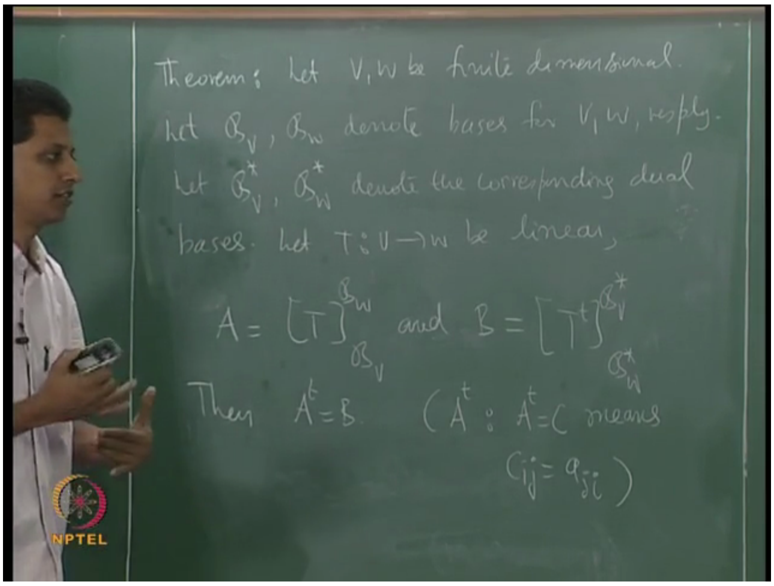

We will now look at derive the two consequences, okay. The first one is not really a consequence I will use this notation let B V, comma B W denote bases for V, comma W respectively. Let B V star, B W star denote the corresponding dual bases of V star and W star so I am following this notation B bases B star dual bases I also need to index using the spaces V and W.

Let A be the matrix of T T is a linear transformation let T be linear and let T be linear A be the matrix of the transformation relative to the bases that I started with B V B W and B be the matrix of T transpose the natural the natural bases that one must consider for T transpose are the dual bases for these two but T transpose remember is a function from W star to V star so T transpose is B W star to B V star, T is from V to W so the matrix of T will be B V B W, T transpose from the other hand is from W star to V star. So B star W to B V star this is the order in which we need to take of course these are ordered bases.

Then I told you that these two matrices are related by the formula that one is the transpose of the other, remember that if the dimension of V is n and the dimension W is m this matrix is of order m cross n this is of order m cross n, this is of order m cross n this is of order m cross n please remember this. I want to show that A transpose is B, I have introduced a notation what is A transpose? A transpose is the matrix, okay let us say if what is the definition of A transpose? A transpose equal to C means C ij equals a ji, C ij equals a ji. So if A is of order m cross n C must be of order n cross m and you know the operation of taking transposes these are definitions. So we will have to show that these two matrices are the same, okay.

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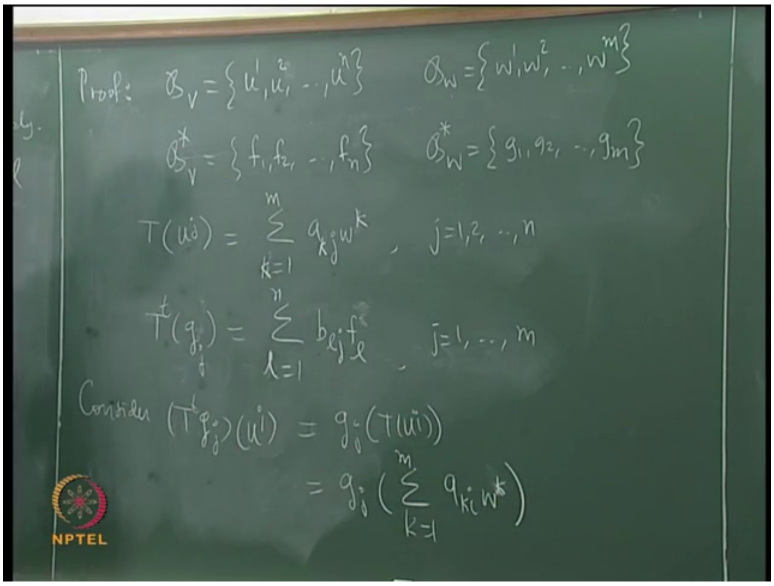

Now you will see that it is essentially the formula for writing down the matrix of a linear transformation relative to certain bases. So for preciseness I need to write down bases so I want to proof this result let us say B V is n dimensional I will have u 1, u 2, etc u n. I will write B star V the dual bases of this I will use the usual notation f 1, f 2, etc f n. Let me use B W the notation will be slightly different from this let us say I have w 1, w 2, etc w m I am assuming the dimension of w as m. The dual bases denoted by B W star that will also have m elements this time let us say I use g 1, g 2, etc g m.

All that I will do is write down the matrix of the linear transformation T relative to these two bases the matrix of the (linear transpose) T transpose relative to these two bases and just compare compare their actions. So by definition this is something we have used earlier T of u j what is the jth column of the matrix A, T of u j equals do you remember I have written summation i equals 1 to j m a i j w i you need to verify quickly and tell me, okay.

Now there is there is an i and a j so the running index this is something we have used before instead of this I will use the running index to be k, okay so let me change this to k, k is the running index j is the free index that is T u j, okay so this is the formula for T u j. This u j as many u j’s I have there are n u j's this is the definition of the matrix of T relative to these two relative to these two bases B u B v any element of u action under T that must be a linear combination of w 1 etc w m.

What is the similarly the formula for T transpose T transpose is a function of W star to V star, for W star I have this, for V star I have this. So I must look at T transpose of I will write similarly g j that must be a linear combination of these these vectors these functional I will write a similar formula summation k equals 1 to n this time because there are n functionals here k equal to 1 to or may be l l equal to 1 to n for T transpose for T the matrix is A for T transpose the notation is b so I must write this in terms of b so this is b lj instead of w I have f f l do you agree with this this is same as this a has been replaced by b, w has been replaced by f, u has been replaced by g, T has been replaced by T transpose matrix of a linear transformation relative to two bases this is what we have this time j runs from 1 to m how many g j's are there I am sorry this must be g subscript j, this is also f subscript l.

The notation that I have been following is that for functionals I will use subscripts, for coordinates I use subscripts, for different vectors I use (super sums) different vectors I use super scripts functional subscripts coordinates will not appear here I think, okay. Then I must show that A is equal to A transpose is equal B. So let me look at consider T transpose g j of u i consider T transpose g j of u i this is by definition j g of T of u i where the definition of the transpose operation T transpose g j u i g j of T of u i for T of u i I will use this formula T of u i I use this formula. So this is g j of summation k equals 1 to m instead of j I must use i a k i w k instead of j I must use i I am looking at T of u i.

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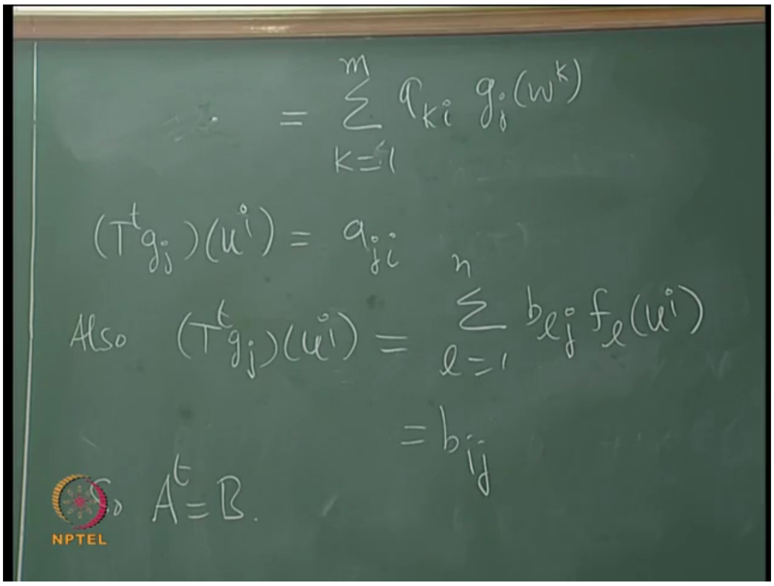

Now g is g j is linear so this is equal to summation k equals 1 to m a k i g j of w k summation k equals 1 to m a k i g j of w k but g j and w are related they are dual g this is the this is the dual bases corresponding to this w ya corresponding to this we know how these are related they are related by the delta j delta i j g g i of w j is delta i j. Let us remember that this j is fixed please understand this for a fixed j and for a fixed i I am doing all this this summation k varies from 1 to m, for a fixed j among those values that k take k will take the value j k will take the value j so for that j g j w j is 1 all the other terms it is 0 there is only one number then this with k equals j a ji do you agree me there is only one term because g and w are related by the formula for the dual bases.

So what I have done is to show that T transpose g j of u i I have shown that this is equal to a ji I will look at different formula for the left hand side so that it is equal to b ij then I am true I have used the formula of the I have used the definition of T transpose to get a ji, I will use another formula for the left hand side so that it is equal to b ij, okay.

So let us consider this again also T transpose g j of u i I will use this formula the second formula T transpose g j acting on u i so I will come back to this this will be summation l equals 1 to n b lj f l acting on u i T transpose g j formula is this that acts on u i. If you remember T transpose g j is a linear functional on V is a linear function on V, okay. Now again what happens l is a running index, i and j have been fixed in particular i is fixed when l takes a value i this is 1 all other terms are 0, when l takes the value i this is b ij all other terms are 0 that is the proof so this is equal to b ij, okay.

So there is nothing complicated about this proof write down the formula and simply apply the formula formula for the matrix of a linear transformation relative to certain bases, okay. So from this it follows that A transpose is B, so I have not used the previous theorem this is only using the definition of the matrix of a linear transformation you have to ask the right question, what is the matrix of the transpose relative to the dual bases that is the right question, okay.

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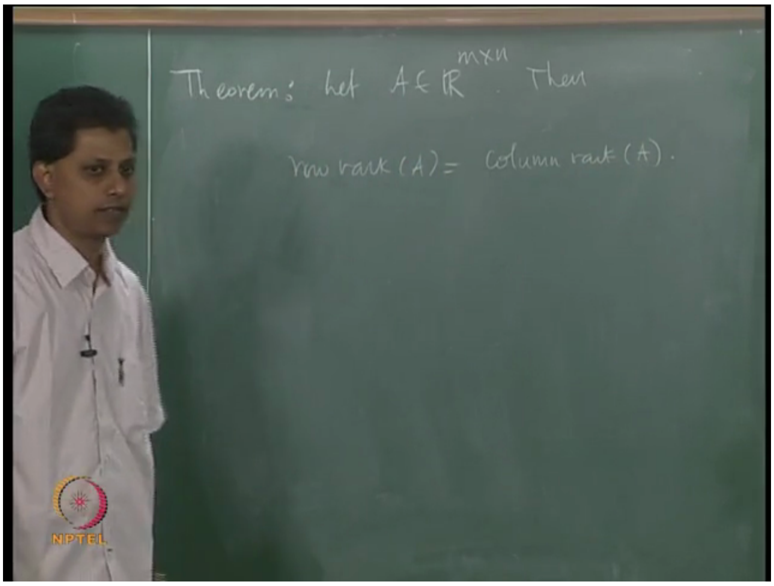

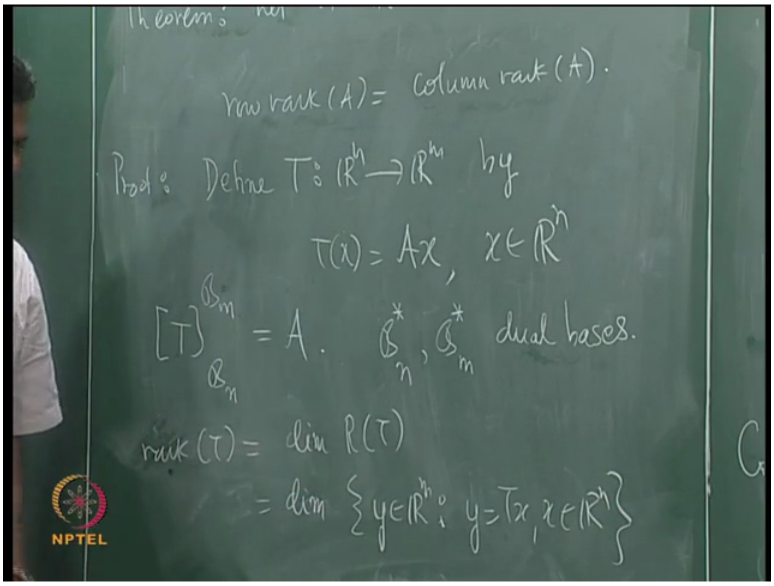

The final result the final result is given a matrix A its row rank and the column ranks coincide we approve this before by using complicated calculations we will use notion of transposes and the theorem connecting the dimension of certain ranges and get a conceptual, row rank of A equals the column rank of A, okay okay.

The proof is we have to use the language of linear transformations so let me define T define T from R n to R m by T of x equals A into x matrix multiplication, x is in R n. What is the matrix of T relative to the standard bases? The standard bases I will use just this B domain is R n so B n B m, I must write B R n B R m but that is complicated so I will just write B n B m, B n is the standard bases of R n, B m is the standard bases of R m so what is the matrix of

T relative to the standard bases? T x equals to Ax, what is A e 1? What is T e 1? A e 1 first column of A so the matrix of T relative to standard bases that is equal to A, okay okay.

Let us look at I will also use this notation B n star and B m star for the dual bases corresponding to B n B m respectively this is notation for the dual bases for B n and B m, okay. What is the column rank? Column rank of A is equal to I start with range of T what is the rank of T, rank of T equals dimension of range of T rank of T is dimension of range of T this is dimension of range of T what is range of T? Range of T is the subspace set of all y in R m such that y is Tx for some x in R n this is the range space but Tx is Ax, okay.

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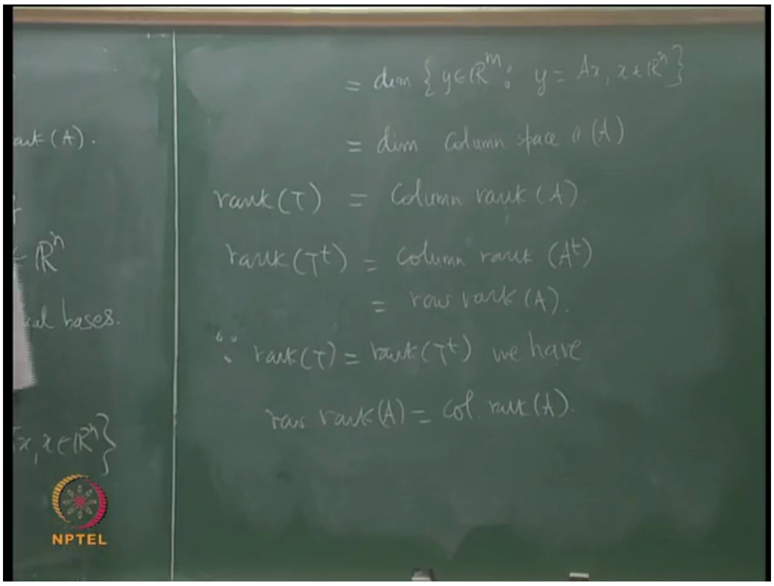

So so this is dimension of set of all y such that y equals Ax, what is this subspace? Ax for any x is in the column space so this is the column space of A dimension of the column space of A, Ax is a linear combination of the columns of A we have seen this before that is precisely the column rank. So what did we start with? We started with rank of T we have shown that it is a column rank of A where T is the linear transformation which is defined through A by the formula Tx equals Ax.

What is rank of T transpose? Rank of T transpose similarly is the column rank of the matrix of T transpose related to the dual bases which we have just now shown is A transpose you have shown that the matrix of T transpose related to the dual bases is the transpose of the matrix of the transformation relative to the initial bases that we started with so this is the column rank of A transpose column rank of A transpose is preciously the row rank of A but we have also shown rank of T equals rank of T transpose this is finite dimension the spaces are finite dimension. We have also shown rank of T is rank of T transpose.

So since rank of T is the same as rank of T transpose it follows that row rank of a matrix is equal to the column rank of a matrix. So we have done two different proofs this is much simpler but remember much simpler more conceptual but you need the notion of the transpose of a linear transformation, okay. Henceforth, we will call this number as a rank of the matrix okay so let me stop here.