Linear Algebra
Professor K.C. Sivakumar
Department of Mathematics
Indian Institute of Technology Madras
Module no 06
Lecture no 24
The Double Dual. The Double Annihilator

So remember, in the last lecture, we were confronted with this question. Given a basis B star for V star, is there a basis B for V such that this B star is a dual for that basis V okay. The answer is yes and we will prove that today. We need the notion of the double dual.

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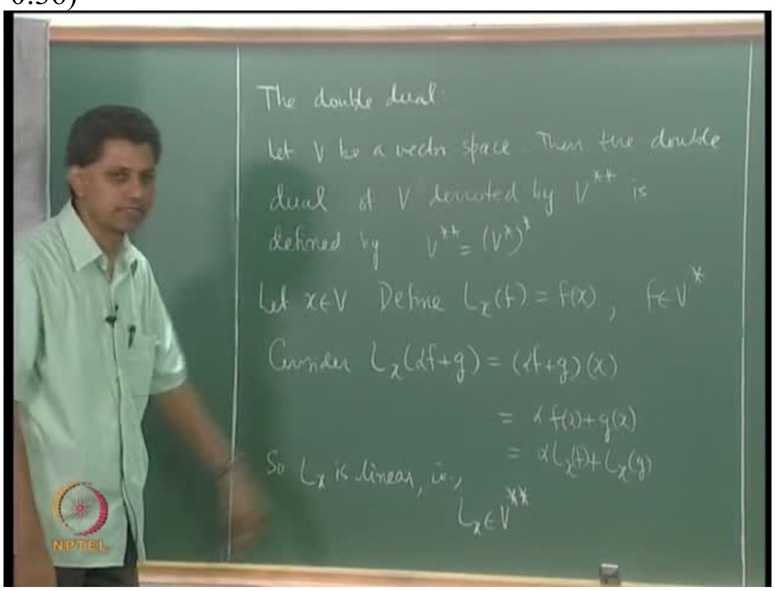

So what is a double dual? The double dual is just V double star okay over R let us say, real vector space. Then the double dual ofV denoted by V double star is defined by V double star is V star star. Okay. That is a double dual. It is the vector space of all functionals on V star. It is a vector space of all linear functionals on V star. That is the double dual. Now for, we will now establish a correspondence between V and V double star as follows. So the object of 1st is to establish a correspondence. Let me take X in V. I will define a functional. I will define LX of F to be equals L of LX of F is FX for every F in V star.

I am defining a functional on V star. So I am defining an element in V double star provided of course, I show that LX is linear. So 1st let me show that LX is linear. Let me show that LX is linear.Then you could follow that this LX of, see this LX acts on V star if I had shown LX is linear, it follows that LX belongs to V double star. Okay. Let us prove that 1st that LX is linear. To prove LX is linear, I must show that LX of Alpha F + G equals Alpha LF + LG. So let us look at LX of Alpha F + G. F and G come from V star. F and G come from V star. ConsiderLX of alpha F + G.

By definition, this is alpha F + G of X. But this is addition of functionals, we know is point wise. So this is alpha FX + G of X. Alpha FX, alpha times FX that is alpha LXF + GX is LXG. So it follows that LX is linear. Is that okay? So LX is linear, that is LX belongs to V double star. Remember that it, the domain of LX is R, see LX of F is FX. F is a linear functional. So F of X is a real number. So LX is a function from V star to R. LX is a function from V star to R. So it is functional. We have shown it is a linear function. Okay. So for every X in V there is an LX. So can it happen that this LX is 0? Do we have plenty of these LX? That will be answered by the next theorem but let us quickly observe that ifX is not equal to 0 then LX cannot be 0.

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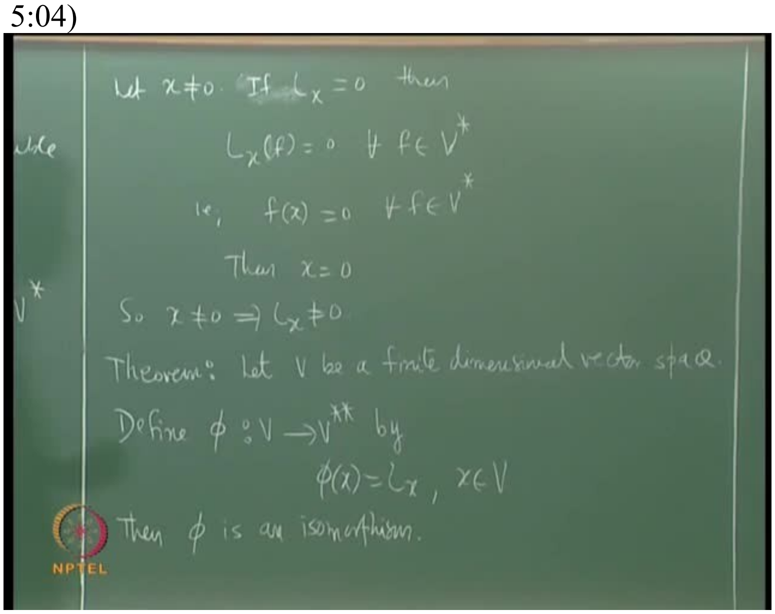

If X is not 0, we will show that LX cannot be the 0 function. X is not equal to 0. Look at if LX is a 0 functional than what it means is that LX of F equals 0 for all F. A function is 0 if it takes the value 0 for every X in the domain. So if LX is the 0 functional, then LXF is 0 for all F in V star. But LXF is by definition FX. FX equals 0 for all F in V star. Can we conclude from this that X is 0? How? Yes? Okay. See there are at least 2 ways. One is a little long winding, the other one is to simply identify that since this is true for all F linear functions,one could take the identity functional. Identity linear functional. Identity linear functional. Identity of X is X.

So this means X equal to 0 contradiction to what we started with. So LX cannot be 0. So X not equal to 0 implies LX is not equal to 0. That is if X is not zero, there is at least one functional F such that LXF is not 0. Okay. We want to show there are plenty of functions okay. Thisthis correspondence between X in V and LX in V double star we will make it formal and prove the following theorem. Our concern is over finite dimensional vector spaces. Let V be a finite dimensional vector space. Let V be a finite dimensional vector space. Define the function phi from V to V double star by means of this correspondence by phi of X equal LX, phi of X equals LX.

See what know now is that this LX indeed belongs to V double star. And the definition of LX for any fixed X is that LX of F for F and V star is equal to F of X. So this is a, this takes the vector X in V to an element in V double star. Okay. Then what is the natural conclusion that one would like to draw? Phi is an isomorphism. Let us state this and prove. Then phi is an isomorphism. Then phi is an isomorphism. It is this result that will give a proof of the fact that for every basis, for every basis V star in V star, there is a basis B for V such that this V star is dual basis of that basis B at least for finite dimensional spaces. Okay.

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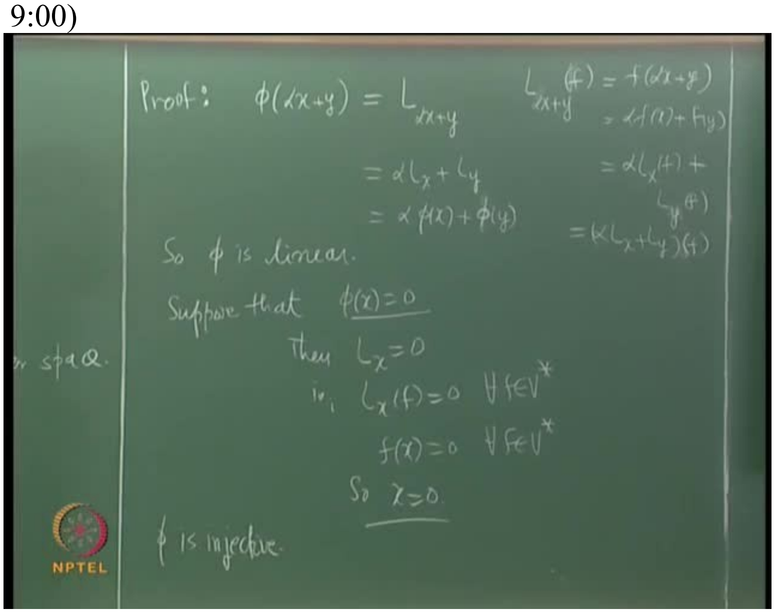

So let us look at the proof of this theorem. I must show that say phi is an isomorphism is the claim, I must show that phi is linear. I must show phi, phi is 121, phi is 12, then it follows that phi is an isomorphism. Phi is linear, that is the 1st thing. Let me consider phi of Alpha X + Y. I must show that this is alpha phi of X was phi of Y. So look at phi of Alpha X + Y. This by definition is L of this, L of Alpha X + Y. I would to know what L of Alpha X + Y is. So what I will do is look at the following. L of Alpha X + Y, that is known if I know the action of this on F in V star. L of Alpha X + Y of F.

This by definition is F of Alpha X + Y. So remember, F isin V star. N is a functional on V star okay. So this is a number, F is in V star, this is a functional on V star V double star. This is a functional on V star, this is an element in V double star. Okay, F is linear, alpha FX + F of Y. This is alpha FX I have LXF + FG is LXFY LYF. Alpha FX + FY because F is linear and then I go back to the definition of L subscript X. LXF + this. So I can write this as alpha LX + LY acting on F. Okay. So now I know what L of Alpha X + Y is. L of Alpha X + Y is precisely alpha LX + LY. So I will use that here.

That is ifT of X equals S of X, then T equals S. T of X equals F of X. T equals X. T equals S. These functions are the same. So L of Alpha X + Y is alpha LX + LY. I again go back, alpha LX is on the right, phi X + LY is phi Y. So I have shown that phi is linear. Phi of Alpha X + Y is alpha phi of X + phi of Y. So phi is linear. I want to show 1st that phi is injective. So Iknow it is linear. So injectivity, I will show null space consists of just single turn 0. So II start with phi of X equal to 0, I will show X is equal to 0.

Suppose that phi of X equal to 0, then LX equal to 0. But LX is a functional on V star. This means if a function is identically 0, it means its action on any element in the co-domain is 0. So LX of F is equal to 0 for all F in V star. Definition. LX of F on the other hand is FXequal to 0 for all F in V star as before. We have considered that here. FX equal to 0 for all F in V star implies X is 0. So X is equal to 0. So I started with phi equal to 0. I have shown that X is equal to 0. So it follows that phi is injective. It follows that phi is injective. To prove that phi is subjective, I will make use of the rank nullity dimension theorem. If I have a linear transformation on a vector space V and it is finite dimensional, then I know that it is 11 if and only if it is 12.

Okay. Okay.phi is a function from V to Vdouble star. What is the dimension of V double star? The dimension of V star is the dimension of V. If V is finite dimensional, dimension of V star is dimension of B. That is, the dimension of the dual space is the same as the dimension of the original space. The dimension of V double star should be the dimension of V star. Because I am taking the same dual, V star is finite dimensional I am using. So dimension of V is equal to dimension of V double star. Dimension of V is equal to dimension of V double star. So and I have shown that this is a phi this is a linear injective map.

So it must be by rank nullity dimension theorem this must be subjective also.phi is subjective also. So phi is an isomorphism. Okay. That is a proof. So phi is an isomorphism. This is what we wanted to prove. Okay. So let us now look at 2 consequences.

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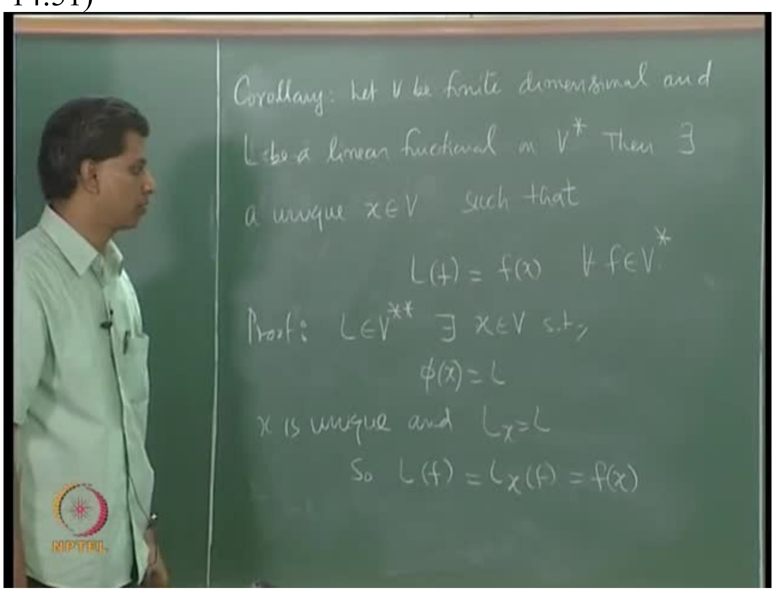

The 1st corollary of this theorem, let V be finite dimensional and L belong to V double star. So let me say L be a linear functional on V star. Then there exists a unique F in V star such that I am sorry, there exists a unique, V is finite dimensional, L is a linear function of V star, then there exists a unique X in V such that L of F equals F of X for all F in V star. Thisis an immediate consequence of the previous theorem. Linear functional on V star, so L belongs to V double star. Okay. Linear functional on V star, so it is an element of V double star. Okay, proof. L is a linear function on V star. So L belongs to V double star.

The previous theorem exhibits an isomorphism, explicit isomorphism. Phi from V to V double star by phi of X equals L of X. It means that for any L and V double star, there exists an X such that phi of X equals L of X. We have established that this phi is an isomorphism. Now look at what we have on the right. This function is 12. So for any L in V double star, there is an X such that phi phi of X is equal to that L. okay. So by the previous theorem, there exists there exists X element of V such that phi of X equals L by the previous theorem.phi is 12. That is what I am using here. So anything in V double Star has a pre-image, pre-image in V.

So that is my X but since phi is an isomorphism, this X must be unique and so X is unique, because phi is an isomorphism. The other thing is, phi X, the definition is LX. So this L is equal to LX. The form of the L is also known. This is equal to LX for some unique vector X in V. So this LX equals L. L is equal to LX. So L is equal to LF. So if you look at the action of L on F, it is the same as the action of LX on F but LX on F I know that is the evaluation functional, that is FX. So what I have shown is LF is equal to FX, that X is unique.

So it is almost a one line proof right from the previous theorem. Okay, so remember that we have made use of the fact that phi is 12 here.phi is 12. So for any L in V double star, there is a pre-image. This premature must be unique because phi is an isomorphism, phi is injective. And so there exists an X such that this happens. But then what is the definition of phi? Phi of X is LX. So this L is equal to LX. Write down what this definition LX means. We have written it.

It is the evaluation functional. So the L that I started with in V double star must satisfy this equation. Okay? This is one consequence. The other consequence is whatI mentioned right in the beginning, this correspondence between a basis for V star and a basis is for V.

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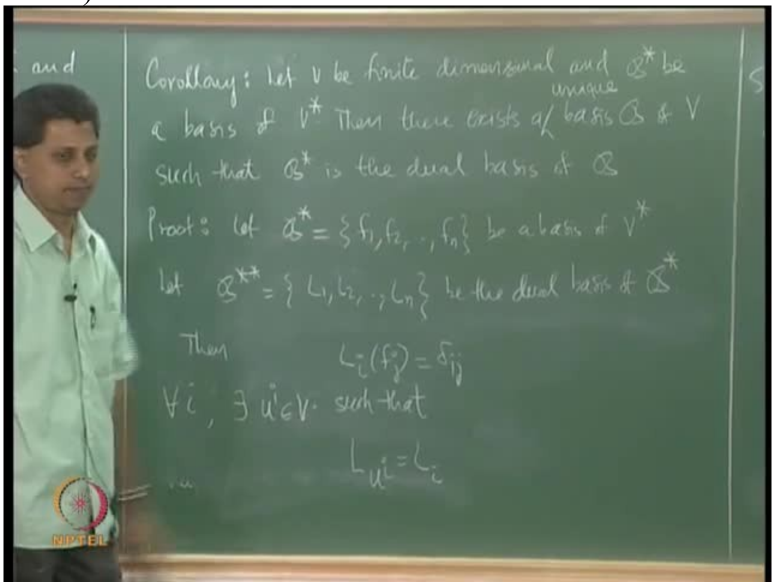

In fact, there exists a unique basis. There exists a unique basis. I can include that also. Given a basis B star of V star, this theorem states that if V is finite dimensional, then there is a basis B of V says that this B star is the dual of the basis B. Okay this is again a consequence of the previous theorem, really the previous corollary. So proof is as follows. I will start with the basis V star. I can write down the elements efficiently. B star is let us say F1, F2,etc, FN. Let B star be a basis of V star. V star is finite dimensional, I have a basis. Corresponding to this, there is a dual basis. Let B double star equal to corresponding to any basis in the finite dimensional vector space, there is a dual basis.

So let B double star equalplease remember that this B double star should not be, this is just a notation okay. See we later identify V double star with V. So please remember this is just a notation. Maybe I can okay I will just leave it as it is see this is I am writing down, this is the notation for a basis in V double star, basis for V double star. So B double star because let me say L1, L2, et cetera, LN. I know that the number of elements in this must be the same as the number of elements in B star.

Let this be the dual basis corresponding to B, sorry corresponding to B star. Then I know how these are related. Then LI of FJ, this is delta IJ. FI of UJ equals delta IJ. UIs are replaced by FIs, UJs are replaced by FJs,Fs are replaced by L. Now look at each LI, each LI is in V double star. This is the basisof V double star okay. This is the basis of V double star. Each isan elementin V double star. By the previous corollary, each element has a pre-image okay. For every I there exists unique UI, that is what the previous theorem says. There exists a unique UI such that unique UI in V. Let us get back to the previous result. For every L in V double star, there exists a unique X such that phi of X equals L.

For every I, there exists UI in V such that LX equals F. Tell me if this is all right. I will use UI. LUI equals LI. Instead of L in the previous corollary, I have LI. Instead of LX, I have LUI. For every I, I do this. Take L1 for instance. Take I equal to 1 for instance. I am looking at 11. L1 is in V double star. There is a unique preimage. I am calling that U1. How are these 2 related? L is equal to LX is the previous theorem. L1 equal to LU1, I do this for all I. LI equals LUI for all I. This is okay? Collect these UIs. It is natural to expect these UIs form a basis for V. We will only provide dependence. Then it follows that these vectors form a basis. Okay. So I will collect UIs. Is the construction clear?

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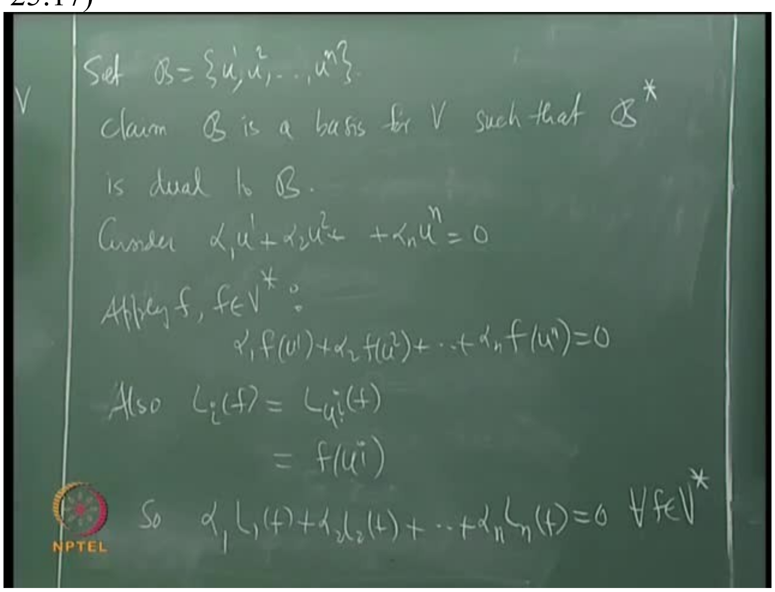

Let us set B as U1, U2X etc,UN. Then we show, so the claim is that this B is a basis for V such that B star is dual basis. These 2 things we have to prove. This B is a basis and the fact that the B star that started wit, is dual to this. Okay is it clear that it is enough to show that this is independent, linearly independent? It is linearly independent, the number of elements is N, the dimension of V is N. So it must be a basis. So we will show independence of this, linear independence of this. So you start with a linear combination, equate that to 0. Suppose alpha 1 U1 + alpha 2 U2, et cetera is 0, we must show that each scalar is 0.

If this is a 0 vector, I applya linear functional F, F of 0, 0. Apply F. F is in V star. Take any linear functional on V and then apply F to this. So I get alpha 1 F of U1 + alpha 2 F of U2, et cetera alpha N F of UN. This is 0. But F of U1, go back to this. F of U1 is L1 of F. Agree? F of U1 is L1 of F. seeI am using this really. Use this. L1 of F is evaluation of F at U1. Right? This equation tells us that LI of F is F at UI because LUI is the evaluation function, LX is FX. LX of F is FX. So LI of F is LUI of F, let me write down. L also something that we have used already also LI of F is LUI of F because each L in V double star, for each L in V double star, there is a unique X.

LUIF on the other hand is F at UI. LX of F by definition is F at X. So FUI can be replaced in the above left-hand side sum by LI of F. So alpha 1 L1F + alpha 2 L2 F, et cetera, alpha N LNF is equal to 0. So remember, you need to apply carefully the appropriate argument. Is this okay? Now this is true for, I started with an arbitrary F. So this is true for all F. This is true for all F in V star. Do I have the required conclusion? I have something like TX equals 0 for all X. So T must be the 0 map.

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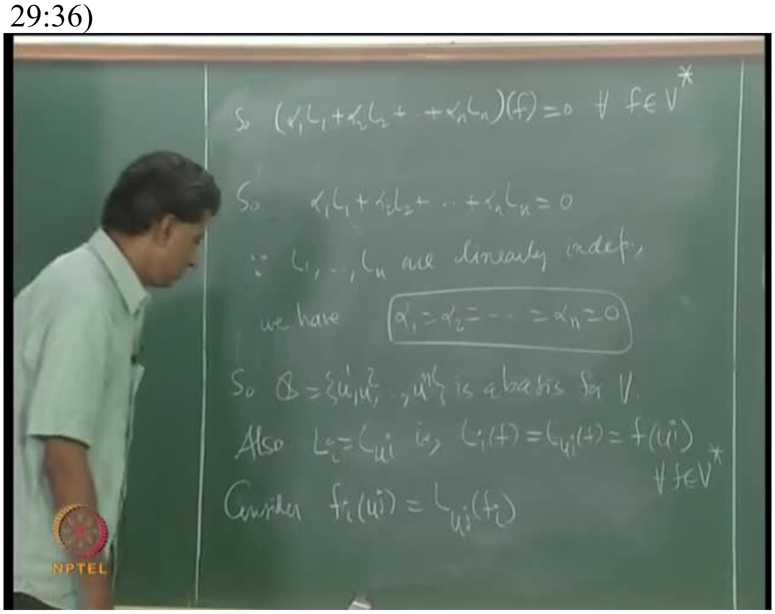

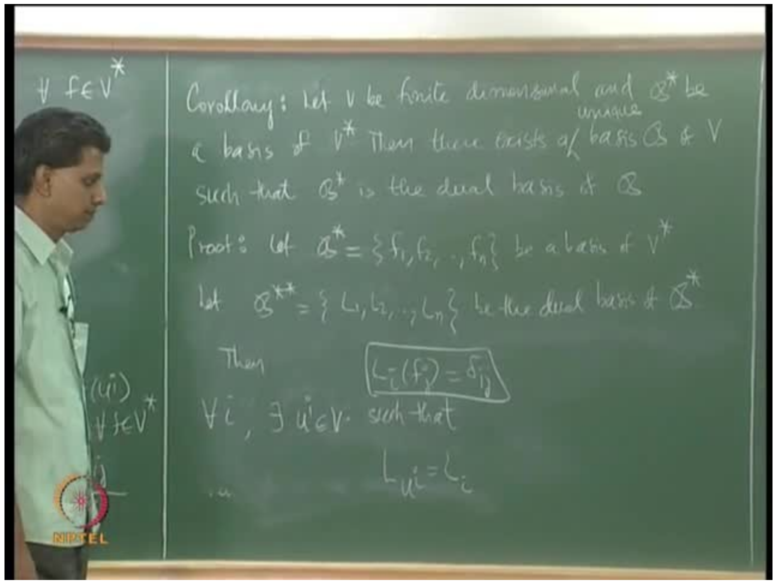

Let me elaborate. So what this means is that alpha 1 L1 + alpha 2 L2,, et cetera alpha N LN, this acting on F, this is 0 for all F in V star. That is what this means. Alpha 1 L1 F + alpha 2 L2 L, et cetera. This is equal to 0 for F in V star means this. Because what is the definition of this is alpha 1 L1 F + alpha 2 L2 F, et cetera. Okay. So I have a function which for every point in the domain gives 0. So this function must be 0. But this is now an equation involving the basis vectors, basis functionals. Since L1, et cetera LN are linearly independent, it follows that each scalar is 0. Okay.

So we wanted to establish the linear independence of these vectors, U1,etc, UN. I started with this combination. I have shown that each scalar is 0. So B is a basisfor V. Why is B star dual to this? We must verify that FI of UJ equals delta IJ. Okay? Also, FI of UJ equals FI of UJ, go back to this equation. FI of UJokay let me write once again what I am using. Also LI equals LUI, is what I want to use. That is, these are elements in V double star these are identical elements in V double star. So the action on F must be the same. Okay. That is LI of F is equal to LUI of F but LUI is F of UI for every F in V star.

Okay, now consider FI of UJ consider FI of UJ. FI of UJ is FI of UJ, instead of I, I have J. LUJ of FI. Be careful with the indices here. I want FI of UJ. Forget FI for the moment. Call it just F. I want F of UJ. F of UJ is LUJ of F.LJ of F but instead of F, I have FI. So it is LUJ FI. Is that okay? See I want to show that B star is dual to this. I must show that FI UJ equals Delta IJ. So I am computing FI UJ. FI UJ, the definition comes from this. F of UJ is LUJ of F. But I want FI UJ so that is LUJ of FI. But LUJ of FI is equal to you can go back to that. LI FJ equals delta. I could have restricted my attention to LI but does not matter. LI FJ is Delta IJ.

So this is equal to Delta IJ. LUJ, maybe one more step, right? LJFI. Delta IJ equals Delta JI.So this is Delta IJ. This is actually delta JI but delta JI is delta IJ. Identity matrix is symmetric matrix. Is it okay then? FI UJ equals Delta IJ. So the basis B star that we started with is in fact dual to the basis U1, U2, et cetera, UN that we have constructed. Is it clear? So for every basis, for every dual basis rather for every basis of V star, there exists a basis B of V such that the basis that we started with, is the dual basis for V. Okay. Okay. Now there is one final sectionon the double annihilator. I will prove maybe one or 2 results on the double annihilator and one consequence.

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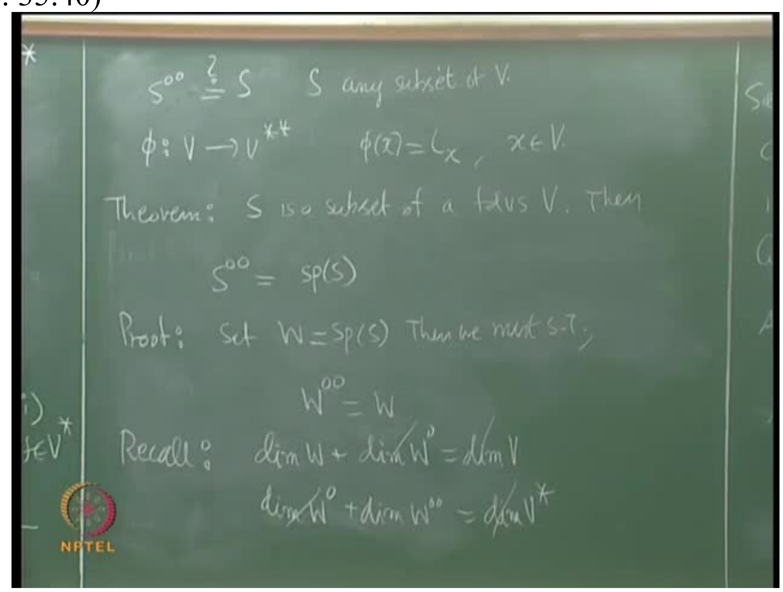

Now before I discuss the double annihilator, see I want to discuss something like S double not. I want to know whether this is equal to S. Any subset of V 1st of all, does it make sense? 1st of all, this is a subset of V, S double annihilator is a subset of V double star. Whether this makes sense, we have to see. For that, you go back and look at this functional isomorphism, phi from V to V double star defined by Phi of X equals LX. See thisestablishes an isomorphism. A one-to-one correspondence between B andV double star. So what we do hereafter is given any element in V double star, we will identify that with the unique element in V.

This identification will enable us to think of V double star as being the same as V. This is identification. We will identify V double star with V. What is the basis for this identification? That is this isomorphism. Any L in V double star, we know can be mapped to unique X in V. So any L in V double star, there is a unique X. So we can think of V double star as being the same as V in the rest of the discussion. That enables us to talk about S, double annihilator being equal to S. Now for any subset, this is not true.

We need to look at subspaces. So for subspaces, we will prove this equation. Okay, that is the 1st result. So what I want to do is to prove this theorem that if you have S is a subset of a finite dimensional vector space V, then thedouble annihilator of S is equal to span of X. The double annihilator of S is equal to span of X. The reason why the right-hand side cannot be an arbitrary subset is that it must be a subspace. See when you take the annihilator, it is always a subspace. So the double annihilator is a subspace. We will show that this subspace is the same as span of S.

The proof is as follows. Okay, so remember that these 2 are subsets of difference spaces but there is an identification that is possible by means of this isomorphism. It is with this identification in the mind that we will prove that these 2 subspaces are the same. Okay. Let me set W as span of S. Thenthis is thesame as this. Okay, this is what we will show. We must show that W double annihilator is W. The proof,We will make use of the fact that dimension of W + dimension annihilator of W, this is equal to dimension of V which we proved last time.

Whenever W is a subspace, dimension W + dimension W not is equal to dimension of V. The annihilator of W, the dimension of the annihilator of W satisfies this. I can do this for the double annihilator. That happens in V star. Dimension of the annihilator of W + dimension of the double annihilator of W, that must be equal to dimension V star, this is happening in V star. This happens in V star and so this is equal to dimension of V star. All these equations make sense. V is finite dimensional, so V star is finite dimensional.

Cancel the right-hand sides. Right-hand sides are the same. So this - this, this - this, soI am subtracting one from the other. It follows dimension W is dimension double annihilator.

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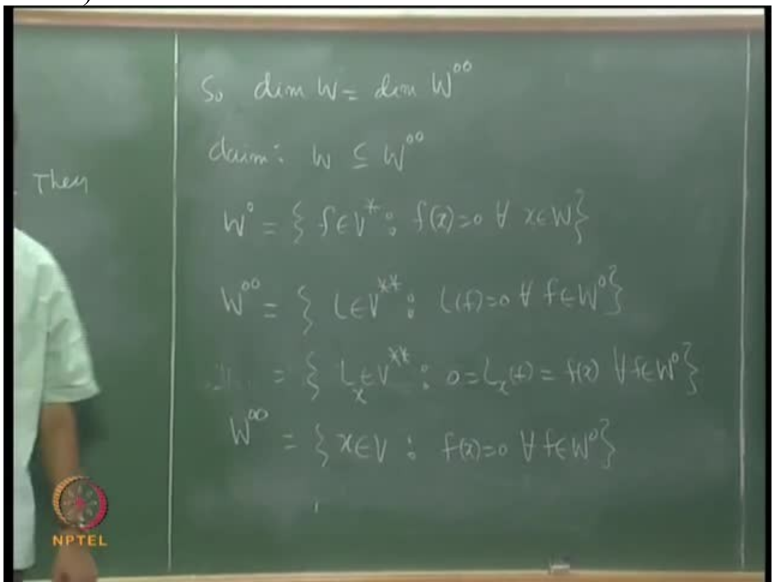

Okay? And ya, so these subspaces are the same dimension. I want to show that W is equal to the double annihilator. Is it enough if I show that one space is contained in the other? Then it will follow that subspaces are the same. Okay. So we will show that W is contained in the double annihilator. It will then follow that W is equal to W double annihilator. Again, this must be understood in the sense of the identification. W is the subset of V. W double annihilator subset of V double star but we must use the identification coming from theisomorphism phi.

Okay. Let me write down W not. W not by definition is the set of all functionals that take the value 0 for all vectors in W. So it is a set of all F in V star such that let us say F of X is equal to 0 for all X and W. This is W not, the 1st annihilator. What is a double annihilator? The double annihilator is theset of all L in V double star such that, see W annihilator is a subspace. It isL in V star such that L of F is equal to 0 for all F in W not. This is the double annihilator.

I have just used the same definition as the previous one, rotation is different. Previously W is in V, so it is a vector. Here, these are linear functionals. Okay, now write this in their expanded form, identify with, this L is in V double star, so there is an identificationwith a vector in V. Use that identification, you will be able to show that W is contained in the double annihilator okay. This is the set of all let us say L in V double star such that set of all LX in I know that each L in V double star is of the form LX.

Each element is of the form LX. Phi is an isomorphism. So it is a set of all LX in V double star such that 0 equals LX of F, that is FX by definition. This for all F, all that I have done is to remember that any L in V double star is of the form LX. But this LX, I told you can be identified with X. So I will use X itself. Set of all X in V double star, I am identifying that with V. So set of all X in V such that, I will just use the last equation. F of X equals 0 for all F in W. This is the double annihilator. Set of all X in V such that Fx equal to 0 for all F in W star.

So I have made use of the identification. LX can be identified with X. And the rest is just use this. I have removed LX now. FX equal to 0 for all F in W not. So there is only one last step. What is that last step?

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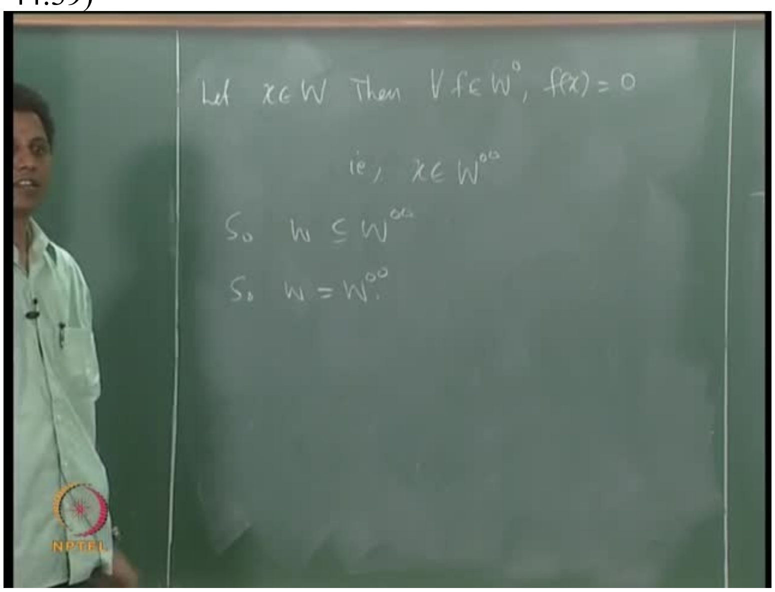

Let X element of W, then for all F in W not, F of X equals 0 by definition. That is, X belongs to W, W perpendicular. W perpendicular, we now know is the set of all X that are sent to 0 by any functional that is in W not right? Is the set of all X that is sent to 0 by any linear functional in W not. If X is in W, then for any linear functional F in W not, FX is 0. So if X is in W, then X must be in W double annihilator. So W is contained in double annihilator. Two subspaces having the same dimension with one contained in the other means they are the same. Okay. This is another consequence.

When you do the double annihilator, you could identify that with the subspace that you started with. Okay. So let me stop here. In the next lecture, I will discuss the notion of the dual linear transformation of a linear transformation and then derived two natural results. One result that we proved earlier that the row rank is equal to the column rank. This wasproved using linear equations, homogeneous equations, then row reduced echelon forms, et cetera. We will give a conceptual proof this time, without calculation.

A conceptual proof of the fact that the row rank of a linear transmission is equal to row rank of a matrix is the column rank of the matrix and also how are the matrices of a linear transformation and its transpose related. If T is a linear transformation, A is a matrix of the linear transformation relative to a basis then what is the matrix of the linear transformation T transpose relative to 2 particular basis? We will observe that it is the transpose of the matrix of the linear transormation that we started with. Okay. So theseresults we will prove next time.