Linear Algebra
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Module no 06
Lecture no 23
Subspace Annihilators II

So we are discussing the notions of dual basis, dual spaces annihilators. Later we will discuss the double dual, the transpose of a linear transformation, okay?

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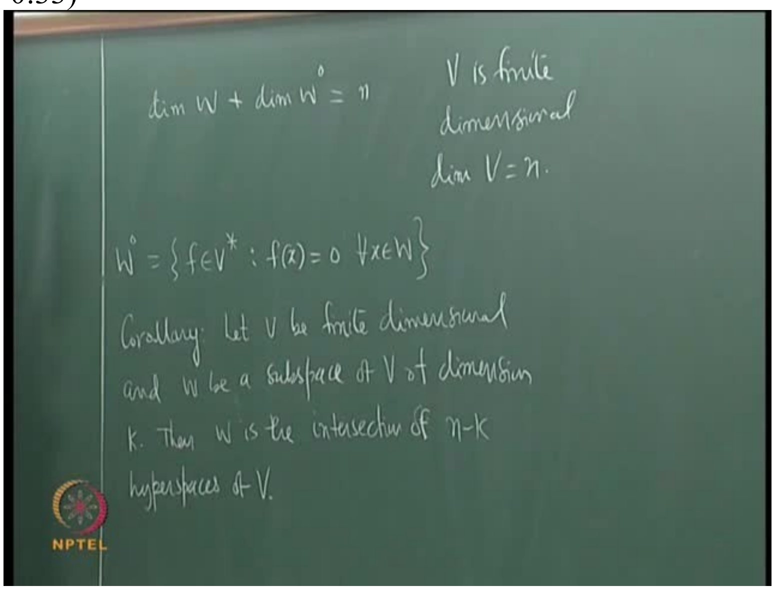

Last time, we had proved the following result that dimension of W + dimension W not equals N. We are assuming that V is finite dimensional. Okay, V is finite dimensional and W space of V. Then this holds where W not is the annihilator of W. So I am just recalling the definition. W not is the set of all set of all functionals, set of all linear functionals on V that take each X and W to 0. This is W not, called the annihilator of W. And I was going to state two consequences, the 1st one, 1st one is the following. Let V be finite dimensional and W be a subspace of V ofdimension K.

Then W is the intersection of N - K hyperspaces. W is the intersection of N - K hyperspace of V. What is a hyperspace of a vector space? Any subspace of thy mention one less than the dimension of the space V. That is called a hyperspace. So one of the consequences of really not this result but the proof of this result okay. So let me only take the 1st few steps of the proof of the previous theorem. From that, we will derive the fact that this W is the interaction of N - K hyperspaces. Okay.

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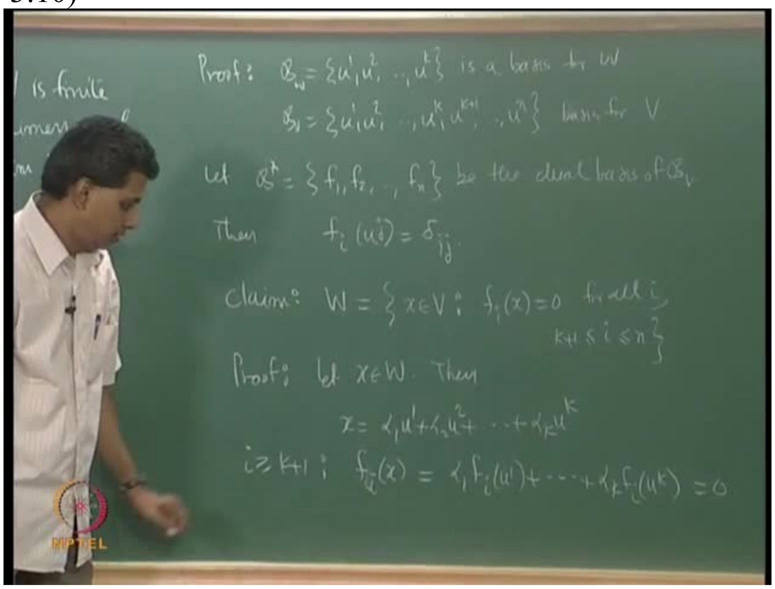

So recall the proof. The 1st few lines is the proof of the previous theorem. Dimension W + dimension W not equal to N. Let us say, W is, W has this as a basis. V equals U1, U2, et cetera, UK. This is a basis for W and what we did was to extend this to a basis for V. I will call this BW BV. BV is U1, U2, et cetera, UK. UK + 1 et cetera UN. The dimension of V is taken to be N. This is the basis for V and then what we did was to construct a dual basis for this. Let B star equals F1, F2, et cetera, FN be the dual basis of BV. Then how are these functional and these vectors related? By definition, we must have FI of UJ equals Delta IJ. Okay? In this notation, I will make the following claim.

So let us look at the following. The claim that I am making here is that the subspace W is the set of all X and V such that FI of X equal to 0 for all I K + 1 less than or equal to I less than or equal to I. Remember, this W is a subspace of V of dimension K. I am trying to describe this W in terms of V. Linear functionals FK + 1, FK + 2, et cetera, FN okay? Suppose, let us say I have proved this claim. Then can you see that this is okay what is this again? This is the set of all X for that FK + 1 X equal to 0, FK + 2 X equal to 0 et cetera, FN + 1 sorry, FNX equal to 0.

This is a set of all X that lie in the intersection, FK + 1, null space of FK + 2, et cetera, null space of FN. That is, if I had proved this, then it would follow that W is intersection null space FI, I equals K + 1 to N. Intersection of null spaces of linear functionals. Each linear functional has a property that the null space is one-dimensional. Okay? Null space is N - 1 dimensional. Range is 1 dimensional, null space is N - 1 dimensional. So each linear functional has a property that the null space is the hyperspace. I have written this as a intersection.

Actually N - K sub K + 1 to N. N - intersection of N - K hyperspace is. Okay? So we need to only prove this. Is that clear? Suppose we prove this, then it follows that W is the intersection of N - K hyperspaces. It is clear that it is the intersection of N - K subspace is but each subspace is null space of a linear functional. Each is a subspace of a null space each subspace is a null space of a nonzero linear functional. Null space of a nonzero linear functional is N - 1 dimensional. And so it follows that this W is the intersection of N - K hyperspaces.

So we need to only demonstrate this. Okay. So let us prove this. One is obvious but I will prove both. I have 2 sets,A equals B. Imus so A contained in B and B contained in A. Okay, let us start with, so we need to prove this claim now. So let me take X and W. I must show that this X has the property that whenever I is greater than or equal to K + 1, FI of X is 0. Let X element of W. Then for W, I have taken this as a basis. So this X can be written as alpha 1 U1 + alpha 2 U 2, et cetera + alpha K UK. Look atI greater than or equal to K + 1 and then FI of X. I greater than or equal to K + 1 FI of X.

Remember, I need to show that FI of X is whenever I want to show left-hand side containing the right-hand side, I must show that if X belongs to W, then FK + 1X is 0, FK + 2X is 0, et cetera, FN of X is 0. So I take I to be greater than or equal to K + 1. I must or, this is 0 okay but FI of X, F is, FI is linear. So this is alpha 1 FI of U1, et cetera + alpha K FI of UK. I have just used linearity of FI okay but remember that this is a F1, F2, et cetera, FN is a dual basis and so whenever I is greater than or equal to K + 1 each of this is 0 because look at this indices. You get from U1, et cetera up to UK.

So when I is equal to K + 1 or more, I is not equal to J. So each term is 0. So this is 0. I is greater than or equal to K + 1. So FI for any UJ when J is less than or equal to K is 0. That is the definition. So FI of X is 0. So what we have shown is that the left-hand side, W is contained in this subset.

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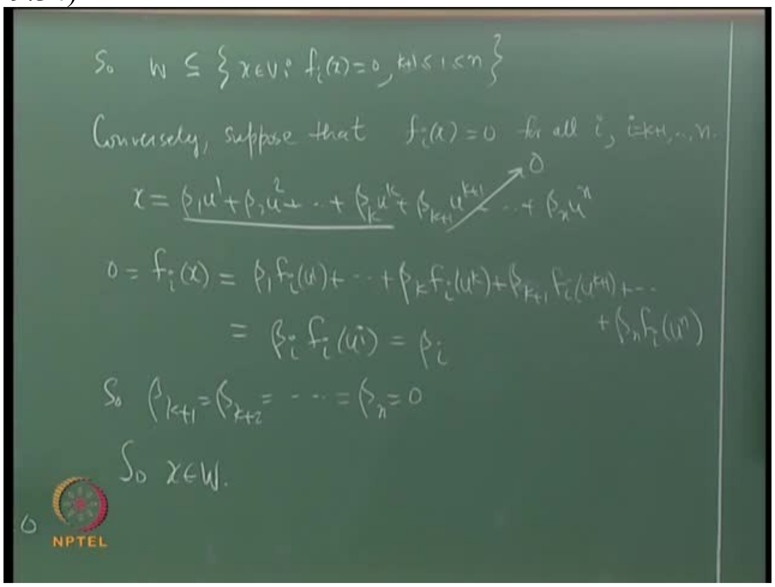

So W is contained in set of all X and V such that FI of X equal to 0 for all I satisfying we need to show the converse. Conversely suppose that FI of X is equal to 0 for all I, I running from K + 1 to N, suppose X is a vector that belongs to the right-hand side subset right-hand side subspace, I must show thatX belongs to the left-hand side W. Now X is an arbitrary element. So I can write X using this basis. So let me usesome other scalars. X is a linear combination of U1, et cetera, UN. So X is beta 1 U1 + beta 2 U2, et cetera + beta K UK + beta K + 1 UK + 1,etc, beta N UN. U1, U2, et cetera, UN is the basis for V and so I can write X in this manner.

FI of X is 0. So I start with 0, that is FI of X. FI of X, FI is linear. Beta 1, FI of U1, et cetera + beta K FI of UK + et cetera. Okay let me write the next term also. Beta K + 1 FI of UK + 1 + et cetera + beta N FI of UN. After applyingFI, I get this. For each I running from K + 1 to N, X must satisfy FI of X is equal to 0. But look at what we have on the right-hand side. On the right-hand side, there is only one term that remains because FI of UJ is that the IT. What is that term? see, I runs from K + 1 to N, I is fixed okay. And so, this is FI of UI, all other terms are 0, all other terms are 0. Beta I FI UI but FI UI is 1.

So this is beta I. So what have we shown? We have shown that if FI X is equal to 0 then beta I is equal to 0 but what are the values that I can take? I takes, I runs from K + 1 to N which is beta K + 1 is 0, beta K + 2 is 0, et cetera. So what we have shown is that beta K + 1 equals beta K + 2, et cetera equals beta N equal to 0. So go back and look at the representation for X. Look at the representation for X. X is beta 1 U1, et cetera. From K + 1th term onwards, they are all 0. So X is just this. This term is 0. The representation of X. The scale is corresponding to BK + 1 et cetera, they are all 0.

That is what we have shown because see this FIX is equal to 0 for all I running from K + 1 to N. So the scale is beta I bring from K + 1 to N R0. So X is a linear combination of U1, et cetera UK but then that is U1, U2, et cetera, UK is a basis for W. So this X must belong to W. So I have started with an arbitrary vector on the right-hand side subset. I have shown that that belongs to W. So right-hand side subspace is contained in the left-hand side subspace. So W is equal to this and hence the theorem, corollary. Let me just write one more step to make the final part transparent.

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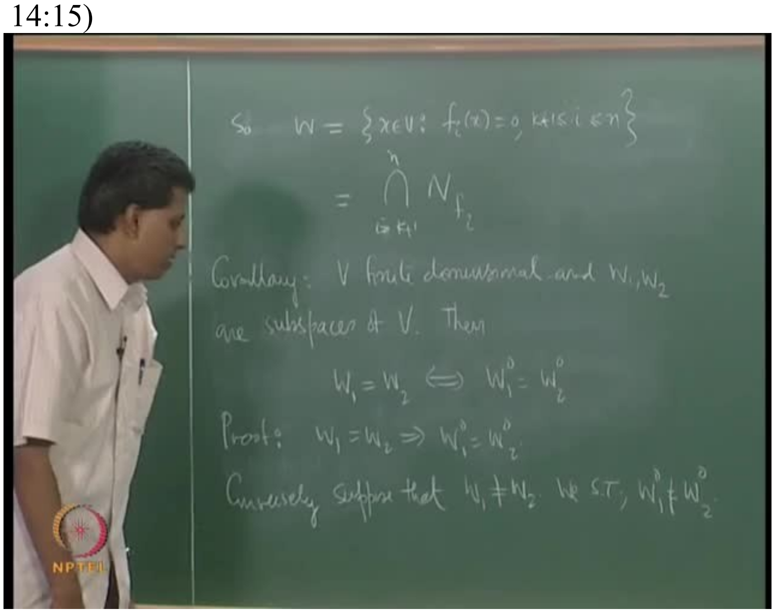

Let me write that here. So what we have done is W is the set of all X and V such that FI of X is 0 for all I running from K + 1 to N. Now you see that this is the intersection of the null space of FI I running from K + 1 to N. Anything on the right-hand side must be in the null space of FI for each I running from K + 1 to N. So this is, and there are N - K, this intersection has N - K terms.

So there are N - K subspaces. Each is a subspace of non-zero linear functional. So each subspace is dimension N - 1, null space of FI, that is N - 1.

So I have written W as an intersection of N - K hyperspaces okay. That is the complete proof of this corollary. There is another corollary which talks about the relationship of 2 subspaces and their annihilators really. So 2nd corollary V is finite dimensional and W1, W2 are subspaces of V, then W1 is equal to W2 if and only if their annihilators are equal. W1 0 is W2 0. 2 subspaces are equal if and only if their annihilators coincide. This again uses the proof of the previous theorem out of which one part is easy. If S is equal to P, then annihilator of S is equal to annihilator of P.

That is easy to see. So W1 equals W2 implies W1 0 equals W2 0. Annihilators must be the same okay. It is a converse that is nontrivial here. To prove the converse, to prove the converse what we will do is assume that W1 is not equal to W2, show that W1 not is not equal to W2 not. Conversely let us suppose that W1 is not equal to W2. We will show that we show that W1 0 is not equal to W2 0. This is really the converse because what is the meaning of this? W1 not equal to W2 implies W1 0 is not equal to W2 0. This is the same as saying W1 0 equals W2 0 implies W1 equals W2. That is a converse. If A implies B then not B implies not A.

That is what we are using. The statement A implies statement B. The negation of statement B implies negation of statement A. So if we demonstrate W1 not equal to W2 implies this, then it follows that if this statement is not true, then negation of this statement is true. So it follows that if this is not true, that is W1 0 is equal to W2 0 implies W1 is equal to W2 which is really the converse okay. So let us prove this. Now W1 is not equal to W2. They are subspaces. So as sets, they are not equal. So one is not contained in the, at least one of them is not contained in the other. For the sake of using this notation, let meassume that W2 is not contained in W1 okay. Okay.

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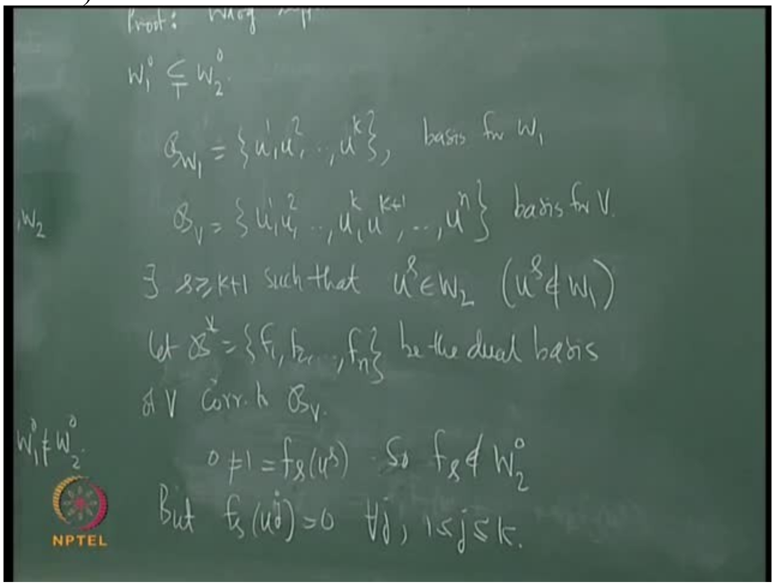

I am saying without loss of generality. So this is what I am going to prove. I am going to prove that W1 not equal to W2 implies W1 0 is not equal to W2 0. Without loss of generality, let us assume that W2 is not contained in W1. Okay? We show that W1 0 is not contained in W2 0. Okay. If W1 were not contained in W2, one could show that W2 0 is not contained in W1 0 by a similar argument. So there is no loss of generality in assuming W2 is not contained in W1. Now I will have to go back to the previous notation. For W1, let me take U1, U2, et cetera, UK as a basis. I am callingthat BW1. This is the basis for W1.

This can be extended to a basis for V. Okay, as before, I am assuming that V is N dimensional. So there are N vectors here. This basis has been extended to a basis for V. If W2 is not contained in W1, what means is that there is a vector in W2 which is not in W1 okay. Is it clear then that there exists S greater than or equal to K + 1 such that US belongs to W2, obviously US does not belong to W1. Do you agree with this? W2 is not contained in W1. W1 has U1, U2, et cetera, UK as the basis. So anything in W1 is spanned by these vectors.

Now you take US where S is not 1 to K. So S is greater than or equal to K + 1. If all of these vectors belong to W1, then W1 is the whole of V in which case this cannot happen. W2 is a subspace. So there is at least one S for which US does not belong to W1 but those UIs that belong to W1 are indexed by 1 to K. So if there exists a US that does not belong to W1, it must be corresponding to an index that is greater than or equal to K + 1. So there exists US where S is, there exists S greater than or equal to K + 1 such that US does not, sorry US does not obviously belong to W1 but it belongs to W2 okay.

Now look at the functional FS. That will do what we require here. Look at the functional FS. I have now constructed dual basis. Let B star equals F1, F2, et ceterabe the dual basis of Vcorresponding to the dual basis BV corresponding to BV. Then as before, FI of UJ equals Delta IJ. I am just calling your attention to FS. Look at FS. FS of US we know by definition must be equal to 1. Okay. And that is not 0. It means FS cannot belong to, see US belongs to W2. And FS is the functional that takes US to a nonzero value. So FS cannot belong to W2 0.

FS does not belong to W2 0. W2 0 is the set of all functionals that takes all elements in W2 to 0. I have produced one element in W2 which is taken to a nonzero value by the functional FS. So FS does not belong to W2 0 but obviously FS belongs to W1. Why? But FS of UI, I will use J. FS of UJ equals 0 for all J such that 1 less than or equal to J less than or equal to K by the definition of the dual basis because J runs from 1 to K, S is greater than or equal to K + 1, so J can never be equal to S. J runs from 1 to K, S is greater than or equal to K + 1. So S is never equal to J.

So this is 0. In other words, FS of U1, FS of U2, et cetera they are all 0 which means FS belongs to W1 0 because anything in W1 0 is a linear combination of these and so if I apply the linear functional FS to that vector, that will take the value 0. Should I elaborate?

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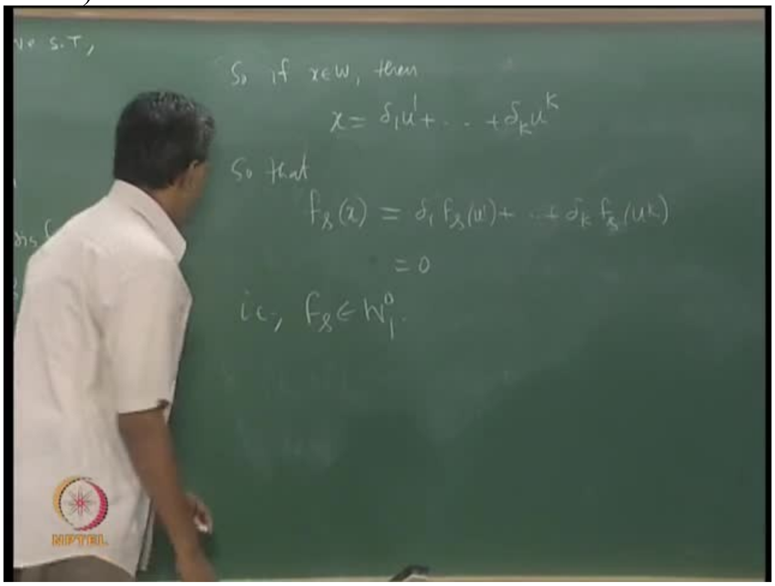

Okay, let me do that quickly. So if X belongs to W, then X is some linear combination of these UKs so that FS of X is what I want. FS of X is Delta 1 FS of U1, et cetera Delta K FS of UK. Each term is 0. So this is 0. That is FS belongs to W1 0. So remember I, that is what I wanted to show. If W2 is not contained in W1, I wanted to show W1 0 not contained in W2 0. I have a functional that belongs to W1 0. I have a vector or a functional that belongs to W1 0 but that is not in W2 0. FS is not in W2 0. And so this holds and so the converse holds.

Okay. So W1 0 equals W2 0 implies W1 must be equal to W2. If the annihilators coincide then the corresponding subspaces, only for subspaces, the subspaces must coincide. By the way, this formula does not hold if you take arbitrary subsets. If W1 is a subset, W2 is a subspace. Then W1 0 could be equal to W2 0 without W1 being equal to W2. The underlying subsets must be subspaces. Okay? Okay. To consolidate, let us look out to example. To consolidate the ideas of dual base annihilators and then determining elements in annual basis.

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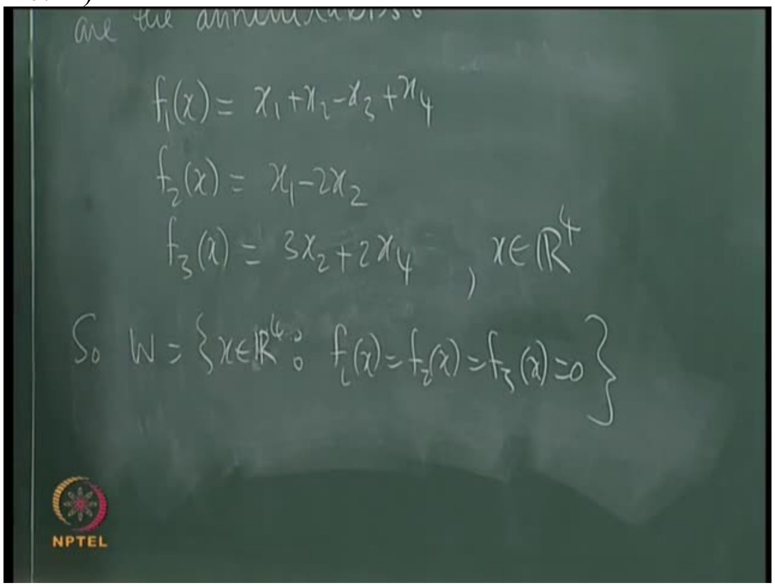

So let us consider the following examples. Let us say I have 1^{st} one, determine the subspace of W, let us say I take R4, determining the subspace W of R4 for which the functionals given below are the annihilators. Okay. I am given let us say 3 annihilators. I will call F1 of X, X is in R 4. So let us say I have X1 + X2 - X3 + X4. F2 of X is X1 - 2X2, F3 of X is 3X2 + 2X4. The question is, you are given functionals. These functionals annihilator a certain subspace, what is that subspace? Okay. Remember, a subspace, just now we have seen. W is equal to set of all X elements of V, I said FI X equal to 0 for all I running from K + 1 to N.

Subspaces can be given by the set of annihilators of that space. Okay. Subspace can be given using the annihilating functionals also. Okay. So we need to solve this problem. So what is the definition of W? W is the set of all X and R4 in this case such that F1 of X equals F2 of X equals F3 of X equals 0. Now what you will see is that again it is elementary row operation. Write down these 3 equations. These are homogeneous equations.

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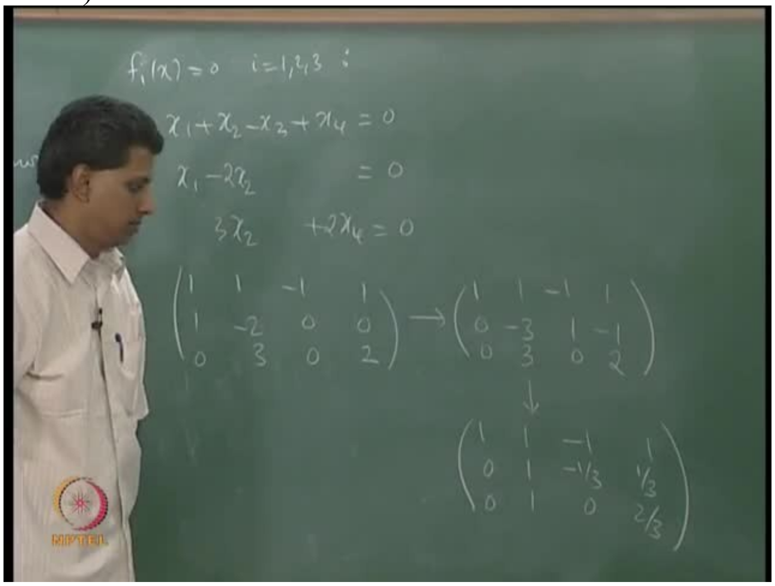

These 3 systems give me the following. Sorry, there is only one system. X1 + X2 - X3 + X4 equals 0. X1 - 2X2 because 0. 3X2 + 2X4 equals 0. Elementary row operations. So I have the matrix 1 1 - 1 1, 1 - 2, 0 3 0 2. I will apply elementary row operations to determine the set of all solutions. I will keep this as a pivot and then straightaway make this 0. Okay. Let me do it like this. I have 0 1 - 1 by 3 1 by 3. I can divide this also by 1. 0 1 0 divide by 3, 2 by 3.

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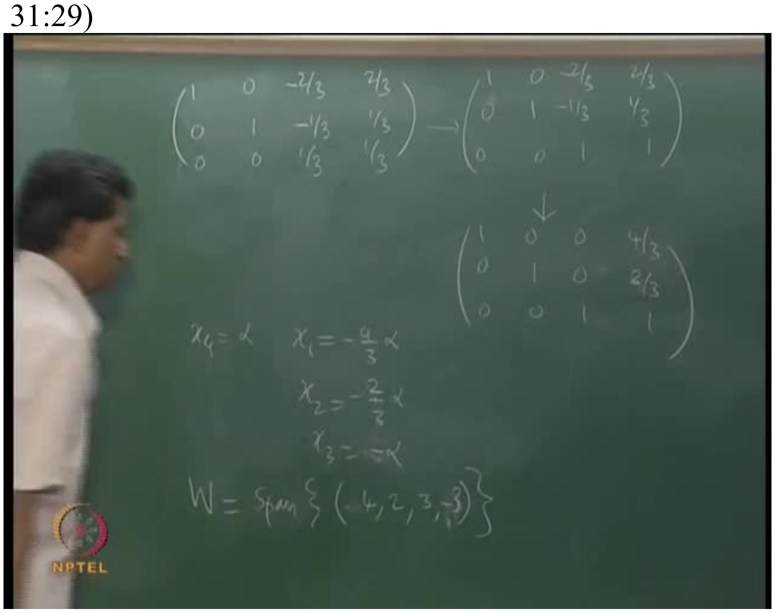

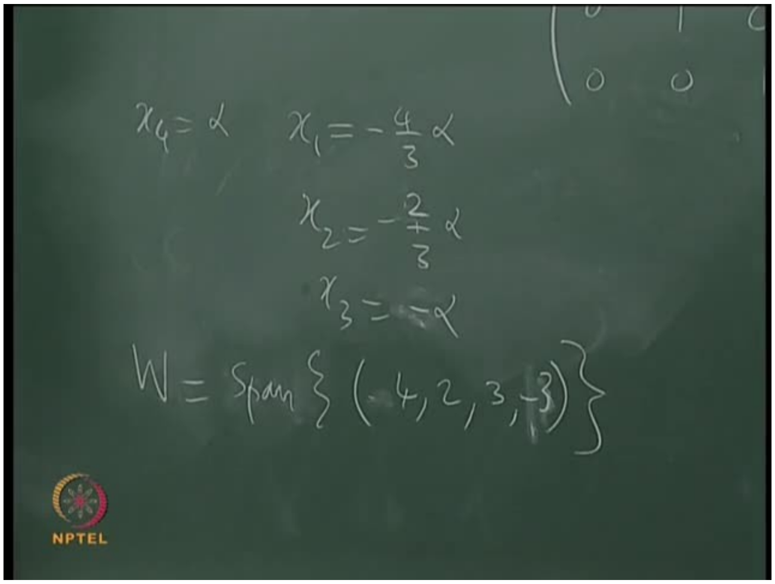

Then keeping the 2nd, next operation will be to okay multiply this by 3. So that was unnecessary. I have 0 0 1 1. 0 1 - 1 by 3 1 by 3 1 0 - 2 by 3 2.Ya, obviously I am not doing it very efficiently but let us look at the solution. Okay so now I have a 3 by 4 system where this 3 by 3 part is the identity part. So what follows is that, remember that this came from this set. There are 4 variables. What this tells me is that I must fix the last variable, X4. So let us say X4 is Alpha, then X1 is - 4 by 3 Alpha, X2 is also - 4 by 3 Alpha, X3 is - Alpha. There is a problem with the solution.

From this step to this step, it is correct. From here to here, see, I am keeping this as the pivot row. Then 1 by 3 times this + this. So these entries, this becomes 0. 1 by 3 times this + this 2 by 3. Let me look at it once again. After this step, it is correct. Okay. So I am keeping this as fixed. Okay. Then multiply this by 2 by 3 or cancel this. 1 0 2 by 3. 2 by 3 + 2 by 3, 4 by 3. That is okay. So this entry is - 2 by 3 Alpha. Please check the calculations. So what is W then? W is one-dimensional. W is one-dimensional because it is any multiple of, so let me just say W is span of this vector. Let me multiply throughout by 3. X1 is - 4, X2 is - 2.

So let us say 4 2, X3 is multiplying throughout by 3 and - 1. I am multiplying by 3. I am multiplying by - 3.- 3, okay. 6,- 3,- 3. X1 - 2X2. So this is the subspace W which is the annihilated by these 3 functionals okay. Again, it is only solving homogeneous systems. One more problem where we will determine the annihilator.

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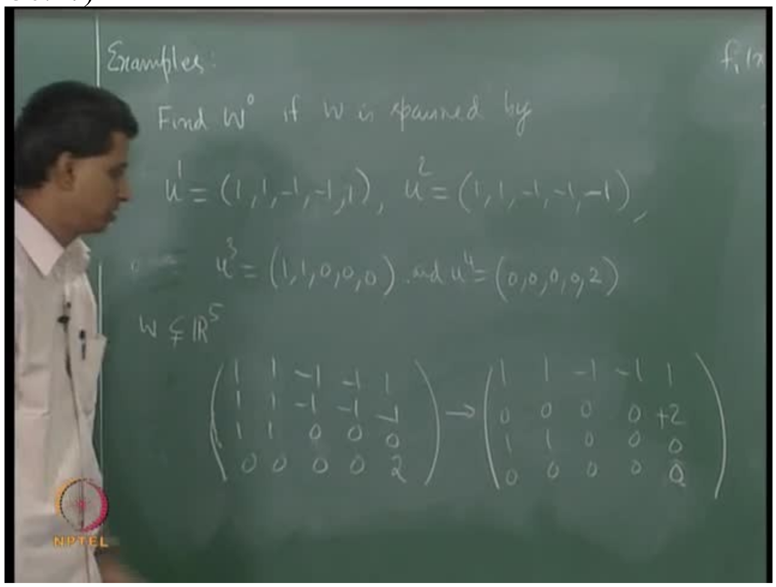

2nd example, find W not. Find W not if W is spanned by these vectors. Let us say U1 is let me take one more. W is a subspace spanned by these 4 vectors. I am not claiming that these vectors form a basis for W. W is spanned by these set of vectors. What is W0, the annihilator of W? Okay. Remember that W is a subspace of R5. Not the whole of R5. There are only 4 vectors here. So W is a subspace of R5. I must find the functionals that generate W not. Okay. So I will determine a dual basis for a basis corresponding to, for a basis containing more for a basis contained in this set.

These 4 may not been dependent. Okay. So let us know what do we need to find? We need to find W not okay. So let us look at the functional F. If F belongs to W not, what is the condition that F must satisfy? Okay? But before that, let us look at these 4. I will again write it in the matrix form okay. 1 1 - 1,- 1 1, 1 1 - 1 - 1 - 1. 1 1 0 0 0, 0 0 0 0 2. And apply elementary row operations. W is the subspace which is the row space of this matrix. W is the subspace corresponding to the row space of this matrix. Okay? Row space does not change if we do elementary row operations.

So I want to do just probably one operation. I have not reduced it to the row reduced echelon form. I will just do one of these. So - this,+ this. I will keep this one also as it is and then observe the last one is 0 0 0 0 2. In the next step, what could be done is these 2 could be made 0. So I will make that here itself. This could be made + 2, does not make a difference. So this, so what is clear is that the 4 vectors are independent, only 3 of them are independent. Dimension of W is 3. Dimension of W is 3. We need to determine W0. Okay.

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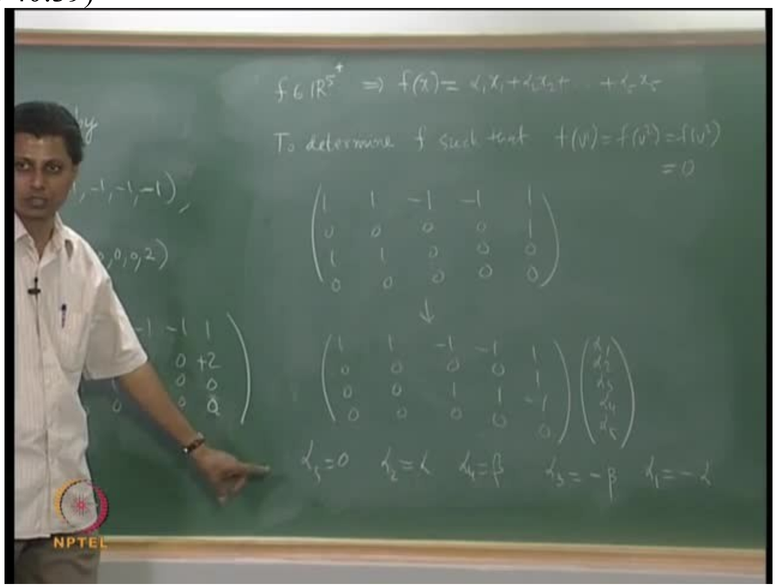

What we need to determine is that any functional F in R5 any functional on R5. If F belongs to R5 star, then F can be written as Alpha 1 X1 + Alpha 2 X2 alpha 5 X5. Any function on RN can be written as A1X1 + et cetera ANXN. So FX is of this form. I need to determine, okay I do not know how many are there. I will just keep I need to determine F such that F of, these 2 vectors, I will call them V1, V2, V3. I will call these row vectors, V1, V2,V3. I must determine all X that satisfy these 3 equations. Again, homogenous equations, okay. So let us write this. It is almost in the row reduced echelon form. So I want 1 1 - 1 - 1 1 into X1, X2, X3, X4, X5 equal to 0.

We will have to interchange. Let us say I make, I keep this as it is. So from here, I get 0. I do not really have to do it but I get a 1 here. This is not the row reduced echelon form. So let me not get into the details there. From this, I can right away tell you what these functions are. See, from this what follows is that remember, I need to determine F. So I need to determine the 5 unknowns here, alpha 1, alpha 2,et cetera alpha 5. Is that clear? I need to determine W not. F belongs to W not if satisfies this equation. These equations give me this into alpha 1, alpha 2, et cetera, alpha 5, that is equal to 0.

Now what is clear is that from this alpha 5 is 0. Is that okay? Alpha 5 is 0. Okay let me write down. 2nd equation tells me, alpha 5 is 0, 3rd equation, okay, so from this I have 3 equations in 5 unknowns. So I need to fix 2 of them. Which one do I fix? It cannot be the 3rd one, it cannot be the 1st one. And so it is the 2nd and 4th. It is the 2nd and the 4th which you get by the row reduced echelon matrix. So let me say I fix alpha 2 equals alpha. See this diagonal entry, thisis 1, 2, this is 3. So these 3, these 2 will not be fixed. Alpha 5 is already 0. 1 and 3, so 2 and 4.

Alpha 4 is beta. Then determine the others. In particular, the 3rd equation gives me Alpha 3. Alpha 3 is alpha 5 - alpha 4 - beta. The 1st one should give me alpha 1. This is gone. Alpha 4 + Alpha 3. Beta + Alpha - sorry alpha 5 is 0. Alpha 4 + Alpha 3, that is 0. So Alpha 1 + Alpha 2 is 0. So alpha 1 is - Alpha okay. So can you see that the basis consists of just 2 functionals because there are just 2 variables, Alpha, beta. All the others are in terms of these.

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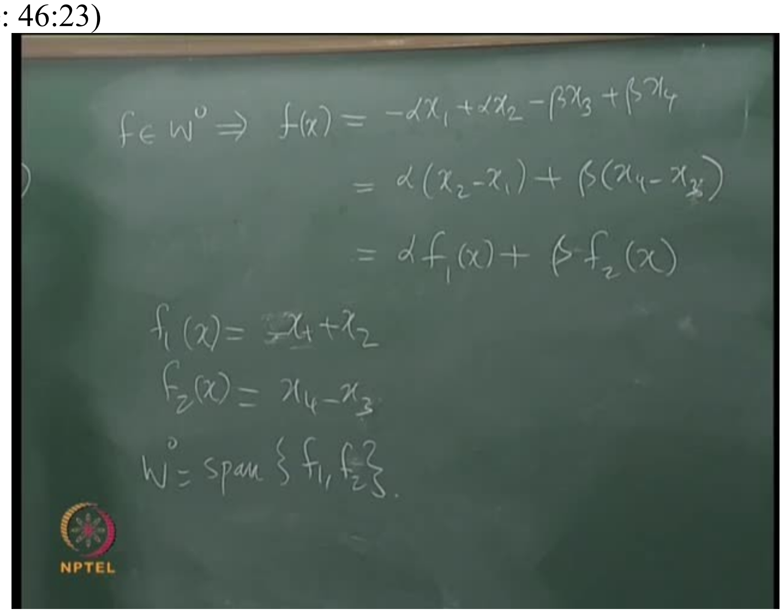

Let me summarise. F belongs to W not implies F is of the form this is what I have.- Alpha X1 Alpha 2 is alpha.+ Alpha X2. Alpha 3 is - beta. Alpha 4 is beta. Alpha 5 is 0. Alpha 5 does not figure here. Now I can write alpha and beta are, can take arbitrary values. In particular, alpha 0 beta 1, alpha 1 beta 0. So I can write this as alpha times X2 - X1 + beta times X4 - X3 which means I can write this as alpha times F1 of X + beta times F2 of X. F1 and F2 are independent functionals. F1 of X is X2, let us say - X1 + X2. F2 of X is the 2nd term, X4 - X5.

Sorry. X4 - X3. F1 is this, F2 is this. Any F and W not is a linear combination of these 2. These 2 are independent. These 2 are independent because you can think of F1 as the vector - 1 0, sorry - 1 1, all other entries 0. F2 is 0 0 - 1 1 0. So these 2 are independent. Okay. Okay finally W not is span of F1 F2. So please verify the calculations. Essentially it is solving homogeneous equations.Even remember the 1st problem when we determined the dual basis. That was solving homogenous equations. Okay. Let me stop here. Next time, let us discuss the notion is ofthe double dual.

Once we have done one go from V to V star, can we go from V star to V double star? Sothis can be done and when we discuss the notion of double dual, we will also consider the following question which we have not dealt with before. What we know is that given a basis for V, finite dimensional case. Given a basis for V, there is a dual basis for V. So there is a basis for V star where there is a natural correspondence between the basis for V that we started with and the basis for V star that we constructed.

The other question is, given a basis for V star, is there a basis for V such that the basis for V star is the dual for the basis of V? That we would consider. The answer is yes. For finite dimensional spaces, the answer is yes. For infinite dimensional spaces, the answer is no. Infinite dimensional spaces will be discussed in functional analysis. For finite dimensional spaces, we will show that the answer is yes. Okay, this is one of the main results that we will prove in the next lecture. Okay. Let me stop here.