**Linear Algebra**
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**Module no 06**
**Lecture no 22**
**Dual Basis II. Subspace Annihilators I**

We are discussing the notion of dual basis okay. Let me start with an example. Example where I show how a dual basis corresponding to a basis of R3 could be constructed, okay?

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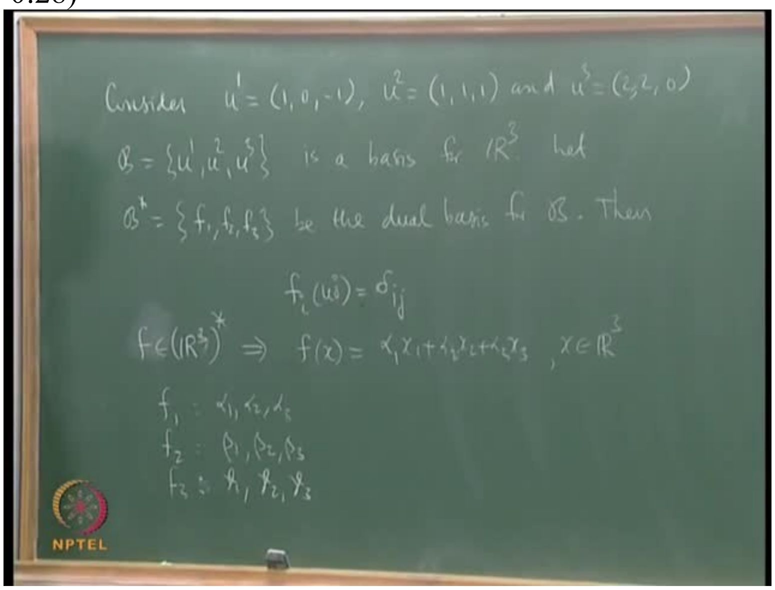


Let us consider the following vectors. Last vector is 2, 2,0. You can verify that these 3 vectors form a basis of R3. What is the basis dual to this basis? Okay? Let us try to construct that simply using the definition of what the dual basis is. We are using B star okay to denote the dual basis. So let me say B Star has the vectors functionals F1, F2, F3 be the dual basis corresponding to the basis that we started with. Then we know that by definition, these 9 equations must be satisfied by these 3 functions. FI of UJ equals delta IJ, IJ vary from 1 to 3, these 9 equations must be satisfied by these functionals.

Using these equations will determine these functionals but you will see that this is where what we learnt for solving systems of equation will come in handy. What are the equations that we need to solve? Let me write it in full and then write the compressed form and then use what we learnt earlier. What this means is for, to begin with F1 of U1 is 1, F1 of U2 is 0, F1 of U3 is 0. Okay, so I have these equations. But okay, before I write, let me also use this observation, let me also use this observation that I made the other day. If we haveF as a linear functional on RN then F has the representation, F of X equal to A1X1 + A2X2, et cetera ANXN.

Okay. Let me use that representation by observing that F element of R4 Star, R3 star in this example, dual space. F belongs to this implies that F of X equals let us say alpha 1 X1 + alpha 2 X2 + alpha 3 X3 where as usual, X is in R3, the notation for X is X1, X2, X3. Okay. So this was proved 1st immediately aftergiving the definition of F, linear functional. We will make use of this now. So these F1, F2, F3 must satisfy these equations where each FY will have certain coefficients. Okay. So let me say, for F1, I will use the coefficients alpha 1, alpha 2, alpha 3, that is F1 of X is alpha 1 X1 + alpha 2 X2 + alpha 3 X3. F2 is another linear functional, beta 1 X1 + beta 2 X2 + beta 3 X3.

So alpha, beta and then let us say for F3, gamma. Gamma 1, gamma 2, gamma 3. Okay. So what is the problem? The problem is to determine these 9 numbers. For instance, if I determine the 1st 3 numbers, I know what F1 is. F1 of X is alpha 1 X1 + alpha 2 X2 + alpha 3 X3. So I need to determine these 9 numbers. There are 9 conditions, 3 into 3, 9 conditions coming from this. So there is a unique solution. Okay. Okay, let us then write down these equations.

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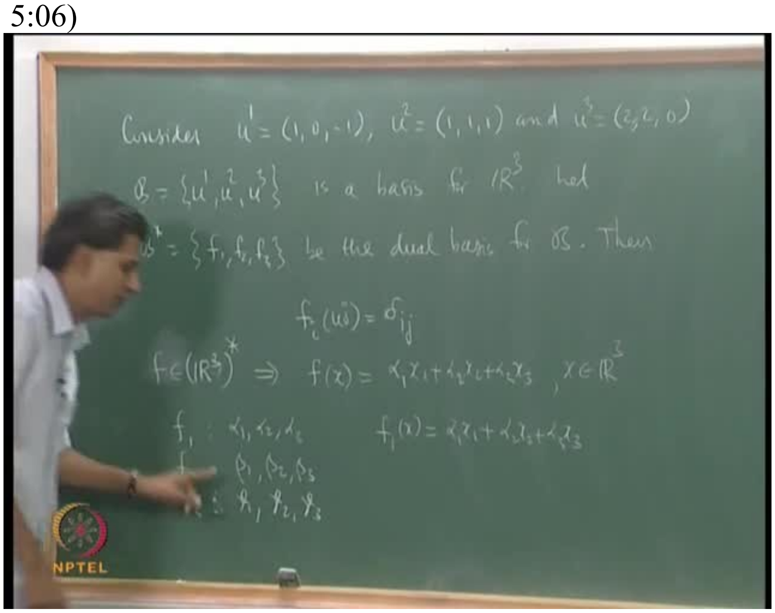

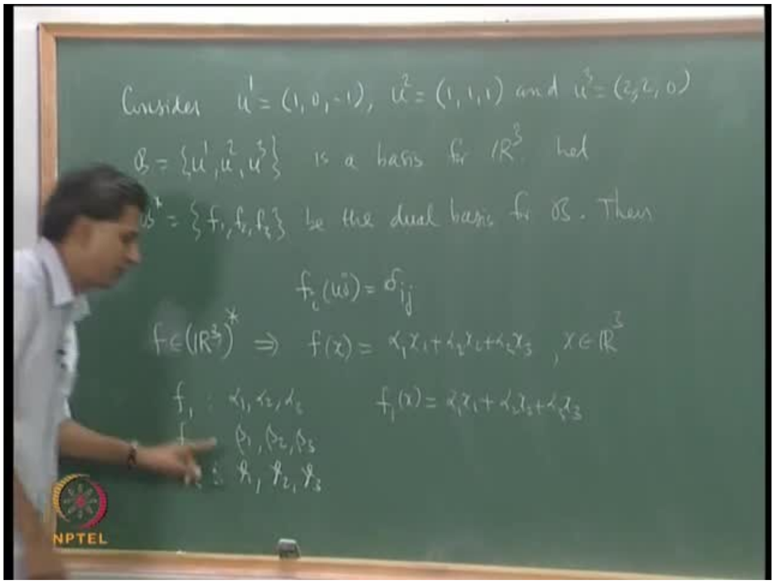


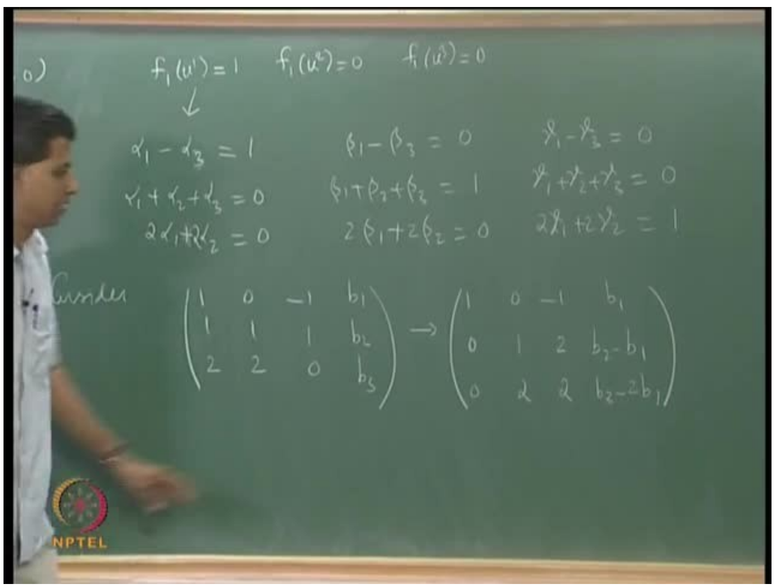


In the light of these representations, I have the following. FI of UJ. So I am going to write down F1 of U1, this is 1. F1 of U2 is 0, F1 of U3 is 0. I am taking 1st, the functional F1. F1 goes with the 3 numbers, alpha 1,alpha 2, alpha 3 okay. So what is F1 of U?le Let me write like this itself. F1 U1 equals 1 gives me alpha 1 - alpha 2 equals 1. Do you agree? F1 of U1. U1 is 1, 0,- 1. What is the, okay, just to confirm, the notation that I am using here is maybe I will go back and write it here. This is a general functional.

For F1, what does this mean? This means, F1 of X is alpha 1 X1 + alpha 2 X2 + alpha 3 X3. For F2, it is beta 1 X1. F2 of X is beta 1 X1 + beta 2 2 + beta 3 X3. Similarly for F3. So I am going to determine F1 of U1, F1 of U2,F1 of U3. This is the definition of the dual basis. I have just written down the 1st 3 equations. So I get alpha 1 - alpha 2 equals 1. That is the 1st equation. F1 of U2 - alpha 3. Yes. F1 of U2, just sum up. Alpha 1 + alpha 2 + alpha 3, that must be 0. F1 of U3, the 1st 2. 2 2 alpha 1 + alpha 2 is 0, that is alpha 1 + alpha 2 is 0. Okay let me write 2 alpha 1 + 2 alpha 2 and then we can simplify it later. This is the system that alpha alpha 1, alpha 2, alpha 3 must satisfy. There is a similar system for beta 1, beta 2, beta 3, similar system for gamma 1, gamma 2,gamma 3. Okay.

In order to solve this system, what do we do? This is like AX equal to B. We need to, we need to reduce A to the row reduced echelon form and probably A will be row equivalent Y. Now the dual base is unique. So in this case, A will be row equivalent Y. In general, it may not be but in this case, the dual base is unique means the solution for this system must be unique.A X equal to B has a unique solution, square system if and only if A is invertible, that is, that gives rise to unique solution. So this will, this should be an invertible matrix. Okay. Instead of solving all these 3 systems one after the other, we would just solve one system and substitute for the right-hand side.

See, it is the same, okay what happens to the next system for beta? You will again get something like this. See, forbeta, I am determining F2. For F2, this is 0, F2 of U1 is 0, F2 of U2 is 1,F2 of U3 is 0. So that will give me beta 1 - beta 3 equal to 0, beta 1 + beta 2 + beta 3 equals 1 this time. Beta 1 + 2 beta 2 equals 0. This must be the system that beta will solve and similarly for gamma. The other system is gamma 1 - gamma 3 equals 0, gamma 1 + gamma 2 + gamma 3 equals 0 this time. Finally, the last equation. 2 gamma 1 + 2 gamma 2 equals 1.

So it is essentially the same coefficient matrix A, the right-hand sides change. So let us do it efficiently than solving each of the systems individually. So I will take the general, a general right-hand side, reduce it to the row reduced echelon form and then substitute these vectors. Okay. In other words, I consider solving the following system.So I write down the coefficient matrix, 1 0 - 1, 1 1 1, 2 2 0. Just the row vectors that I have here. This I want to solve this for a general right-hand side. This time I will take B1, B2, B3, do elementary row operations, reduce to row reducedechelon form where this part will be reduced to the identity matrix.

Then instead of B1, B2, B3, I will substitute these 3 right-hand side vectors to get the solution. Okay. So this I need to do elementary row operations. From here, I get 1 0 - 1 B1 will be.- this+this 1 2 B2 - B1.- 2 times is + this B3 - 2B1. Please check the calculations here. Okay? Next step, I will keep this as the pivot row. I need to make this 0. Okay, I will keep this as the pivot row.

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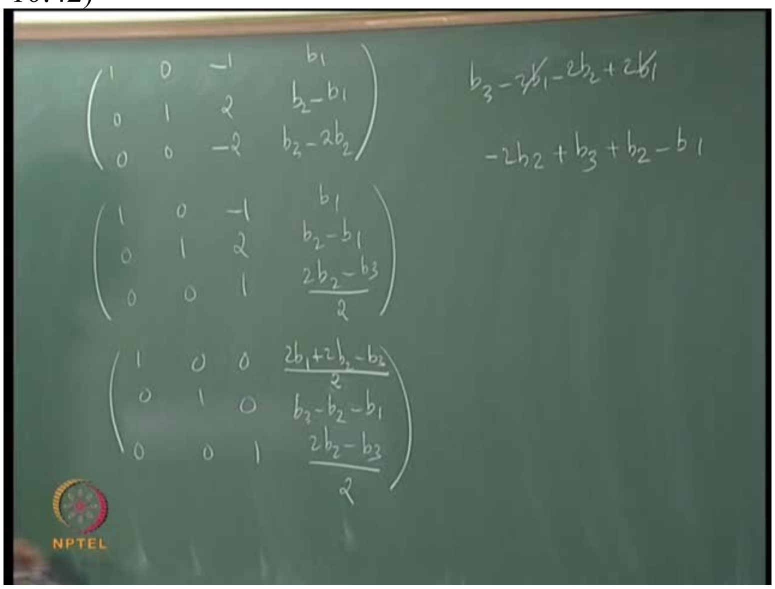


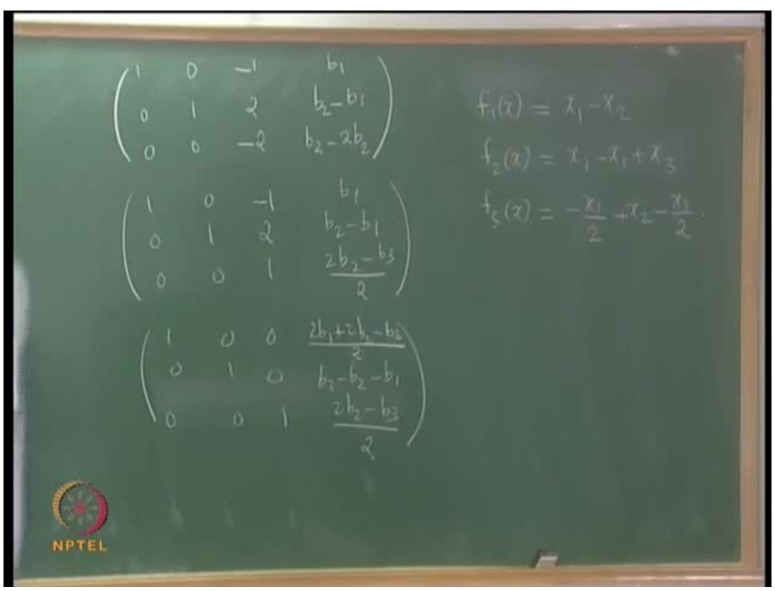


So the next step is I get this matrix. 0 1 2. B2 - B1, this will be, oh the 1st row is also kept as it is. This will be - 2 times this + this.- 4 - 2.- 2 times this + this. B3 - 2B2. Then I will divide throughout by - 2, the next step. Then I make these 2 entries 0. So last row is kept as it is. Just add these 2. Add these 2. 2B2, okay let me write like this. 2B1 + 2B2 - B3. I need some space. I am just adding these 2 rows to replace the 1st row. Then - 2 times this + this. I do not still have enough space. Okay.

- 2 times this + this. This will go with the negative sign,- 2. Okay what are the calculations? - 2 times this.- 2B2,+ B3,- 2 times this + this.+ B2 - B1. Is that okay? - 2 B2.- 2, that becomes + B3. Yes, so that is B3,- B2,- B1. That is this entry. Okay. Let us now, it is clear that this is the row reducedechelon. This is the identity. So the system has a unique solution as we expected. Let us not determine the solutions, alpha 1, alpha 2, alpha 3, beta 1, beta 2, beta 3, gamma 1, gamma 2,gamma 3.

For determining alpha 1, alpha 2, alpha 3, right-hand side is the vector 1 0 0. B1 is 1, B2 is 0,B3 is 0. So what is F1 then? So let me write F1 right away. Alpha 1, so that is the 1st vector. B1 is 1. So I get a 1 here. So it is X1 - 1. This is 0. So F1 of X is X1 - X2. That is alpha 1 is 1, alpha 2 is - 1, alpha 3 is 0. Is it clear what I am doing? This is for a general right-hand side vector B. To determine the solution corresponding to this system, I will replace the right-hand side vector by this vector. This is B1, this is B2,this is B3. Just substitute there. And then, these are diagonal entries.

The 1st equation gives you X. The 1st equation gives you alpha 1 equals this, alpha 2 equals this, alpha 3 equals this where B is 1 0 0. Okay. What happens to the 2nd one? 2nd vector. So B2, so that is again a 1 X1.- 1 + 1. Is that okay? I get a 1 here, I get a - 1 here,I get a 1 again. So that is F2. F3 corresponding to 0 0 1. 0 0 1,- 1 by 2.- X1 by 2. 0 0 1,- X2. 001,-…

Student: B3.

Student: B3.

Professor:- X3 by 2.

These are the 3 functionals. Now you can verify that these 3 functionals satisfy these 9 equation. FI UJ equals Delta IJ. For instance, look at F1 of U1. U1 is 1 0 - 1. So F1 of U1 is 1. F2 of U1, 1 0 - 1. X1 + X3, 1 and - 1. That is 0. F2 of U3, 2 2. That is - 1 + F3 of I am sorry, I am calculating

F1 of U1, F1 of U2. F1 of U2 is 1 - 1. F1 of U3 is 2 - 2. Okay. So just to verify. So this is one method of doing it. Okay. And we have done it little more efficiently than solving 3 systems.

Okay this is an example where we have constructed a dual basis. Compare this with the example that I gave you last time. I have given the dual basis 1st and then I have asked you to find the basis corresponding to which this is the dual basis. Okay? That corresponded to the lagrange interpolating polynomials for the nodal points, T1, T2, T3. Okay, let me move on. I want to discuss the following problem. A formula which somewhat remains as of the rank nullity dimension theorem. Okay, we will derive a formula and then look at 2 corollaries of this formula. So what is the formula? Or I state this formula, let me make the following observation.

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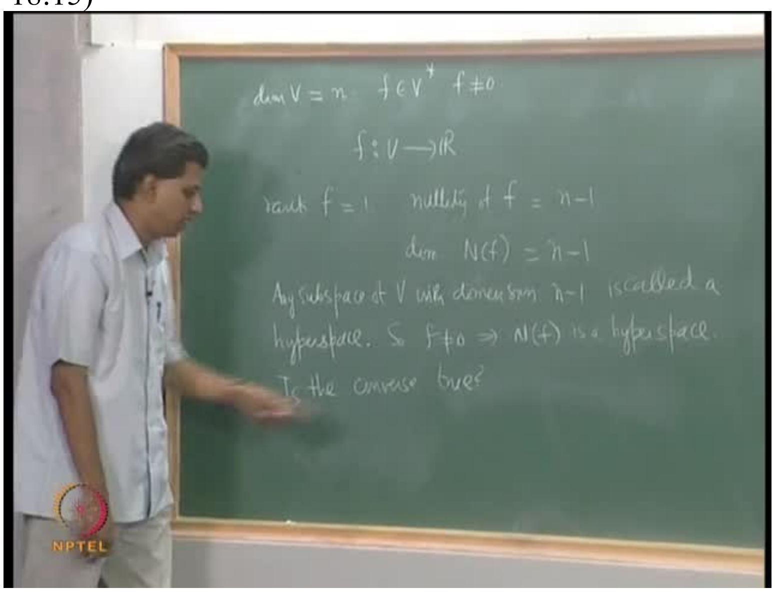


Suppose you have dimension of V is N. Okay, so V is a finite dimensional vector space, F is a linear functional. Let us say F is not the 0 functional. Then what is the rank of F? What is the rank of S? Range of F. See Fis a function from V to R. R means R over R as a vector space.Look at range of F. Range of F if F is not 0, is one-dimensional. So rank is 1. Rank of F is 1. If F is not the 0 vector, 0 functional, then it means that rank of F, rank of, range of F is the subspace of R. Okay. If it is not 0, then the subspace will be the whole of R. And so rank of F equal to 1.

If V is finite dimensional, then by rank nullity dimension theorem, it follows that the nullity of F equals N - 1. Okay, rank + nullity is the dimension of the domain space. Nullity of F is N - 1.

Remember the null space nullity of F is what? Dimension of the null space of F. That is N - 1. Null space of F is the subspace of V. Range of F is the subspace of the co-domain, null space of F is the subspace of V. This subspace has dimension one less than the dimension of the whole space. Such subspaces are called hyperspaces.

Any subspace of V of dimension N - 1with dimension N - 1 is called a hyperspace. Dimension one less than the dimension of the original space. That is called a hyperspace. So what follows is that if F is a nonzero linear function, then null space is the hyperspace okay. So F not equal to 0 implies null space of F is a hyperspace, hyperspace of V. The question is whether the converse is true. This is what we will try to answer. What is the converse? I have the subspace of dimension N - 1. Is there a functional corresponding to this subspace?

That is, is there a functional F such that nulls this subspace is null space of that function? Is the converse true? What is the converse? Given a subspace of dimension N - 1, is there a functional F such that null space of that functional F is equal to the subspace. Okay? We will see that the a little more general question can be answered, affirmative, okay. In order to answer this question, we need the notion of the annihilator. So let me give this definition 1st. The annihilator of a subset of a vector space.To answer this question, we need the notion of the annihilator of a subset.

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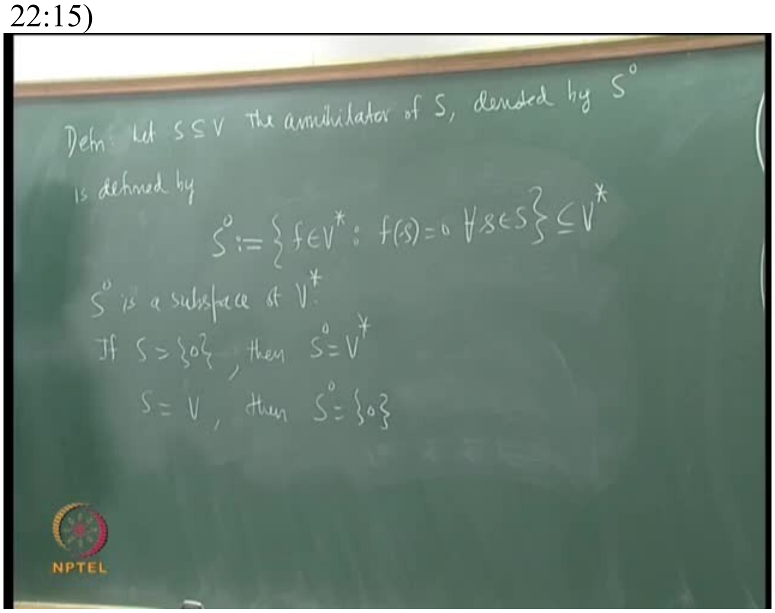

So let me give this definition. Let S be a subset of V. The annihilator of S, let us denote it by S0. Let us denote it by S0. It is defined by S0, this is the set of all functionals, set of all linear functionals on Vthat have the property that F of S equal to 0 for all S element of S. Set of all linear functionals on V which take every element in S to the 0 number, 0 real number. So this is a subset of V star. The annihilator is the subset of V star. So it consists of linear functionals, it consists of certain linear functionals. Now it does not matter what S is, S0 is always a subspace.

S0 is the subspace of V star. Now that is easy to see because if F, G belong to S0, then I must show that alpha F + beta G also belongs to S0. But if F, G belong to S0, then alpha F of F is 0, beta G of S is 0. Their sum is 0. F is a linear functional, so this is a subspace. So S not is a subspace of V star. Our interest is in determining the dimension of S not given V is finite dimensional. S not is a subspace. Let us look at 2 extremes for this S not. If S is single turn 0. What is S not? 2 extremes I said. Are you sure? All functionals, all linear functionals. A linear functional must satisfy the condition that F of 0 is 0. If S is equal to single turn 0 then S not is a whole of V star. If S is equal to V, then S not is the 0 functional. S not is the 0 functional. Okay. Let us now lookat the dimension of a subspace and the dimension of the annihilator of that subspace. They are related and this is the formula that will lead to an affirmative answer for this question, for the converse question.

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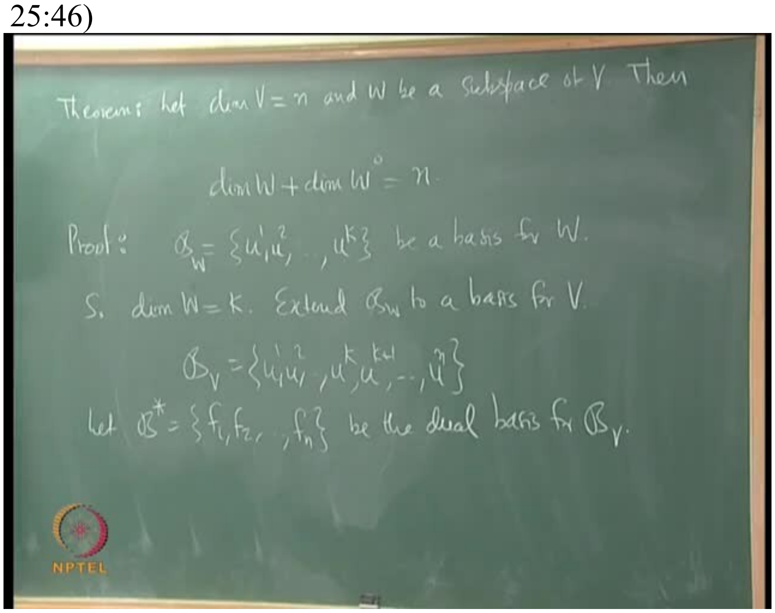

So I want to prove the following theorem. Let V be finite dimensional. So I will assume dimension of V is N and W be a subspace this time, not just the subset. Subspace of V. Then the following formula which I told you must remind you of the rank nullity dimension holds. Look at the dimension of W + dimension of W not. This is the dimension of V. Okay. Now you will see that the proof is also somewhat similar to that theorem. Okay. So remember, thisis not a straightforward result because this connects two numbers, 1 for the vector space V, the other one for the vector space V star okay.

So proof let us start for the basis for W. That means I have B equals U1, U2, et cetera, UK. Let me emphasise that it is a basis for W. This be a basis for W. So W is K dimensional. Any basis for that matter, any linear independent subset can be extended to a basis of the whole space. So I can extend this, extend BW to a basis for the whole space V. V is finite dimensional. So let me write BV for basis for B. BV contains BK, so it contains the vectors U1, U2, et cetera, UK and also other vectors. I will call them UK + 1, et cetera, UN. U1, et cetera UN because dimension of V is N.

This is the basis for V. I can construct, there is a dual basis for this basis. So let B star equal F1, F2, et cetera, FN be the dual basis for the basis BV. This is the dual basis. What we will show is that you kind of remove the 1st N functionals, the remaining N - K functionals we will show forms a basis for W not. Similar to rank nullity dimension theorem. The 1st vectors form a basis for the null space, the remaining vectors form a basis for the rain space. Okay. So let us now look at FK + N etc FN. We want to show that, the claim is that these vectors form a, these functionals form a basis for W not. See we want to determine the dimension of W not. We will be able to explicitly write down a basis for W not.

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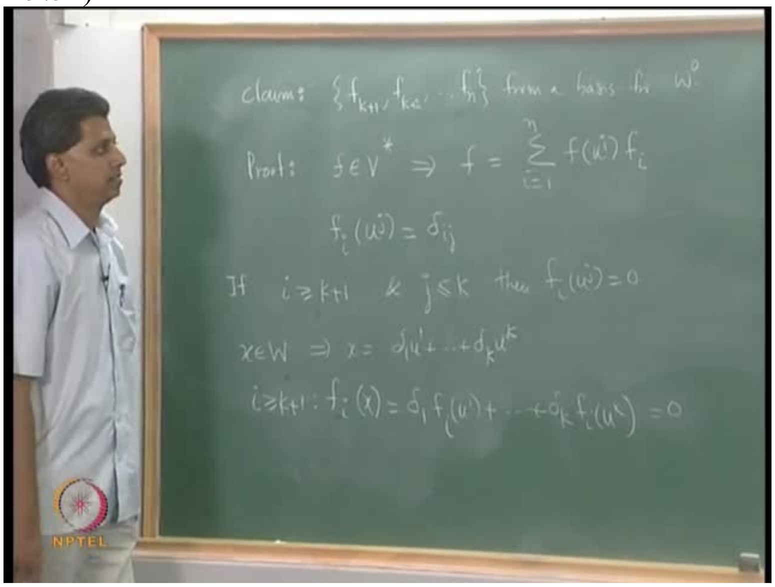


So this is what we will prove. Let me just list these functionals. FK + 1, FK + 2, et cetera FN. These functionals form a basis for W not, this is the claim. Suppose we have proved this claim, then the result follows because the number of functionals here is N - K. There are N - K functionals. So if I have proved that this is the basis, then W not is N - K dimensional, W is K dimensional. So K + N - K is N. So it is enough if we prove this. Okay.

Let us recall the following formula which was proved in the last lecture. Proof is as follows. Proof of the claim. If I have any functional on B, then this any linear functional on B, then this linear functional is a linear combination of the dual basis functionals, F1, et cetera, FN where the coefficients also can be given explicitly. That is F is summation I equals 1 to N, you look at F of UI and then multiply that with FI. F is a linear combination of F1, F2, et cetera, FN because F1, F2, et cetera, FN form the dual basis.

So any linear functional can be written in terms of those. Inaddition, we also have information on the coefficients. The coefficients are FUI. Okay. If, okay what is it that we want to show? That these vectors form a basis for W not. 1st of all, are thesefunctionals in W not? Otherwise there is no sense. Are these functionals in W not? What dowe need to show? Let us just keep this aside for a while. I want to show that these functionals belong to W not. That is, these functionals must satisfy the property that they take the value 0 for all the points, for all the vectors in W, okay?

But look at this. What we know is the dual basis satisfies this condition, FIUJ equals Delta IJ. So if I is, see I am looking at functionals from K + 1 to N. So if I is greater than or equal to K + 1 and J is less than or equal to K, there is no way these 2 can become equal. So you will get the value 0. So if I is greater than or equal to K + 1 and J less than or equal to K, then FI of UJ is 0. For all the I and J, that satisfy these conditions. FI UJ has to be 0. In other words, is that clear? In other words, FK + 1 of U1, FK + 1 of U2 et cetera, FK + 1 of UK, they are all 0. FK + 2 of U1, et cetera, FK + 1 UK, they are all 0.

So what is it that we have proved? We have proved that these functions take the, you take, okay you now take X and W. See, I want to show F is 0 on W. Then it follows that if I want to show F is 0 on W, I must show that F of X is equal to 0 for all X. Okay. You get X belongs to W, then X is a linear combination of those basis vectors. So let me take some Delta 1 U1, et cetera + Delta N UN Delta K UK. Now lookat FI. I am again considering I greater than or equal to K + 1. I am again considering I greater than or equal to K + 1.

That is let us say FK + 1 is a functional that we are using now. Look at FK + 1 applied to this. That will be Delta 1 FI of U1, et cetera, delta K FI of UK but you observe that these scalar, thesesuper scripts, 1 to et cetera K, they are always distinct from K + 1. And so this has to be 0. And so what is it that I have shown? Whenever I is greater than or equal to K + 1 and X belongs to W, FI of X is 0. So in the 1st place, these are functionals that belong to W not. These are functionals that take 0 that take the value 0 for any vector in W.

Okay. Now we need to show that this forms a basis. We must show, they are linearly independent and they spend W not but linear independence is obvious because this is just a subset of the dual basis. Subset of a linearly independent set is independent. So linear independence is easy. We only need to show that span of these is equal to W not. So we will show that W not is contained in span of these. That is the last step.

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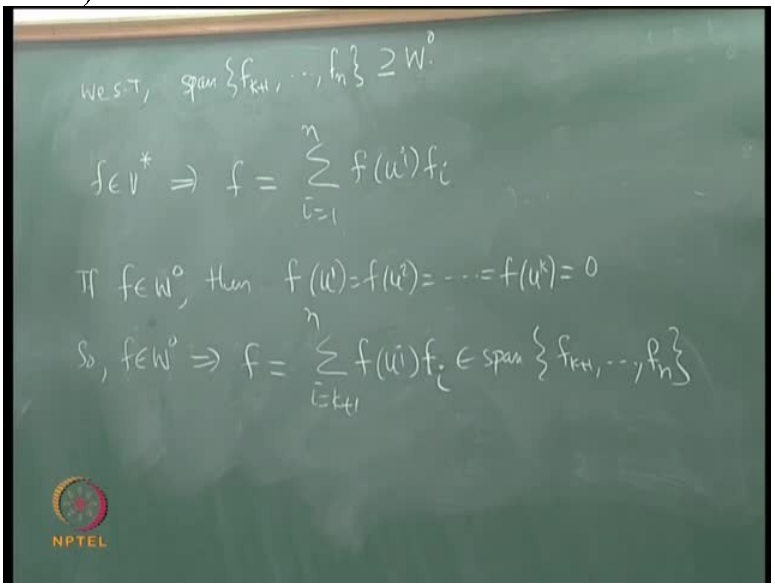


FK + 1 et cetera, FN, they are in W not. So span of those vectors will again belong to W not, there is nothing to see. W not is a subspace. It is only the other way round we need to show. W not is contained in span of these. Okay? Okay let us now look at these representations. I have written down, if F belongs to V star then F equals I equals 1 to N F of UIFI. W not is a subset of V star. So W not consists of functionals. W not consists of linear functionals, let us remember. So if F belongs to W not, if F belongs to W0, that is what I want to show is F belongs to W not implies F is a linear combination of these functionals.

So let F belong to W not. This means what? Then W not is a annihilator of W. So any F will take the value 0 for anything in W. So F of U1, F of U2, et cetera, F of UK, all this must be 0 becauseU1, U2, et cetera, UK belong to W. W not is a set of all functionals that take every vector in W to 0. In particular, that must take the basis vector to 0. So if F is in W not, then these are 0. Go back to this equation. So F belongs to W not implies, this representation has RN terms but if F belongs to W not, the 1st K terms are 0.

So there are only N - K terms. Summation I equals K + 1 to N F of UIFI. Forget about these numbers. This F is a linear combination of FK + 1, et cetera, F N which is what we wanted to prove. This belongs to span of FK + 1, et cetera, FN. Okay. So what we have done is if F belongs to W not, then F is in the span of this. So that proves this part and so it follows that if we go back, this claim has been proved and so it follows that W dimension W + dimension W not is N okay. W is K. That is the dimension of W. Dimension of W not is N - K which we proved just now. That is N. That is a dimension of V. Okay. Let me stop here.