Linear Algebra
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Module no 06
Lecture no 21
Linear Functionals. The Dual Space. Dual Basis I

In the next few lectures, I want to discuss the notion of linear functionals, dual, double duals and the transpose okay. Maybe about 3 or 4 lectures. So let me write down the topics.

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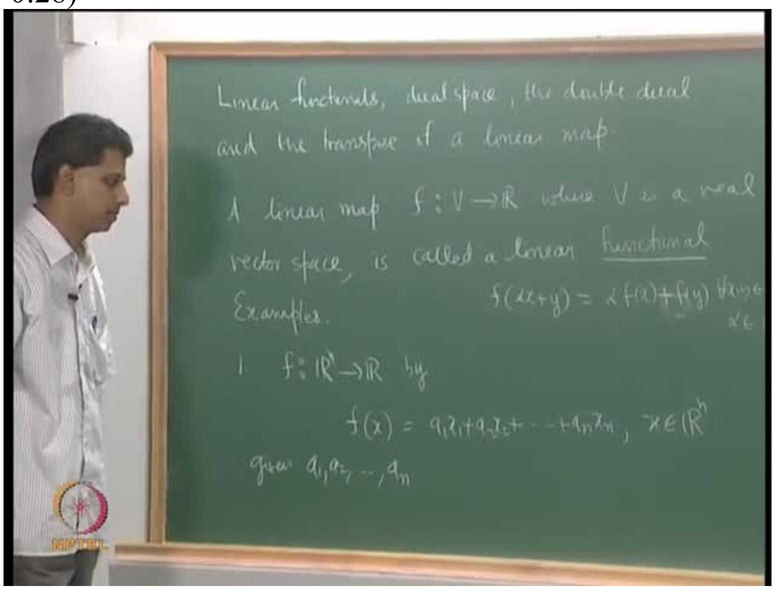

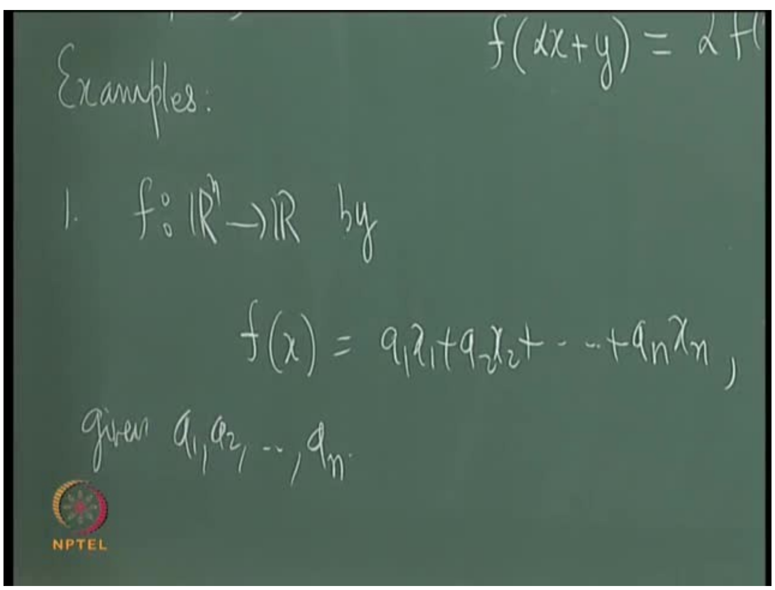

Linear functionals, dual spaces, the concept of the dual space, the double dual and the notion of the transpose of a linear transpose. Okay. Okay so let me start with linear functionals. A linear functional is a special case of a linear transformation when the co-domain space is the underlying field okay. So the definition of a linear function. A linear map, F from V to R where V is a real vector space is calleda linear functional. So what is important is the word functional. Linearity, we have encountered before, throughout the course, we will encounter this notion of linearity.

We will in particular look at what are called linear functionals. So linear functional is a linear map from V to the underlying field, in most of the cases for us, the underlying field will be R. So this is a real linear functional. Okay. Let us look at some examples. Okay by the way what does it mean? Linear functional F means, we always use T. T from V to W. For functional, I will use the word F, the 1st letter of functional. Sowhat is the meaning of this? This means F of alpha X + Y. This alpha FX + Y where XY comes from V and alpha comes from R only to emphasise what a linear functional is. F of what you must remember is that on the right I have addition of real numbers okay. This is addition of real numbers.

The addition on the left is happening in B. Okay. Let us look at examples. The 1st example is I can take any FN and define a functional over F where F is a field, real complex, et cetera but again, I look at a specific case. I will define F from RN to R as follows. F from RN to R, any vector must be mapped to a real number. I will define that as F of X equals A1X1 + A2X2 + et cetera + ANXN where I am given the numbers A1, A2, et cetera, AN. X is in RN. I am given the numbers A1, A2, et cetera AN. This is obviously a linear function. That can be verified immediately here. In fact, this is for my FX equals AX where A is just the row vector. Row vector consisting of the elements A1, A2, et cetera AN. So row vector into the column vector X1 et cetera XN. This is the functional. This is a linear functional.

Now what is important is to see that any linear functional on RN can be given by this representation. So we that gives the complete understanding of the linear functionals over RN,In fact over any FN. Solet me prove that quickly. This is the linear functional, all right? But if you take any other linear functional, that must also be of this form. Then we have completely understood what linear functionals over RN R okay? So let me prove that quickly.

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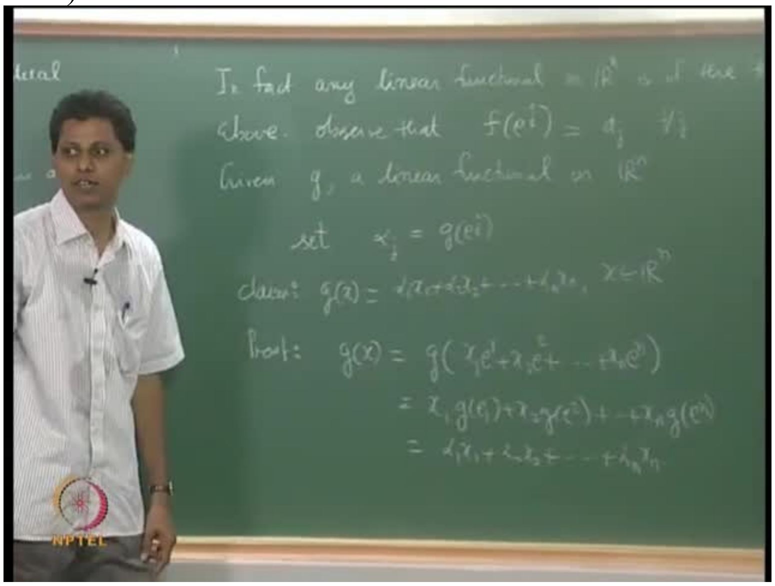

In fact, any linear functional on RN is of the form above. Let me prove that quickly. Now in order to prove this, let me make the following observation. Observe that F of EJ, EJ is the J standard basis vector, then F of EJ equals AJ. Is that clear? This is my F. F of EJ, say this X is in RN. What is F of EJ? F of EJ must be the vector, EJ is a vector whose Jth coordinate is 1, all other entries 0. So it is only AJ on the right. So F of EJ equal to AJ. Let us make use of this. So what is it that I need to prove? Given F, let me say given G, given G, a linear functional on RN, I am given a linear functional on RN, I want to that G has this representation. I want to show that G has representation.

So will just make use of this, that is given G, a linear functional on RN, I have G of EJ to be some number. G of EJ is some real number. I will call that alpha J. So I call the number alpha J as G of EJ. This is a real number I know. Okay. Then I want to show that G of X is of this form. I want before that G of X is of this form. This time, it will be alpha 1 X1 et cetera, alpha N XN, that is a claim. So let us consider okay what is the claim? Claim is G of X equals alpha 1 X1 + alpha 2 X2 +, et cetera alpha N XN. Okay. That is easy to see. Let us do that quickly. Proof of this G of X. X is in RN, I use the standard basis G of summation okay let us say X can be written as X1E1 + X2E2 + et cetera + XNEN.

This is a unique representation of any vector XNRN in terms of the standard basis vectors, E1, E2, et cetera EN, G is linear, so this is X1 G of E1 + X2 G of E2, et cetera + XN G of N. Now I will take these numbers and replace G EJ by these numbers. So this is alpha 1 X1 + alpha 2 X2 + et cetera + Alpha N XN. So I have proved what I wanted. G of X for me is alpha 1X1 + alpha 2 etc alpha N XN. So G of X has this form. Okay. So linear functionals on FN are completely determined. I know how they look like. Okay, let us look at other examples. This is my 1st example.

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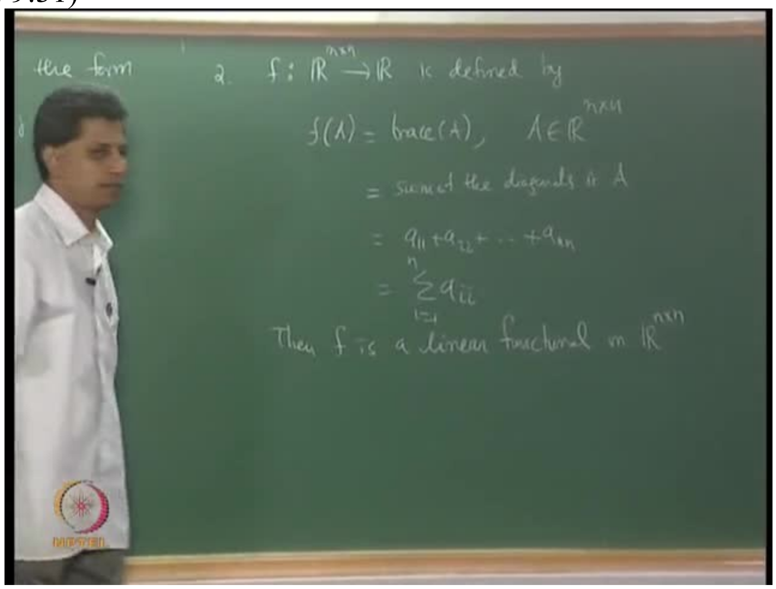

Let me look at other examples. One of the important examples of a linear functional is the so-called trace functional. Let me define F from RN cross N to R. F of A equals trace of A. F of A equals trace of A. What does the trace of square matters? It is some of the diagonal entries. So this is equals sum of the diagonals of A. We will usually denote the entries of A by AIJ and so this is equal to A11 + A22 + ANN summation I equals to 1 to N AII, that is the trace functional. It is easy to see that the trace functional is a linear functional on RN cross N. Okay? 2nd example.

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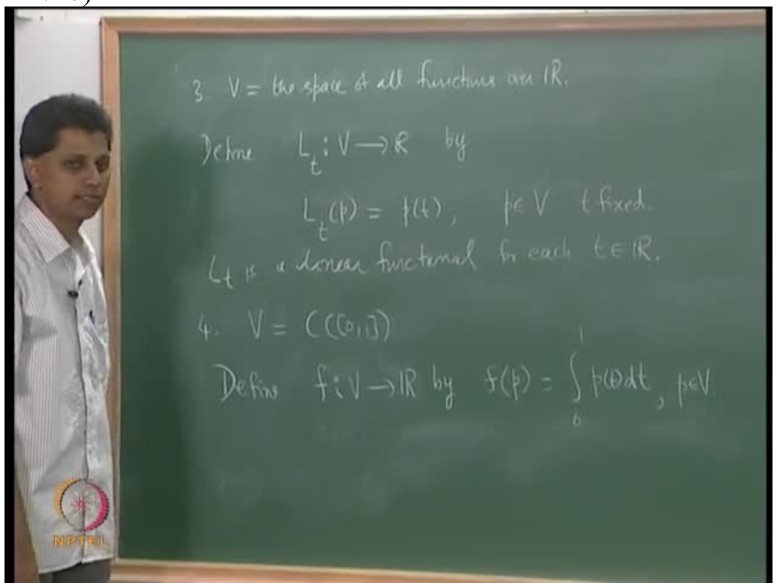

3rd example, let us look at the 3rd example we have so 3rd example is a special case of what we call as the evaluation map. Let us say I have the space of all functions V equals the space of all functions over R space of all functions over R, functions of a single variable, that is what it means. Functions over R. I will define F, okay let me use some other rotation this time. Define L from V to R by LT of P. LT of P equals P of T. This P belongs to V. This is called the evaluation functional. I take afixed T and then define R and then define LT like this.

T is a real variable. See T is a function, so T is a real variable. P belongs to V. V is the space of all functions over R. So T is the real variable, I fix T. Let us say T equal to 0 and then I define L0 as L0 of P is P at 0, the value of P at 0. This is called evaluation map. You can easily show that this is a linear functional. LT is a linear functional for each T. See, this evaluation functional will be discussed once again when I give an example of a dual basis. That is the reason why I wanted to include this example also. Then, let me give you one more example. The last but not certainly the least, linear functional is the following. For me, V this time will be C01 let us say. C01 this time is the space of all real value continuous functions over the interval 01. C01 is the space of all real valued continuous functions over the interval 01.

I define F from V to R by F oflet us say function F function P, that will beintegral 0 to 1, P of TDT. Remember that P is a continuous function, so this (())(14:49) this exists. Therefore P is integral 0 to 1 P of TDT. This exists and sothis is a number. It is a definite integral. This is a number. You can easily verify from the properties of Riemann integralsthis is a linearmap, so linear functional. Okay? F is a linear functional on the space of continuous real valued functions okay. Okay. Now let us look at the collection of all linear functionals.

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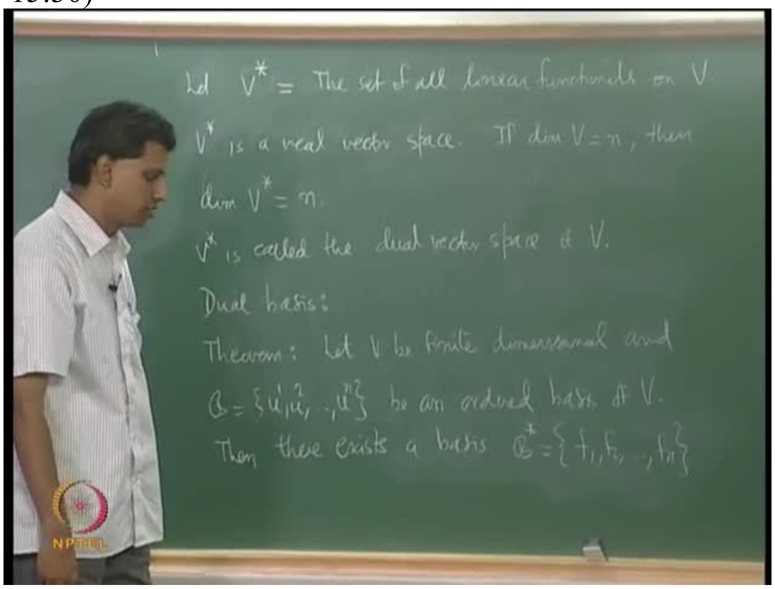

Let us look at the collection of all linear functionals. Let me use a notation V Star. Let V Star equals theset of all linear functionals on V. Now remember that if V and W are vector spaces then the set of all linear transformations T from V into W, set of all linear transformations forms a vector space and we have also determined the dimension of the subspace when V and W are finite dimensions. Okay? So the set of all linear functionals is a vector space because it is a particular case of a linear transformation. The co-domain space is theunderlying field. So the set of all linear functionals is a vector space. So V Star is a vector space and what is the dimension of V Star?

If V has dimension N, V Star is a real vector space. Vector space over the same underlying field. It is dimension ofV equals N, then what is the dimension of V Star? Dimension of the vector space LVW is M times N where M is the dimension of V okay, N is the dimension of V, M is the dimension of W. What is the dimension of R over R? 1. Any field over itself is one-dimensional. So the dimension of V Star is N, same as dimension of V. Okay? So we have a certain information on V Star. V Star has the same dimension as V for finite dimensional spaces. Okay, for finite dimensional spaces, V Star has the same dimension as V.

The question is, can I write down the basis for V Star in such a way that this in some sense corresponds to a basis for V that I start with? That is a question. Okay? Can I write down a basis for V Star in such a way that there is some natural correspondence between the basis for V Star that I write down and the basis for V that I start with? But before that, let me tell you, this V Star is called the dual space. This V Star is called the dual vector space of V. It is a vector space that is dual to V. Dual vector space of V. Set of all linear functionals, the space of all linear functionals. Okay, so this is the question that we would like to address. The answer is yes.

Given a basis of V, there is a natural basis that one could associate with V Star, that basis will be called the dual basis. That is one of the topics that I have written down. So how to determine what is the dual basis? How to determine the dual basis? So we would like to discuss the notion of a dual basis. The notion of a basis, dual to a basis of V, a basis of V Star dual to a basis of V. Okay. We have in fact the following theorem that will define the dual basis. Let V be finite dimensional and I will write down the basis of V explicitly. U1, U2,etc, UN. This is at this B in orderedbasis of V.

Then there exists a basis. I will call this basis B Star, script B Star. And I will call the elements of B star as F1, F2, FN. F will stand for linear functionals. F1, F2,etc, FN, see what I know for sure is that the number of elements in B star must be the same as the number of elements of B because if V is finite dimensional, because we know that dimension of V Star is equal to dimension of V. Then there exists a basis, V Star, this has certain properties. Well, the most important propertythat we are interested in is the following.

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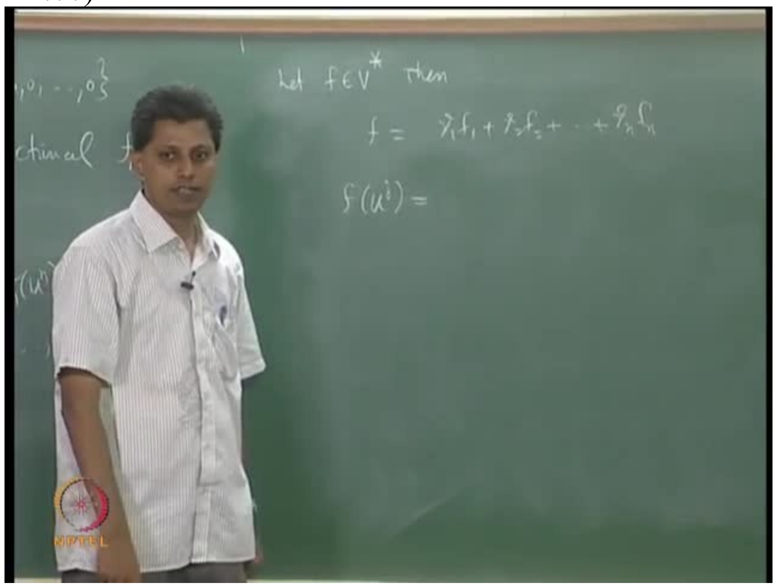

Then there exists a basis B star of V Star such that FI of UJ is equal to delta IJ. This basis F1, et cetera, FN is related to the basis U1, U2, et cetera, UN by means of these equations. Remember, theseare N square equations. Delta IJ is the Chronicle delta. It takes N square values. I and J vary from 1 to N. So these are N square equations. We also have further properties. Take any of the linear functional, any linear functional on B, that is F belongs to V star then I can write down explicitly, F of X in terms of these basis vectors F1, F2, et cetera. See, this F must suppose that we have proved that this is a basis of V Star then any FN V Star is a linear combination ofthese vectors, these functions. So any F can be written as like the alpha 1 F1 + alpha 2 F2 et cetera + alpha N FN okay.

In the case when we have the dual basis, the numbers alpha 1, alpha 2, et cetera can also be written down immediately. FX is nothing but F of U1X1 + F of U2X2 + et cetera + F of UNXN. Okay? This is the formula for F. Any X can in turn be written in terms of these functions. Any XNV can be in turn be written in terms of these functions as follows. Okay, I want this representation. Again come back and look at this X. X is in V. V has this as a basis and so V isso X is a linear combination of these vectors. What are the coefficients? What are the coefficients?

The coefficients are F of U1 U1+ F of U2U2, I need to just check this. See then, X can be written as I want to write F1 of X. Yes what I want is this. F1 of X, see these numbers depend on X. F1XU1 + F2XU2, et cetera FN, this is the explicit representation of any vector X in terms of the dual basis vector. So this is what I wanted to say. Any X is a linear combination of U1, U2, et cetera, UN. In the case of dual basis, once you know the dual basis, you can give the coefficients of X. You can give thecoefficients here, U1, U2, et cetera, UN explicitly in terms of the dual basis vectors, F1, et cetera, FN okay.

So we will prove all these representations. This is the representation for any functional on V, any vector XNV can be written in this manner. Foremost is that F and U, FIs and Ujs are related by means of these N square equations okay? So let us prove this. Any X is a linear combination of U1, et cetera, UN. The question is what are the coefficients? We are claiming that the coefficients can be given in terms of the dual basis vectors, dual basis functions. Okay, let us prove this now.

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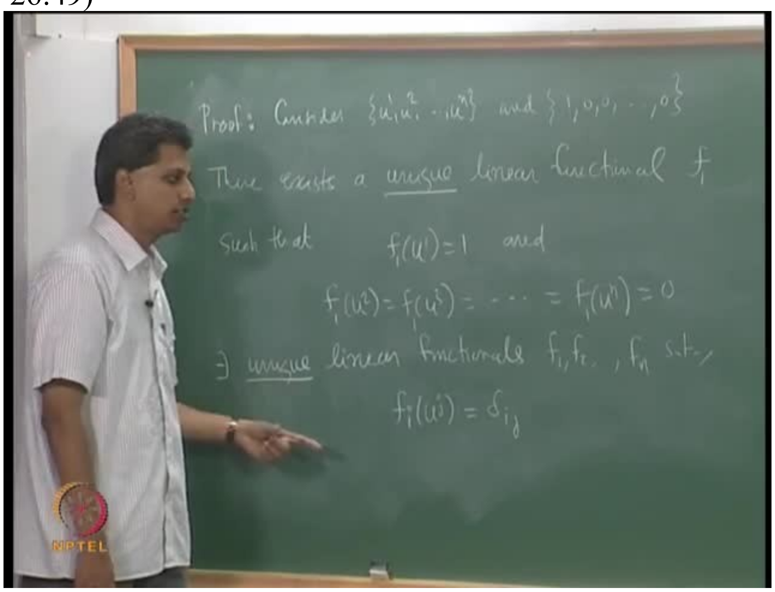

Okay let us start with U1, U2, et cetera, UN. This is the basis of V and the 1st step, I will take these N numbers, 1, 0, 0, till 0. See, this comes of N elements okay. The 1st entry is 1, all the other entries are 0. N - 1 0s, one 1 okay? This is the set of vectors in the vector space R. This is the set of, this is the basis for the vector space B. So these are basis vectors for V, this is any set of vectors in the co-domain vector space. I know that there is a unique linear transformation that takes U1 to 1, U2 to 0, et cetera, UN to 0, this result we have seen. Given U1, U2, et cetera, UN, a linear transformation is defined determined completely by its action on a basis and I know that given a set of vectors, U1, U2, et cetera, UN basis vectors, W1, et cetera WN, absolutely no conditions on these vectors. WN, et cetera WN in W absolutely no conditions.

There is a unique linear transformation that takes UJ2 WJ. Okay. So for this, consider this, there exists a functional, linear functional I want to emphasise. There exists a unique linear functional in fact. Let me write it again. Thereexists a unique linear functional, unique is important. I have not stated that in that year but this is a unique basis. This basis is unique given the basis U1, U2, et cetera, UN. There exists a unique linear functional,I will call it F to begin with. Unique linear functional F such thatthe F of U1 is 0 and all other vectors are mapped to 0.

Yes. There exists a unique linear functional F such that it FU1 is 1, all the other vectors are mapped to 0. I can do this for us of what I do next is, take this basis and then take the numbers, 0, 1, 0, 0, 0, et cetera. I can keep doing this up to N steps. Now for each step, I have a unique linear functional. So what I will do is now, index these by F1, F2, et cetera. Okay. So I will say that there is a unique linear functional F1 such that F1U1 is 1, F1U2, F1U3, et cetera, F1UN is 0. In general, I have there exists unique linear functionals F1, F2, et cetera, FN with the property that F1 of okay FI of UJ equals delta IJ which is the 1st part of the theorem.

That is, F1 of U1 is 1, F1 of all other vectors is 0. F2 of U1 F2 of U2 is 1, F2 of all other vectors is 0, et cetera. So unless I equal to J, I will not get 1. When I is not equal to J all of them are 0. When I is not equal to J, all of them are 0. I is equal to 0, I get one. I equal to J corresponds to F1U1 equal to 1, F2 U2 equal to 1, et cetera FNUN equals 1. So this proves the existence, in fact the unique, existence of unique set of linear functionals that satisfy these N square equations. Okay, that is the 1st part. We need to verify the other 2 representations. Okay.

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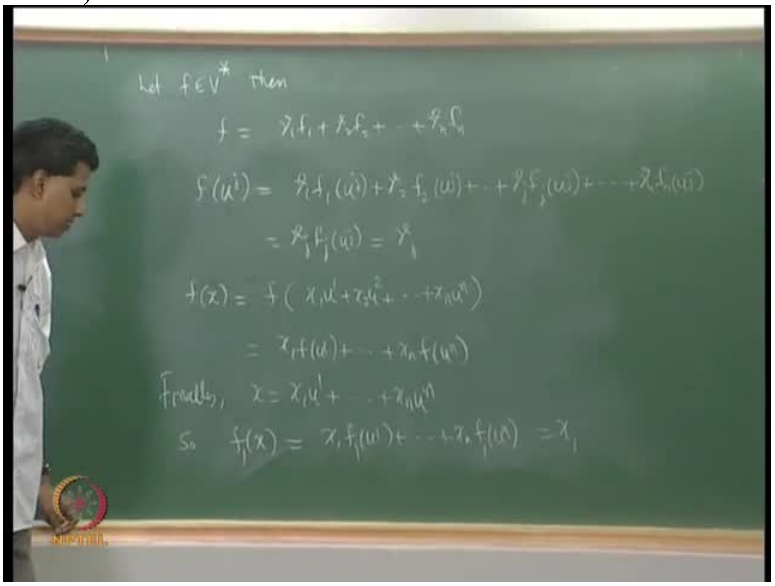

Let F belong to V Star then I have just now proved that, by the waywhat aboutlinear independence of these functionals? I have not proved, I have simply said these functional satisfy those N square equations. Why is it a basis? Why is it a? By the way,are these functionals distinct? Are these functionals in the 1st place different? They are different because F1 takes the value 1 for U1, F2 takes the value 0 for U1, et cetera. So by definition, these are different. That is, there is at least one vector for which the values that F1, F2, et cetera, FN take are different. So these aredistinct to begin with. What about linear independence of them?

Suppose I prove linear independence, then it follows that they must form a basis because dimension of V Star is N. Linear independence of these functionals, that is also easy to see. Maybe I will leave that as an exercise. Okay, linear independence of these functionals I am going to leave it as an exercise. Simple, straightforward. So this forms a basis. V Star is a basis which is related to the basis B that we started with by means of these capital N square equations okay. We also need to show that the other representations hold. Representation for F and representation for any vector X.

Okay, so this F is in V Star, F is a linear functional, we know that F can be written as a linear combination of the dual basis vector. I have not yet defined dual basis vectors. F is equal to let us say something like Gamma 1 F1 + Gamma 2 F2, et cetera Gamma N FN. This is because F1, et cetera FN, these functionals form a basis for V Star. Okay, what is F of UJ? F of UJ is gamma J.

Is not it? F of UJ is gamma J. It is gamma 1F1 of UJ + Gamma 2 F2 of UJ, et cetera gamma J FJ of UJ + et cetera + Gamma N FN of UJ. So that is gamma J FJ of UJ, all other terms are 0, that is just gamma J.

So we have the representation of this FX. Look at FX. F of X equals F ofthis representation is rather incomplete. So I have got to go back, what do I mean by this? What I meant by this is that see, when I write down this, I must know what X1, X2, et cetera, XN are. What is the visit is that X1, X2, et cetera, XN are the coefficients of X in terms of when I write X in terms of U1, U2, et cetera, UN, I mean that the coefficients of X will be X1, X2, et cetera, XN. Okay. In other words, FX, I will write it as F of X1U1 + X2U2 + et cetera XNUN. That is I am assuming that the representation of X in terms of the basis of vectors go with, goes with these coefficients, X1, X2, et cetera, XN.

Okay, so now this is X1 F of U1, et cetera, XN F of UN. F of U1 is gamma 1, I think I have what I want right away. There is no, perhaps what I wanted right away, yes what was the need for this? Ya, there is no need for this. This comes straight away. Okay, the representation of F, the action of F on any vector X, this is the representation. That is what I have written down here. Okay? This part maybe I will use in the next part. Is it clear? This is the representation. That is, if I want to know the action of F, F is any linear functional, I want to know the action of F upon X, then it is enough to find out the action of F on the basis vectors. What I must know is the action,ya what I must know of course is what is the representation of X in terms of U1, U2, et cetera, UN. What are those coefficients? Okay. The last one, X has this representation.

I will go back and look at this representation. Finally, X is my X1U1 et cetera XNUN. So that F of X, F is linear. F of X is X1 F of U1 + X2 F of U2, et cetera. I have determined. Is that okay? What I really want is F1 of X. F1 of X is X1 F1U1 et cetera XN F1UN. Now here, F1U1 is 1, so this is X1. Okay. What is F2 of X? F2 of X is U2, sorry F of X is X2, et cetera,FN of X is XN. I go back and substitute, I get that representation. Okay?

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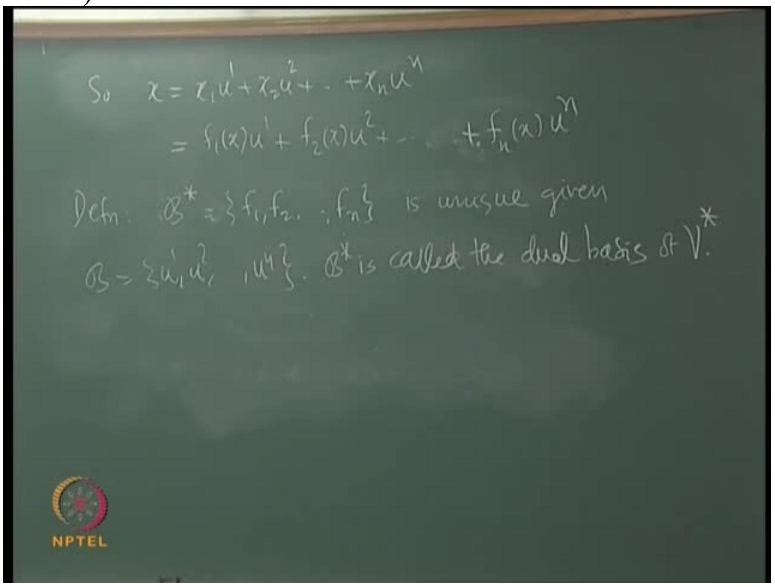

So I go back to this equation. I go back to this equation. X equals X1U1 + X2U2 et cetera XNUN. X1 for me, I have just now determined, is F1 XU1 + X2 + F2XU2 et cetera XNFN sorry, just FNX. F1 XU1 + X2 + F2XU2 et cetera + FNXUN. This is a representation of any vector X in terms of the dual basis vectors, dual basis functions. That is what I do is, given a basis, I determine the dual basis and then compute these numbers, F1 X, F2 X, et cetera, FNX. I know X, then those are the coefficients corresponding to the representation of X in terms of U1, U2, et cetera, UN. Okay?

So let us observeonce again the fact that this B the elements of B star is unique given B. Okay? This B star is called the dual basis. B star is okay let me write it fully, B starF1 et cetera FN, B star is unique givenB. This is the basis of V, this is the basis of V star and they are related by means of those N square formulas. This B star is called the dual basis of V star. We have determined the basis of V in terms of the basis of V that we started with.

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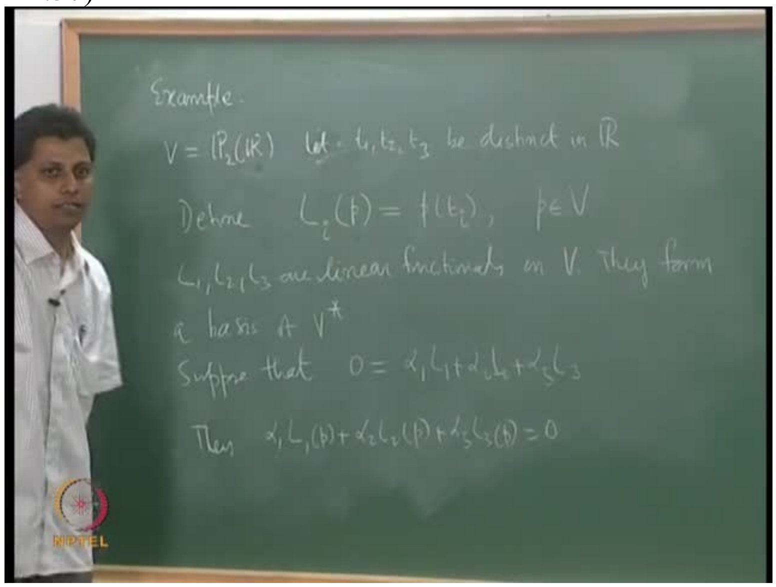

Let us look at an example. Let us look at V as P2 of R. The real vector space of all polynomials have degree less than or equal to 2. This has dimension 3. The dual basis V star will also have dimension 3. I will determine 3 dual basis vectors corresponding to the usual basis of P2, 1TT square. I am sorry, that is not what I will do. I will determine a dual basis and then try to determine what is the basis for which this dual basis is the dual. In this example, I will determine a basis for V star and then determine which is the basis of V for which this is the dual basis. Okay? So for this, we need this evaluation function.

So let me define let us take 3 numbers, T1, T2, T3 as distinct. I have 3 distinctive numbers. Let me define 3 functionals, L1 of P, L2 of P, L3 of P. LI of P is P at TI. Given a polynomial of degree less than or equal to 2, I will compute the value of this polynomial at T1. That is my functional L1 of P. L2 and L3 are determined similarly. I have 3 functionals here. Each is linear. L1, L2, L3 are linear functionals on V. My claim is that this forms a basis of V star. They form a basis of V star. V star is the space of all functions, space of all linear functions on B. I have been claiming that these 3 functionals, linear functionals form a basis of V star. It is enough if I prove that these are linearly dependent. Okay?

So let me prove that these are linearly independent. Suppose that you functional is a linear combination of these 3. Okay? So let us say alpha 1 L1 + alpha 2 L2 + alpha 3 L3. I must show that alpha 1, alpha 2, alpha 3 are 0. Okay. What this means is that these are functionals, so what this means is that the action of this functional on any polynomial of degree less than or equal to 2, gives the value 0. Alpha 1 L1 of P + alpha 2 L2 of P + alpha 3 L3 of P, this must be the 0 number okay, this must be the 0 number here, this 0 is a linear functional, 0 functional. I will look at specific choice of P, 3 specific choices of P.

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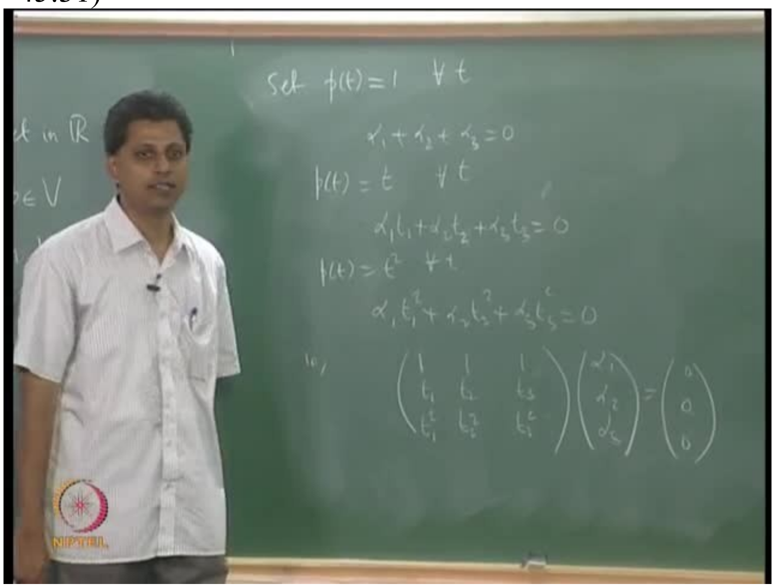

Take the case when P of T is 1. Okay? P of T is 1. Constant polynomial. So what does this equation give me in that case? This is 1, so alpha 1 + alpha 2 + alpha 3 equals 0. Take the case P of T equals T. P of T equals T. Then what is alpha 1 of T? Alpha 1 of T. Alpha 1 L1 of T. T1. Do I get alpha 1 T1 + alpha 2T2 alpha 3 T3 to be 0? L1 of T is T at T1. That is T, okay. I get this for the next polynomial, T square. I must get alpha 1T1 square, alpha 2T2 square + alpha 3 T3 square. That is 0. These are the 3 questions that must be satisfied by the numbers alpha 1, alpha 2, alpha 3 okay? I get a homogeneous system. I get a homogeneous system, the 1st row is 1. 2nd row, T1, T2, T3. 3rd row, T1 square, T2 square, C3 square, alpha 1, alpha 2, alpha 3.

Okay? T1, T2, T3 are distinct numbers. It can be shown that this matters is invertible. You can do elementary operations on this and reduce this to the identity matrix. See this is called the so-called vandermonde matrix. Okay? If T1, T2, T3 are distant, then this matrix is invertible. Homogeneous system with an invertible coefficient matrix as 0 as the solution, so alpha 1, alpha 2, alpha 3 are 0. L1, L2, L3 are linearly independent. Okay? They form a basis for V star because V star has dimension 3.

Dimension of V star is 3. These form a basis. I will leave, okay I will just I will probably answer this question. What is the basis of V? What are the 3 polynomials which form a basis of V which give this dual basis? What are the 3 polynomials which form a basis of V which for which this is the dual basis? That can be determined by looking at the defining equations of the dual basis, FI of UJ equals Delta IJ. The answer is so-called lagrange interpolating polynomials. You will study this in numerical analysis. The polynomials in V corresponding to which this is the dual basis, L1, L2, L3, are the lagrange interpolating polynomials at T1, T2, T3, okay. So please try to determine these 3 polynomials for which this L1, L2, L3 is the dual basis. So let me stop here.