Linear Algebra

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Module no 05
Lecture no 20

Matrix for the Composition and the Inverse. Similarity Transformation

So today we will discuss some properties of the matrix of a linear transformation. Okay. Let me recall the important result that we proved last time towards the end.

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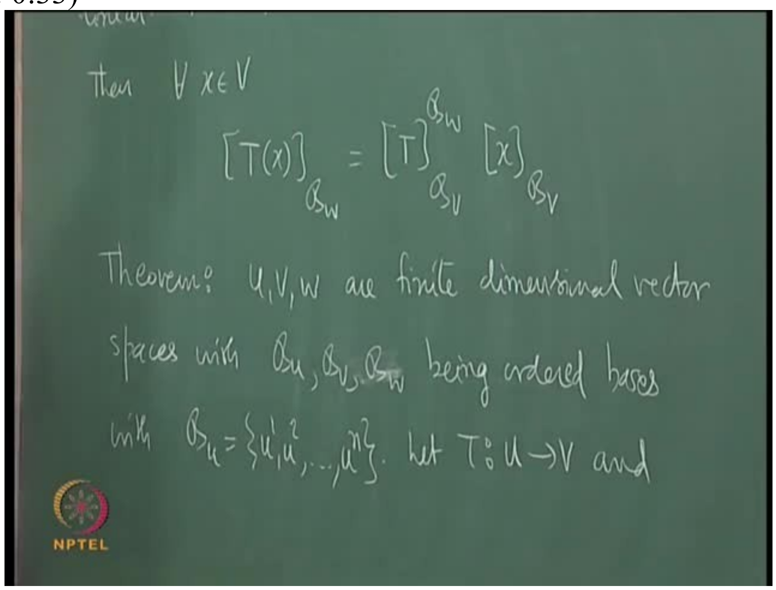

V and W are finite dimensional vector spaces. Over the same field, let us say real field. T from V to W is linear. I have 2 ordered basisof V and W respectively. I amusing this notation for that, BV BW. Then we had seen last time that for every X and V, if you look at the matrix of TX, TX is in W. The matrix of TX related to VW. This is the matrix of T the relative to the basis BV BW into the matrix of X relative to BV. This is what we said is the converse of the statement that if A is a matrix, then TX equal to AX is a linear transformation, okay. Let us look at a few more properties of the matrix of a linear transformation,specially we will look at 1st how it behaves. What is the matrix of the composition of linear transformations?

What is the matrix of the composition of linear transformations? What we will see is that this is the multiplication of the corresponding matrices and this is really the defining place for matrix multiplication, something that we should always remember. Matrix multiplication, the unique, peculiar way it is defined, comes really by looking at matrices as linear transformations okay. So the next result really defines matrix multiplication. Then we will also look at the question, if T is invertible,how can you compute the matrix of the inverse transformation T inverse okay? The natural answer is, it will be the inverse,the matrix ofT inverse relative to the same basis will be the inverse of the matrix of the transformation T okay.

And also finally establish relationship between matrices corresponding to different basis okay. These 3 results we will discuss today. So the 1st is composition. So this is a framework for me. The framework isI will use a slightly different notation for this theorem. You will see itis because of its simplicity. I have UVW, finite dimensional vector spaces. Real vector spaces say with I will write down the basis for U. So these are ordered basis. I will write down the basis BU explicitly. I need to talk about composition of maps. So I have 2 maps. T is from U to V.

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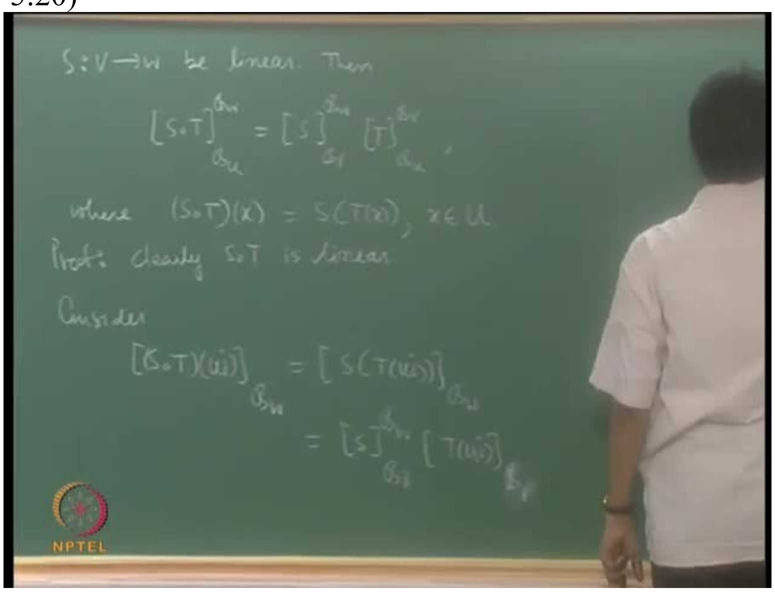

T is from U to V and S is from V to W. Suppose these are linear. Suppose T and S are linear, then I am looking at the composition. 1st I must define the composition but before that, I want to write down the formula for the matrix of the composition of data. Remember, a circle T. A circle T is a map from, see T is from U to V. So T takes a vector X from U to V. S takes that vector TX to W. So this is a matrix from U to W. So I must write DUBW. Okay, this is a linear transformation, this is a function in the 1st place from U into W. What we would like to demonstrate is that this is S, now is S is from V to W. So BV BW into T.

T is from UV BU BV where I have not yet defined the composition where a circle T of X, the composition, a circle T of X, this is equal to S of T of X where X is in U. This is the formula for the composition. For X and U, a circle T of X is S of T of X composition. Okay. So let us prove this result and you will see that this really defines, sees if S and T are linear transformations, we will now show that S it is easily see in that S circle T is a linear transformation, so we know what this matrix is. What this formula says is the matrix multiplication of the matrix of this transformation and the matrix of this transformation, the product is defined by the left-hand side matrix okay?

So this defines matrix multiplication really. Given 2 matrices I can always find linear transformations S and T such that the 1st matrix A is this, the 2nd matrix B is this, then I would like to know what is AB? That is given by the composition a circle T relative to these 2 bases. Okay? So this is really the definition of matrix multiplication. Okay. So let us prove this. Beforeproving this formula, I must show that a circle T is linear but I am going to leave that as an exercise. So proof, clearly composition of linear transformations is again a linear transformation. We need to observe that a circle T is a linear transformation from U into W. Okay.

Remember, this is a formula connecting to, showing that 2 matrices are equal. Let us say, I want to show matrix T is equal to matrix Q then I will have to show that the corresponding entries are the same. A little more general, I will so that the Jth column of the matrix MT is equal to the Jth column or the matrix Q. Then it follows that T is equal to Q. That is what I will do. Okay. So let me start with, I have written downthe basis BU explicitly. I am going to exploit that. Consider a circle T of UJ, a circle T of UJ is a vector in W, I want to look at the matrix of this relative to BW. Want to look at this is a matrix relative to BW.

This is an element in W. This is a vector in W. I will appeal to this, I will appeal to this result and also you said the definition of composition. So this is equal to let me write this as it is, S of T of UJ, that is composition and then keep this BW. I have simply expanded what is inside this bracket. Now I have S of some vector. What is the matrix of S of some vector? What is the matrix of T of some vector? It is a matrix of T into that vector. Just write the appropriate basis. This is the matrix of S, oh the matrix of S. S is a function from V to W. So the matrix of S will be BV BW into the matrix of T of UJ. Okay.

Now matrix of T of UJ, T of UJ is from U into W. So I must remember to write this as BW. It is, is that correct? BV. T is a function from U into V. So this is BV. TUJ BV. Apply this formula once again.

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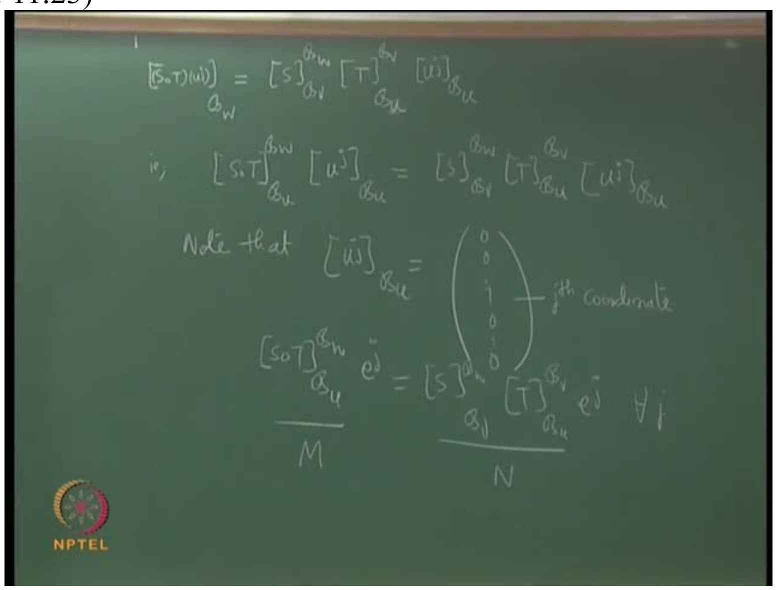

So this is equal to matrix of T. T is from U to V DU BV and UJ is a vector in U, BU. Is that okay? This is U. Is this okay? So what I have done is on the left I have a circle T of okay, now I will expand the left-hand side. This will be, this is another linear transformation. I can call that R if you want. So this is a circle T, a circle T is now a matrix,is now a transformation from U to W. UJis in U BU, this is my left-hand side and the right-hand side, I will write it as this. Do you agree with this? A circle T is a linear transformation, so I can call that as R if you want. Then it is left-hand side is R of UJ. R of UJ, I will appealto the same formula. R of UJ is matrix of R into matrix of UJ. What is R? A circle T. This is because R is a circle T, that is linear. Okay. The only thing that you need to observe is what is the matrix of UJ to the basis BU. Method observation, we are through.Ith component 1, all other entries 0. That is a column vector. UJ, matrix of UJ relative to BU, how do you write that? You must write UJ as a linear combination of U1, U2, et cetera, UN. The only unique linear combination of UJ is 0 times U1 + 0 times U2, et cetera 1 into UJ + 0 UJ + and etc, 0 UN.

So the matrix of UJ relative to BU is the column vector EJ. This happens in the Jth coordinate. This is what we call as EJ, the standard, the Jth standard basis vector of RN, the Jth standard basis vector of RN. This is the matrix of UJ relative toBU. With respect to some other basis, you will not get this. Are we through with the proof? You need to make one more observation. This observation was made much earlier. This is a matrix. On the left, this is a vector, it is really EJ I have written down. So this left-hand side is a circle T BU BW EJ is equal to S T EJ for all J. This is true for all J as J varies from 1 to N.

What you observe is that if A is a matrix and EJ is the Jth standard basis vector, A is of order M cross N, then A EJ is the Jth column. EJ is the Jth standard basis vector, A is an M cross N matrix,then A EJ is the Jth column of A.Jth column of A on the left is the Jth column of this product. If A EJ equals B EJ for all J, then A is equal to B. That is what we have. So if you want, you can call this M, you can call this as N. Then I have M EJ equal to N EJ for all J.Jth column of M is equal to Jth column of N. J arbitrary. So M is equal to N. So these 2 matrices must be the same and so I have this formula.

A circle T BU BW, on the right, BV BW S BU BV T. Okay? Okay. I told you this defines matrix multiplication and we know that matrix multiplication is associative. That can be shown by using this result and one of the previous theorems. So let me just give this as a corollary. To fill up the details is an exercise for you. Matrix multiplication is associative, is a corollary of this result. Okay?

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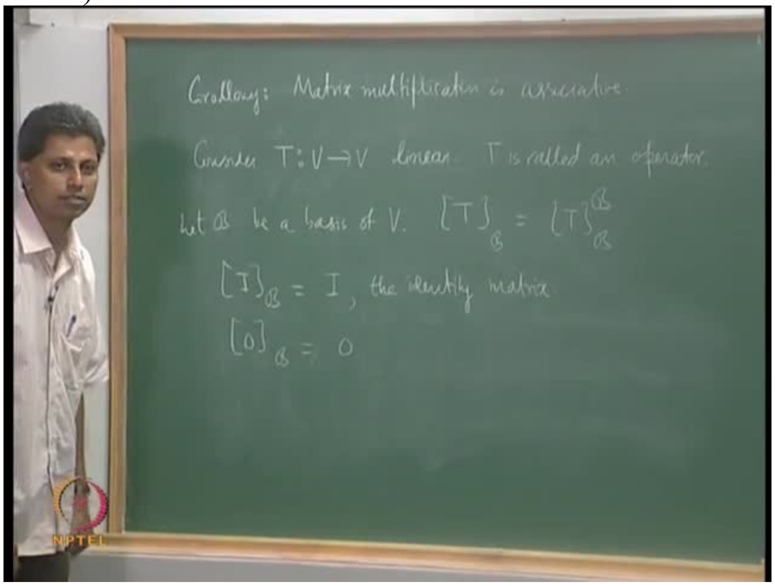

One of the consequences, matrix multiplication is associative.Just a few lines of this proof, given3 matrices, matrix multiplication A, B, C, given 3 matrices A, B, C such that multiplication is possible, the product A, B, C is possible, then product AB, product BC will be possible.

So given 3 matrices such that the product ABC is possible. A into BC is AB into C, that is what matrixmultiplication associativity, means this. We want to show A into BC equals AB into C. You are given A, B, C. Construct 3 transformations, TA, TB, TC such that TAX equals AX, TBS equals BX,TCX equals CX. What is the matrix of the transformation TA corresponding to the standard basis? That will be A. Matrix of TB corresponding to the standard basis, that will be B. Matrix of TC corresponding to standard basis, that will be C. Standard basis in the appropriate spaces. See, the product ABC must be defined. So the number of columns of A must be the same as the number of rows of B, the number of columns of B must be the same as the number of rows of C.

The order of A, B, C will be the number of rows of A times the number of columns of C. So you need to choose appropriate basis and appropriate spaces. K, L, M, whatever. Then use the fact that this formula holds and show that you have AB into C is A into BC okay? So just take matrices, write down the obvious linearnatural linear transformations defined through this matrices and look at the matrices of these linear transformations in turn with relative to the standard basis and apply this here okay? You can show that matrix multiplication is associative.

Okay.One of the other consequences is what is the inverse matrix of a linear transformation which is known to be invertible. Okay?

The answer has been given. Let us provethis result. But before that, let us look at the specific case, see we are talking about inverse transformations, in particular we will look at a linear transformation over a vector space, that is from the vector space to itself. Consider T from B to V linear, such a transformation will be called an operator. If the domain and the co-domain are the same, then T will be called an operator. If thespaces V and W are different, then you have to obviously choose different bases but if the spaces are the same, then it is comfortable to deal with only one basis okay?

So let us say script B, I will not use V. There is only one vector space. Let script be B, a basis of V. I will not use BV. There is only one space here. Then I will use this notation, TB. There is only one notation. So I will use this notation to denote TBB. See I know how to write down the matrix of a linear transformation when 2 bases are given. In particular if the bases coincide, then I know what this right-hand side is. Instead of writing BB, I will simplify the rotation by writing TB. Now this I can do when I know that T is an operator. The space, from the space V to itself. So I will use this notation.

This is just a notation. Okay, this is just a terminology. TB will be this matrix which we know how to compute. Let us look at the particular case. So what is the matrix of the identity transformation on V? What is the matrix of the identity transformation on V? Can you see that it is identity matrix? But if it is between 2 different bases, then it is not identity matrix? Why is this identity matrix? That is because you must look at the 1st basis vector, write it as a linear combination of the same basis. The only choice is 1st co-ordinate is 1, all othercoefficients are 0. So the 1st column is 100, 2nd column is 010, et cetera.

So the identity matrix sorrythe linear transformation I on V with respect to a particular fixed bases B, reading that as 2 different bases, this is the identity matrix. The identity matrix of order the same as the dimension of V. So for the identity transformation, if the 2 bases are the same, then it is identity matrix. If the basis are different, then it is not identity. You can verify easily by simple examples. Also, what is the matrix of the 0 transformation relative to a single basis? That is a 0 matrix. Okay. Left-hand side 0 transformation, right-hand side is a 0 matrix. This is an equation involving matrices. Okay, in this case let us go back to that formula that we derived just now.

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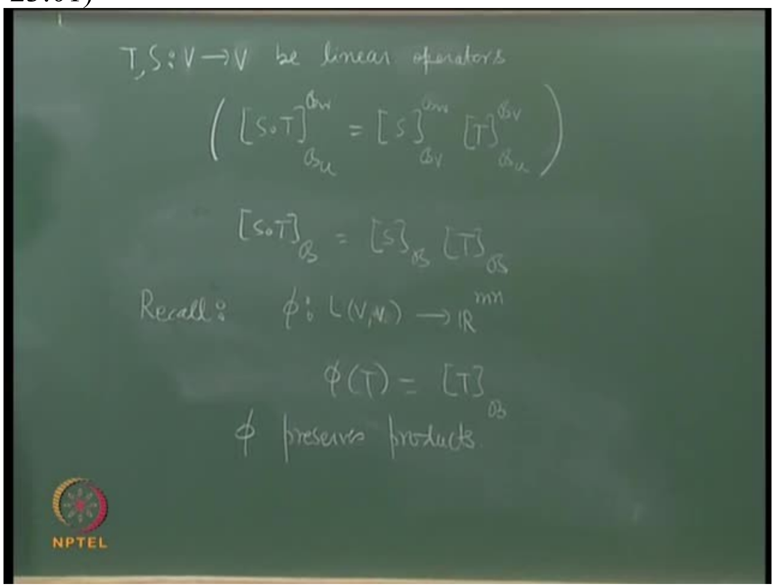

I have T, S from V to itself be V linear operators. This I have proved earlier. I have proved this earlier. What happens with this in this example in this particular situation? a circle T, there is only one basis. Okay, so this is simplified formula when you are dealing with linear operators. Now what it actually means is that little more abstraction can be brought here. I defined the function phi from TL, recall I defined the function phi from, on a linear transformation T. So this is in LVW to R M cross N, RMN or R M cross N. By phi of T equals the transformation T relative to 2 bases. This time I will choose in the case when W is V, I will have only one basis. So this will be with respect only one basis. One be the only base that I started with. LVV can be shortened to LV but I leave it as it is.

We had observed that this is an isomer. This is linear 1 to 1 and down to. And so it is an isomersism and we use this formula to compute the dimension of LVW. If M is M dimensional, W is N dimensional then we computed the dimension of LVW by using this formula, this isomersism. Nowin the light of this formula whatalso follows is that this phi preserves products.phi preserves products. What is the meaning of this? phi of a circle T equals phi of S into phi of T. It is like FXY equals FX into FY. That is called multiplicative function. Phi preserves products.

What is the consequence of this? Consequence of this formula really, the formula that I told you for the inverse transformation when you know that the inverse exists. So let us derive that next.

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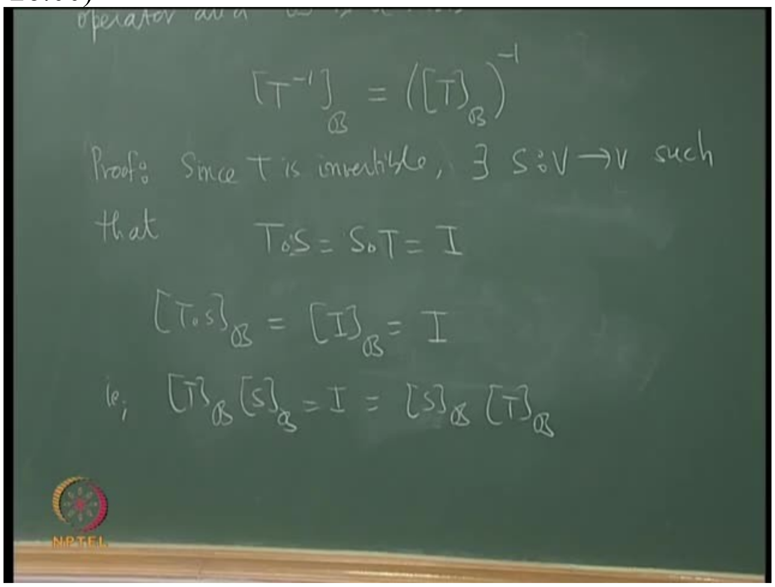

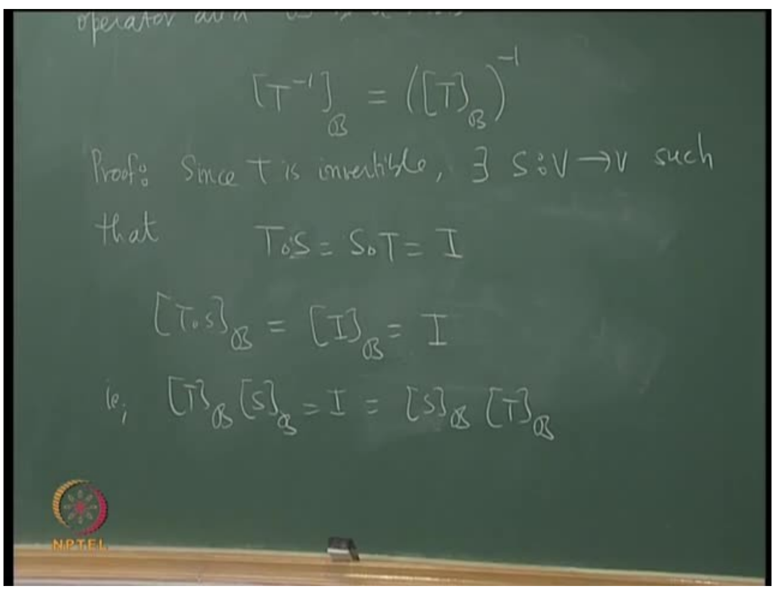

So I want to show this result that let T from V to V be an invertible transformation an invertible linear operator and a script B be a basis of V, then T inverse is also a linear transformation from T inverse is also a linear operator. We had seen this before. What is the matrix of T inverse relative to B? What we want to show is that this is equal to the matrix of T relative to B, take the inverse of that.

So I will introduce a bracket and write this - 1 outside. Okay? Remember this is an equation again involving matrices equation involving matrices. What is inside the brackets are linear transformations. Okay, proof. I will make use of what we had seen just now. This formula and the fact that the identity information with respect to the matrix of the identity transformation relative to a fixed bases is the identity matrix. Since T is invertible, there exists S from V to V such that T is invertible, there exists S such that T is by which I mean T circle S. So T circle S is a circle T equals the identity transformation.

This isthe formula for transformations. There are no matrices here. S and T are linear transformations. On the right-hand side, I is the identity transformation. So I am using the same notation. The context must make it clear whether it is a matrix or a linear transformation. So I will apply this formula to this equation. So if you look at T circle S relative to the fixed bases B that I started with, that will be equal to identity relative to B. I am using the 1st equation. T circle S equals that. I am using just that. Identity related to B is the identity matrix. This time, it is an equation involving matrices.

The right-hand side, I is the identity matrix. T circle S B I will invoke this formula. That is the matrix of T relative to B into the matrix of S relative to B. This is equal to the identity matrix. I can do a similar thing for the 2nd equation and just write down on the other side. This is equal to S into T, matrix of S into matrix of T. This is matrix of S into matrix of T coming from this equation. So I have 2 matrices, let us say A and B. A into B equals identity, that is equal to B to A. In fact one of them is enough because it is a square matrix, rank nullity theorem could be used.

Okay, in any case, I have an equation like A into B equals identity.A and B are matrices. So since A square and B squared,A and B, both must be invertible. What it also means is that this SB is the inverse of this matrix. Okay? This matrix must be the inverse of this matrix. So let me write that.

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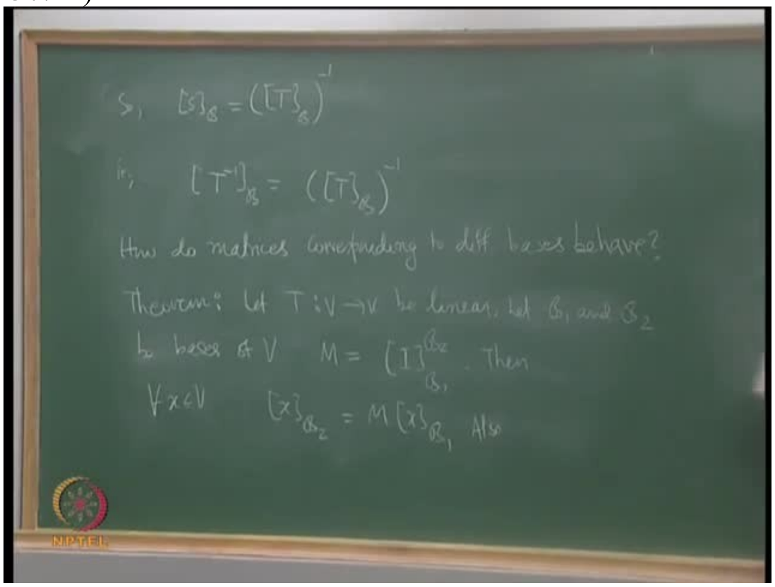

What in particular this means is that S, this is the matrix remember, this must be the inverse of this matrix by definition. AB equals identity A and B are square. So B equals A inverse. That is what I have written down. But S is T inverse and so I have this. So you do not have to compute the matrix of the universe transformation if you know the matrix of the original transformation. You take the inverse of the matrix of the transformation that you started with, that will be the matrix of the inverse transformation relative to the same basis that you started with okay. If you change the basis, then this changes. Okay.

That brings us really to the next probably that most crucial question, how do matrices corresponding to different basis behave? how do matrices corresponding to different basis behave? Okay? So let us answer that question. How do matrices corresponding to different basis for the same transformation behave? The answer is given in the next result. Let me state this theorem. So I have a single linear transformation, a linear operator really. Oh. I have 2 bases. Let T be linear,let B1 and B2be basis of V. Look at the identity transformation and then look at the matrix of the identity transformation relative to these 2 bases.

Let me call that as the matrix M. I am looking at the identity linear transformation relative to I am computing the matrix of this identity linear transformation relative to these 2 bases. We know that this is not I, this is not identity matrix. I am calling that as a matrix M. Then we have the following. For every X and V the matrix of S relative to B2, the 2^{nd} basis, is M times the matrix of X relative to B1. This really gives us the other formula. How are matrices of a particular linear transformation corresponding to different basis related?

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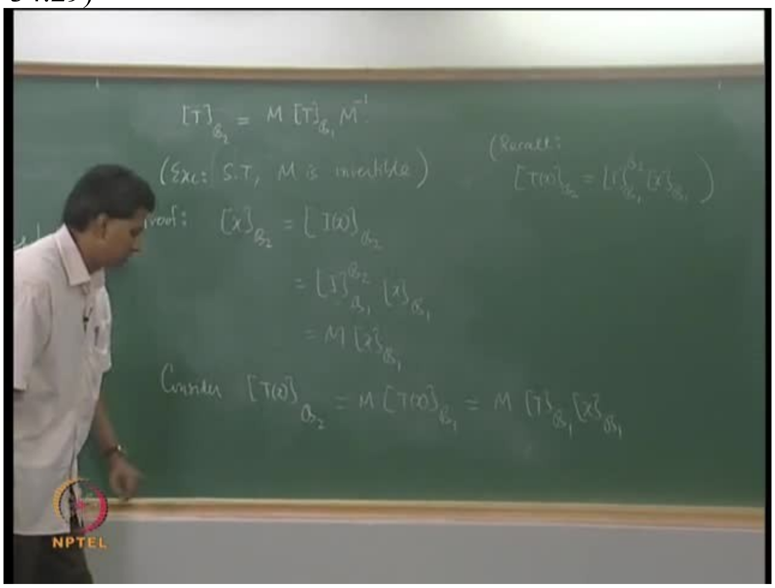

Let me write that formula here. The matrix of T relative to B2 by which I mean B2 B2 is M into the matrix of T relative to B1 B1? That is matrix of T relative to B1 times M inverse okay? This is the important probably the most important relationship. Remember that this involves M inverse, so we must show that M is invertible, okay. But I am going to leave that little part as an exercise. This is similar, this is similar to what we did earlier. This is similar to what we did earlier. Use the previous result, composition, rather, okay use the earlier result to show that M is invertible.

So I am going to leave this part as an exercise. Exercise, show that M is invertible. Remember, M is a matrix. To show that M is invertible, one could for instance show that the system MX equal to 0 for XNRN if M is the dimension of V, has X equal to 0 as the only solution. MX equal to 0 for XNRN have X equal to 0 is the only solution. Now that, you can use this idea to prove that M is invertible. So I assume that M is invertible and derive this formula but this formula is really a consequence of this formula. So we need to prove this 1st. So let us prove this.

Let us start with the matrix of X relative to B2. I can write this as the matrix of identity X, identity transformation. X relative to B2. And then use this formula that we proved earlier. Let me recall that here. Matrix of TX relative to B2, BW actually, that is the matrix of T relate of B1B2 into matrix of X relative to B1. This is what we proved earlier. We used BV BW. This is BW. BV BW BV. I am using B1 B2 here. Vector space V is the same. Two different basis now. So let me use this result here. This is the matrix of I relative to B1 B2 into the matrix of X relative to B1. I have to 1st formula immediately.

This is what we are denoting by M. So XB2 is M XB1. That has proved the 1st formula. Okay? XB 2 is identity transformation I am applying and then I amappealing to this formula. Matrix of I relative to B1 B2 and then matrix of X relative to B1. This is what we are calling as M. So I have the 1st formula. XB 2 is M XB 1. That is the 1st formula. I need to prove this.So let me now start with consider TX B2I am going to appeal to the previous formula. Let me call TX as Y, then I am looking at Y relative to B2. Y relative to B2 is M times Y relative to B1. Y relative to B2 is M times Y relative to B1.

Instead of TX, I have Y. Y relative to B2 is M times Y relative to B1. TX B1, I will apply this formula. I will apply this formula with B2 replacing B1 with B1 equals B2 really. I will apply this formula B1 equals B2. This formula holds for any 2 bases in particular B1 equals B2. So I am going to look at this formula. Tell me if this statement is clear? I will write this as M. Do you agree with this? That is Iam looking at a single basis now. I am looking at a single basis and then I am looking at this formula for a single basis. M into T D1 into XB1. That is, it is actually matrix of TB1 B1 into matrix of X B1.

The formula for a single basis, I am using that. But matrix of T B1 B1 is what we are denoting as T B1, matrix of T relative to B1. On the left-hand side, I have T XB2, I will use that formula for B2. See, I have really got what I wanted on the right-hand side. So I want this. I have sort of gotwhat I want really here.

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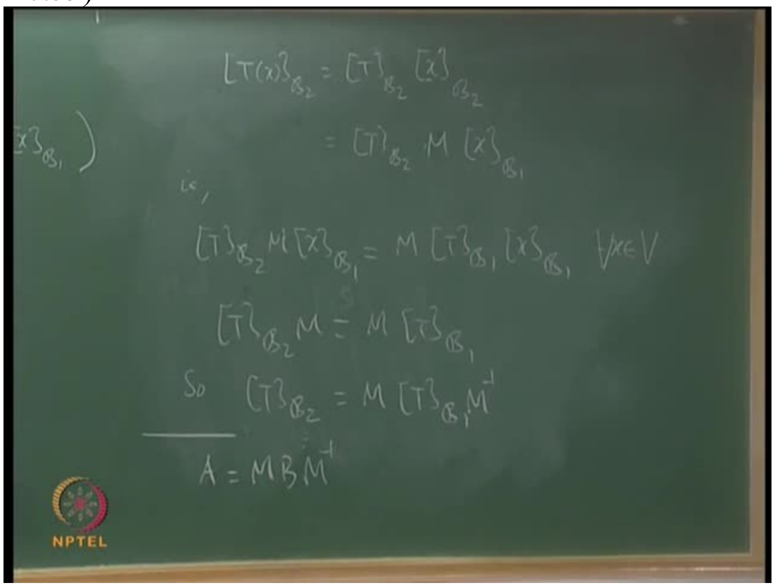

I will now expand this TX B2 for a single basis B2 will be TB2B2 that is just TB2. Is it okay? Same thing. What I did here? I have, what I have done here, I used for B2, then invoked the 1^{st} formula. TB2 into the 1^{st} formula is XB2 is M XB1. I hope it is clear. I started with TX B2, apply this formula foras though there is a single basis. So as a single basis, I get this formula, B2 and then formula 1, XB2 is M times XBN, I proved that. So I invoke that here. So finally what do I have? That is, this is on the left. TB2 MX B1, this is on the left.

That is expanded form of this. On the right, I have this. So let me write this. MTB1 XB 1. I want to show two matrices are equal. I will show that their Jth columns are equal. This is true for all X in V that I started with. This is true for X in V, in particular, seeI am looking at XB1. In particular, take B1, right down explicitly, let us say U1, U2, et cetera, UN and then look at X may be placed by U1, U2, et cetera, let us say X being replaced by UJ. If I replaced X by UJ and write the matrix relative to that same basis, then this is a Jth column. This is something that we did just now.

Instead of X, replace instead of X, replace them by the basis elements that are present in B1. Then I will get TB2M is MTB1. This is true for all X. Apply it to the basis elements, so I get this. Again, something like PEJ equals QEJ. So Jth column of P is equal to Jth columns of Q. So P is equal to Q. This whole thing is P is for me. This is a matrix. So remember M is a matrix. This is a matrix. This is a product. I am calling P. This whole thing is again a matrix, I am calling that Q. I will apply a stand, the basis elements in B1 to this equation.

Then this will be the 1st column of the identity matrix, 2nd column of the identity matrix, et cetera. So these 2 matrices are the same. Invoke the fact that M is invertible, post multiply by M inverse, I get the required formula. Since M is invertible, I can post multiply this equation by M inverse. This is now an equation involving matrices. So I have TB2 equals M TB1 M inverse. Okay? That is ifP is okay, let us say if A is a matrix of a linear transformation corresponding to one basis and B is the matrix (())(44:40) transformation corresponding to the other basis, then A and B are related by the formula, A equals M times B times M inverse for some invertible matrix M.

If A is a matrix of a linear transformation corresponding to one basis, B is the matrix of the same linear transformation corresponding to another basis, then A and B are related by the formula A equals M times B times M inverse for some invertible matrix M. How one could determine M is a different but there is an invertible matrix M that satisfies this equation.Can you see that if A equal to MB M inverse, then B is equal to M inverse AM. That is, M inverse AM inverse inverse. Instead of M inverse, let us use N. Then if A is equal to MB M inverse, then B is equal to NAN inverse, okay? So we can if A is related to B by means of the formula we say that B is similar to A.

If A is related to B by means of the formula we say B is similar to A. By the observation that we have seen just now, it follows that if B is similar to A, then A is similar to B. Okay. A is similar to itself. A is equal to identity A identity inverse. If A similar B, B is similar to C, product of inverses you can use to show that A is similar to C. So this is an equivalence relation. Similarity of matrices is an equivalence relation. What does it preserve? Now that is something that we cannot discuss right now. It preserves what are called eigenvalues. It preserves eigenvalues. Okay that is something we will discuss later. Let me stop here.