Linear Algebra
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Lecture no 19
Module no 04
The Matrix of a Linear Transformation

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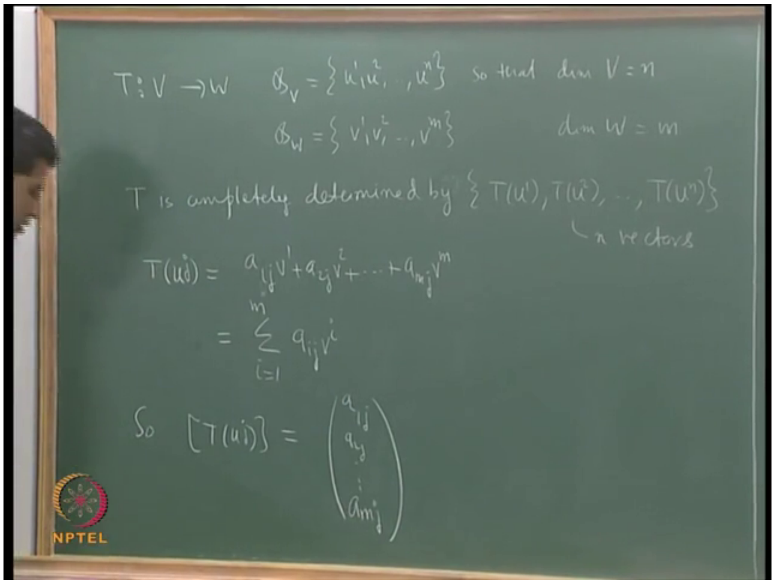

Let me recall, we are discussing matrix of a linear transformation let me give the framework let us say I have T from V to W, V and W are finite dimensional vector spaces, T is a linear transformation I have the notation B v for an ordered bases of V, I remember I have given u 1, u 2, et cetera, u n so that dimension of V is n this is the bases of V and I have a bases for W denoted by B w that is let us say v 1, v 2, etc, v n so that dimension of W is m okay. Then we are going to define the matrix of the linear transformation T that is done as follows, T is completely determine if T u j's are completely determined okay let me say T is completely determined by these vectors T u 1, T u 2, et cetera, T u n, these are n vectors.

These are n vectors in W, each of these vectors is determined by m scalars that we know so this is where I want to introduce the following notation, T u j is in W, W is this is a bases for W and so this T u j is a linear combination of these so let me write it as follows; I will write this as a j okay let me write it as a 1 J; a 1 j v 1 + a 2 j v 2 + etc + a n j v n, there are n vectors so there are n terms, a 1 j, a 2 j, etc a n j. These numbers a 1 j, a 2 j, etc, a n j are unique corresponding to T u j, unique with respect to these bases okay we have fixed these 2 bases now we want to determine the matrix of T related to these 2 bases okay. So these numbers a 1 j, a 2 j, a n j are unique for T u j, let me write the submission notation and then introduce the matrix of the linear transformation.

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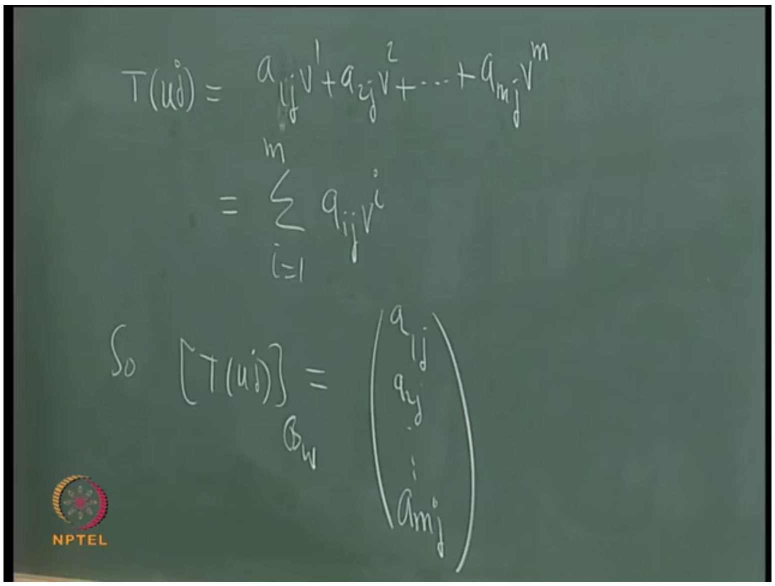

So the notation is as follows the first subscript is what varies so this isi equal 1 to n, a i j v i on the right-hand side i is the running index i is the running index the free index is is j that corresponds to T u j. In other words, T u 1 is a 1 1 v 1 + a 2 1 v 2, etc a m 1 v m okay now this is the submission notation that will be useful in some of the computations later. Okay now if this is the case then we know that the matrix of this vector relative to this bases, matrix of a vector relative to bases was discussed in the last lecture, what is the matrix of this vector relative to this bases? you just take these coefficients and arrange them in a column, so it is this column vector a 1 j sorry yeah a 1 j, a 2 j, etc, a m j, this is the matrix of the vector T u j relative to B w, I emphasise by writing this subscript then the matrix of T relative to B v, B w.

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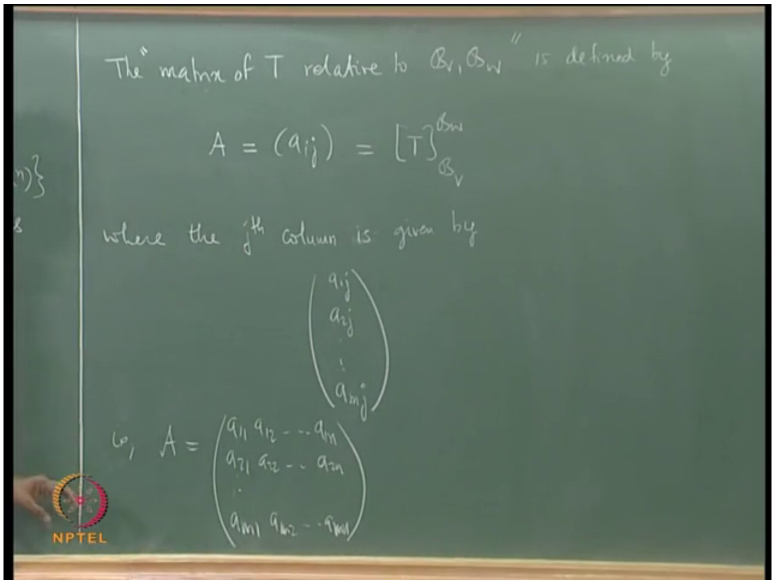

Remember I am talking about ordered bases okay the matrix of T relative to B v, B w that is given by it is defined by... Okay let us recollect I told you that these n vectors uniquely determine T okay complete information of T is given by these n vectors. Each of these n vectors in turn depend on these numbers a 1 j, a 2 j, etc, a m j, in general a i j so T is completely determined by m times n numbers okay. So let us remember T is determined completely by m times n numbers, we arrange them as a matrix and that is what I am going to call as A so A is a i j, ith row jth entry is denoted by a i j that is my matrix A and what is this, this is the notation that we will use for the matrix of a linear transformation, where the jth column is given by a 1 j, a 2 j, etc a m j, this completely defines the matrix that is the matrix of linear transformation T relative to these 2 fixed bases fixed order bases.

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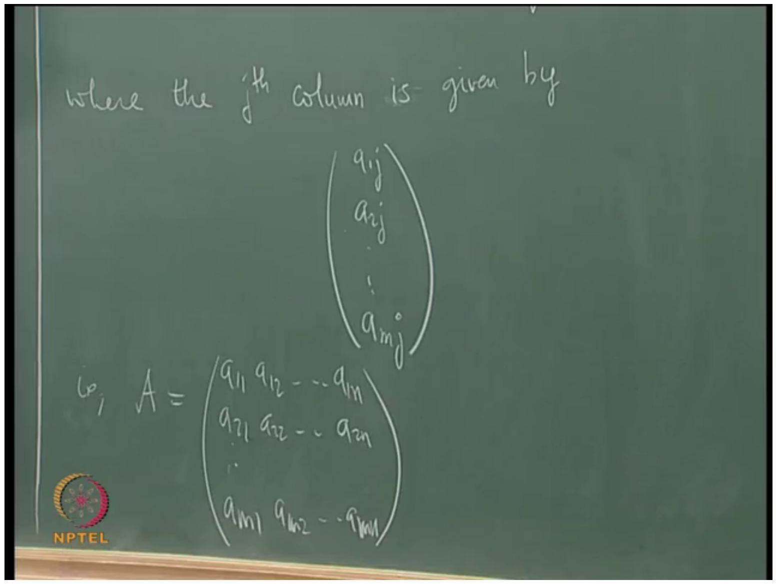

Now this is in conformity with the usual notation of writing down a matrix a 1 1, a 1 2, etc a 1 n, a 2 1 , a 2 2, etc a 2 n, a m 1, a m 2, etc, a m n okay so this is the matrix for linear transformation. To summarise how do you construct this matrix? you take 2 bases; one for V, one for W, 2 ordered bases, look at the action of p on each of the elements of this bases and then write it in terms of the bases vectors in W, collect the coefficients, arrange them as columns that is the matrix of linear transformation, what is important is this formula. What is important is this formula, the T of u j is submission i equal to 1 to m a i j v i, if this formula holds then a i j is the ith row jth entry of the matrix of the linear transformation okay, if this Formula holds then a i j is ith row jth entry of the matrix of linear transformation that completely determines the matrix okay.

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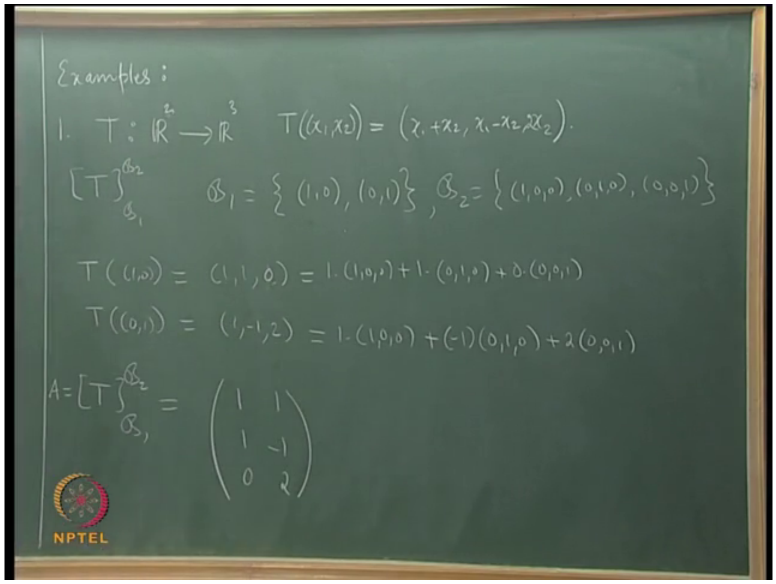

Let us look at some example, let us say I have T from R 2 to R 3 defined by T from R 2 to R 3 so let us say I have T of x 1, x 2 is x 1 + x 2, x 1 – x 2 and x 2 2x 2 okay, I am going to consolidate by doing one or 2 examples. I want to determine the matrix of this linear transformation, (()) (9:01) standard bases let us do that first, you will see that it is very simple. The question is what is the matrix of T relative to the standard bases of R 2, standard bases of R 3, let me denote B 1, B 2; B 1 is the standard bases of R 2 that is 1 0, 0 1, B 2 is standard bases of R 3 1 0 0, 0 1 0, 0 0 1 okay, this is my u 1, this is my u 2, this is my v 1, v 2, v 3 so I must look at T u 1 write it as a linear combination of these 3 vectors, collect the coefficients that be the 1st column, determine T u 2 write it as a linear combination of 3 vectors again write it as a second column that is my matrix.

So T of 1 0 is 1 1 2 I must write this 1 1 2 as a linear combination of those 3 vectors. Now you see it is a standard bases so okay T of 1 0 is 1 1 0, I must write this as a linear combination of the standard bases vectors, since I have standard bases vectors the coefficients are easy to see, this is one time 1 0 0 + 1 into 0 1 0 + 0 into 0 0 1, this actually gives me the 1st column, I will write the 1st column after completing the 2nd volume also. T of 0 1 is 1 –1 2, okay you can see that this is again an easy linear combination okay so I have written down the first and the 2nd columns. So what is the matrix of T relative to these 2 bases? Collect the coefficients 1 1 0 that is my 1st column, collect the coefficients 1 -1 2 that is my 2nd column this is the matrix of a linear transformation corresponding to the standard bases okay.

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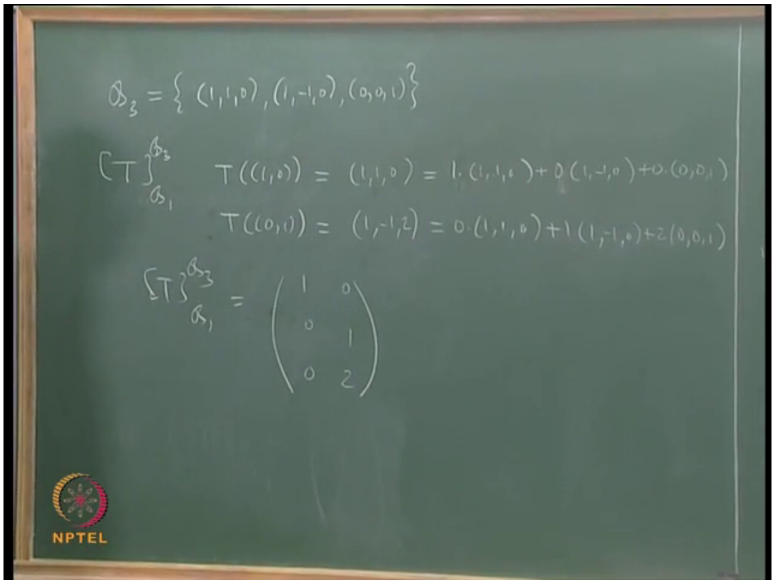

Okay so let us redo this problem with B2 being changed to the following, I will call that B3 now okay this is only to illustrate the fact that the matrix of linear transformation will change if it changes the bases okay so let me change B2, so instead of B2 I have B3, let us say I have these vectors l 1 1 0, 1 -1 0 for instance 0 0 1, it is clear that this is the bases, these 3 vectors are linearly independent and dimension of R3 is 3 so this must be a bases okay. I want to determine the matrix of T relative to B1, B3, I will use the calculations that we have done earlier T of 1 0, B1 has not changed so T of 1 0 is 1 1 0, earlier it was written in terms of standard bases vectors now I must write this in terms of these vectors okay.

But you can see that by inspection this will be 1 by 2 times 1 1 0 + 1 by 2 times 1 – 1 0 this is 0 so this is 0 times 0 0 1 is that okay half + half is 1, half – half is okay so... oh this vector is same as this yes this is same as the 1st bases vector so that has made matters simple. Okay so 1 0 0 is the 1st column of the matrix of T relative to B 1 and B 3, T of 0 1 has been done before what is that, 1 -1 2 so what are the coefficients? okay 0 1 2 okay this is 0 the contribution comes from these 2 this is 1 -1 and 2 okay so it has turned out to be simpler than the previous one for the matrix of T relative to the bases B 1, B 3 is this time 1 0 0, 0 1 2 okay this is obviously different from the matrix of T relative to the bases considered earlier.

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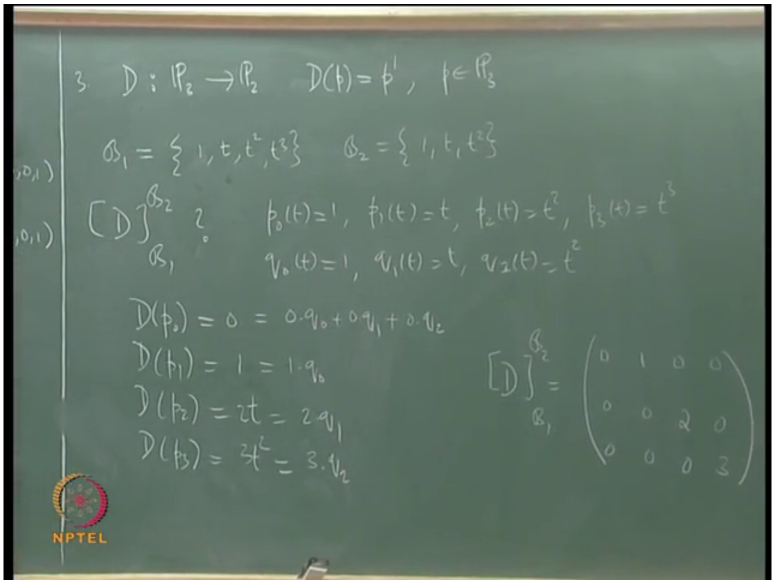

Let us look at one more example, let us say I have the operator D from p 3 to p 2 the differentiation of it defined by D of p is p prime, p prime is derivative of p, p is a polynomial, p prime is derivative of this polynomial okay. Let us take the standard bases for these 2; B 1 let us say is p 3 so I will have 1, t, t square degree less than or equal to 3 this is the bases for p 3 dimensions 4, p 2 I will use B2; 1, t, t square dimensions 3, I want to determine the matrix of D is relative to these 2 bases, what is the matrix of D relative to these 2 bases okay. Okay, look at the 1st one let us call these as p 0, p 1, p 2, p 3; P0 of p is 1, p 1 of p is t, p 2 of p is t square, p 3 of p is t cube, let us call this as Q0, q 0 of t is 1, q 1 of t is t, q 2 of t is t square okay.

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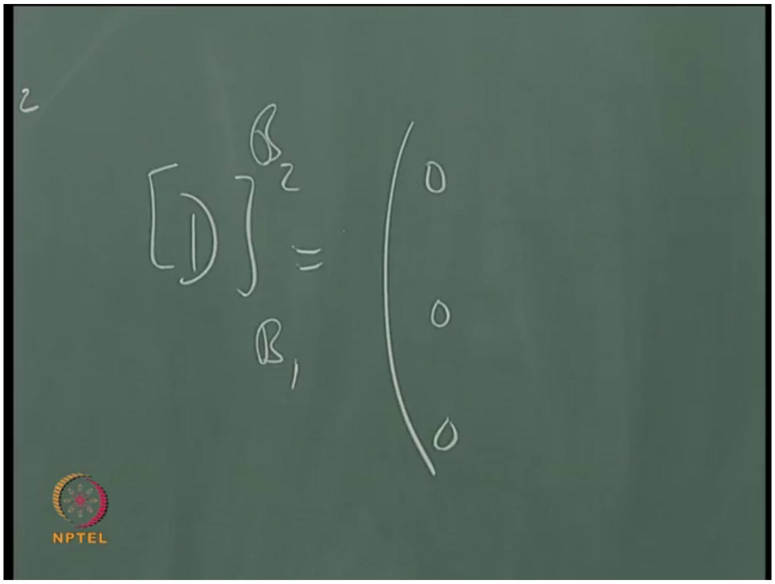

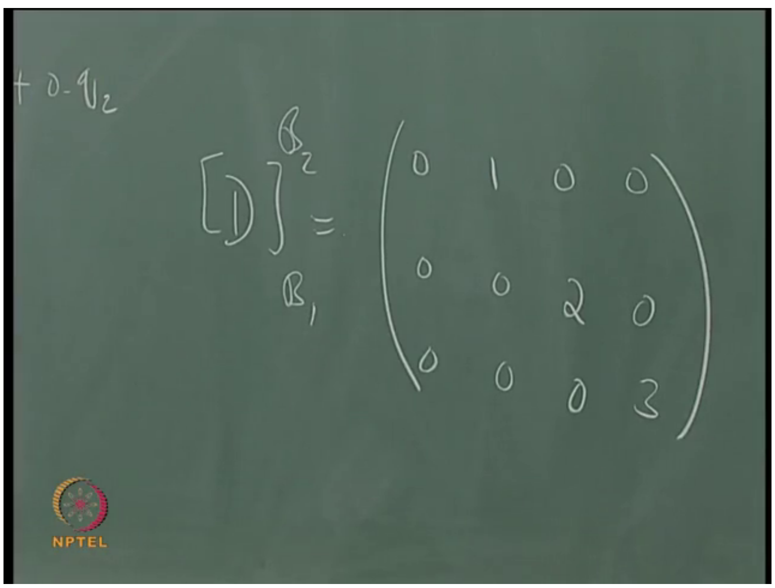

I need to make this calculation, what is D p 0, p 0 is a constant, D p 0 is 0, how do I add 0 as a linear combination of independent vectors in a unique way is 0 time q 0 + 0 times q 1 + 0 times q 2 so that determines the 1st column. Let me write it on this side the matrix of D relative to these 2 bases, the 1st column is 0 0 0. Remember the order of D, what is the order of D, it is from a 4 dimensional vector space to a 3 dimensional vector space 3 cross 4, D is a 3 rows 4 column matrix okay, D p 0 is this, D p 1... p 1 is t, derivative is 1, 1 unique 1 times q 0 + 0 + 0. D p 2 is 2 t in terms of this unique 2 times q 1, I must write + 0 times q 1 + 0 times q 2 I am leaving the details here, 2 times q 1, q1 is the 2nd vector so it is 0 2 0 that is the 3rd column. D of p 3; t cube 3 t square 3 times q 2 all other coefficients are 0 so that is 0 0 3 so this is the matrix of the linear transformation the derivative transformation with respect to these 2 bases okay.

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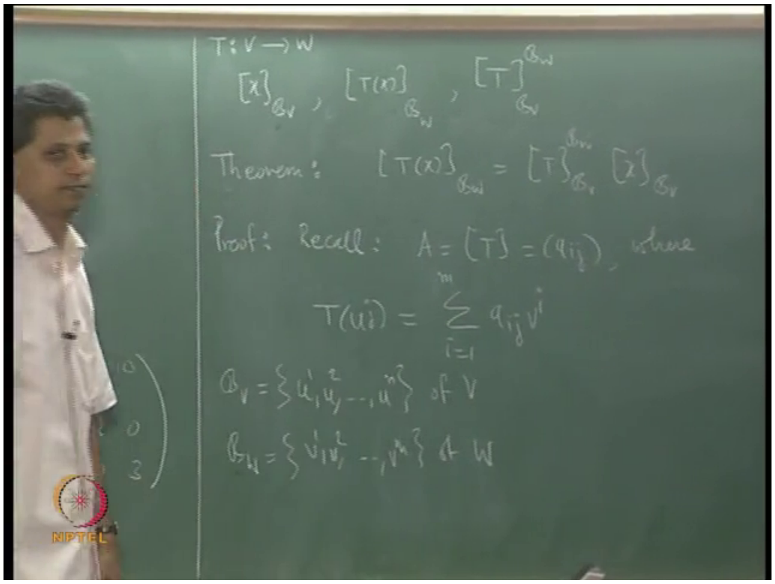

Okay let us get back to the general ideas, the question is how are... See we have defined the matrix of linear transformation but remember that there are 3 matrices that we have actually defined; 1 is the matrix of x relative to B v, see the framework is as before, T is from V to W is finite dimensional spaces, B v is the bases of V, B w is the bases of W, T is linear. There are 3 matrices that we have actually defined, the matrix of the vector relative to the bases B v. There is a matrix of the vector T x that is in W that is relative to the bases B w and finally the matrix of T relative to B v, B w fixed bases. There are 3 matrix really, how are these 3 related? If I look at the relationship so we will derive relationship between these 3 matrices, these 3 vectors really and if you look at that relationship it will tell you.

Why I made the statement that any linear transformation between finite dimensional spaces is like multiplication by a matrix. So we have the following results connecting these 3 matrices, I have given the framework as before let me only emphasise the bases vector it is not necessary, right away I will write down the formula connecting these 3 it is the following. If you look at the matrix of T x relative to B w, it is the matrix of T relative to B v B w into the matrix of x relative to B v, this is the relationship connecting these 3 matrices or vectors. Okay so now can you see the matrix of T x is equal to the matrix of T into the matrix of x, this is like A according to our rotation for the matrix of T x is like A times the matrix of x, the matrix of T x is like A times the matrix of x okay, so any linear transformation between finite dimensional vector spaces is like T x equals A x like multiplication by matrices that is A times x where x is in V okay.

Let us prove this remember this is an equation involving vectors... proof; for the proof I will make use of the formula that I had written down earlier. Let me recall, the matrix of the linear transformation T relative to these 2 bases I call that as A and this is ith entry is a i j where the jth column of this matrix jth column of this matrix is related to the linear transformation T by means of the following formula; i equal to 1 to n a i j v i, i is the running index, j is the free index, this is how we write the matrix of the linear transformation T relative to the bases v 1 v 2 so let us recall, this is the bases for V, this is only to emphasise my notation v 1, v 2, etc v n this is the bases of W okay so from this relationship we will establish this formula okay, let us start with the matrix of x relative to B v.

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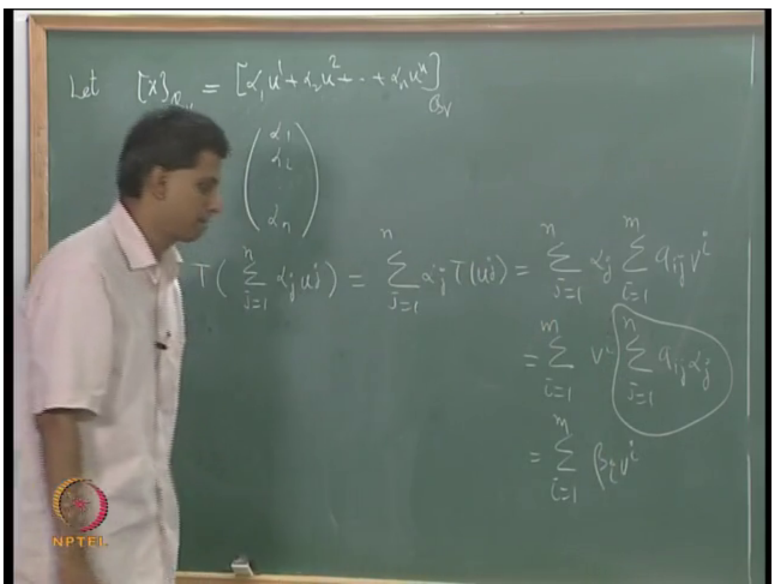

Let us say that this is if x has its representation Alpha 1 u 1 + Alpha 2 u 2, etc, Alpha n u n okay I have simply written x and then I want the matrix of this relative to B v then what I know is that this is just the column Alpha 1, Alpha 2, etc, Alpha n, this is the matrix of this vector x relative to B v okay. I leave this formula, this is the formula for representation for x okay what is T of x? T of x is T of x is given here, let us say I write submission j equals 1 to n Alpha j u J; x is Alpha 1 u 1 + Alpha 2 u 2 + et cetera Alpha n u n that is this. I want T of x, I know T is linear so I write this as submission j equals 1 to n Alpha j T of u j just use linearity of T, plug in the formula for T u j from the previous step, this is submission j equals 1 to n Alpha j T u j the submission index there is i. Submission i equals 1 to m a i j v i okay but you see that this is a finite sum I can interchange the integers i and j without a problem.

So this is submission, first I write i, i equals 1 to n, I will take v i outside and then club the other 2 sets of scalars as submission j equals 1 to n a i j Alpha j. What I have done is to uhh exchange the submission, let me call look at this uhh this is a number for every i this is a number for every i this is a number okay, whereas v i are vectors so forget about this for a moment. This is the number and the running index is j let me call this number as Beta i right the free index is i, let me call this Beta i so this is i equals 1 to m Beta i v i okay submission Beta i v i where what is Beta i? Beta i is this number.

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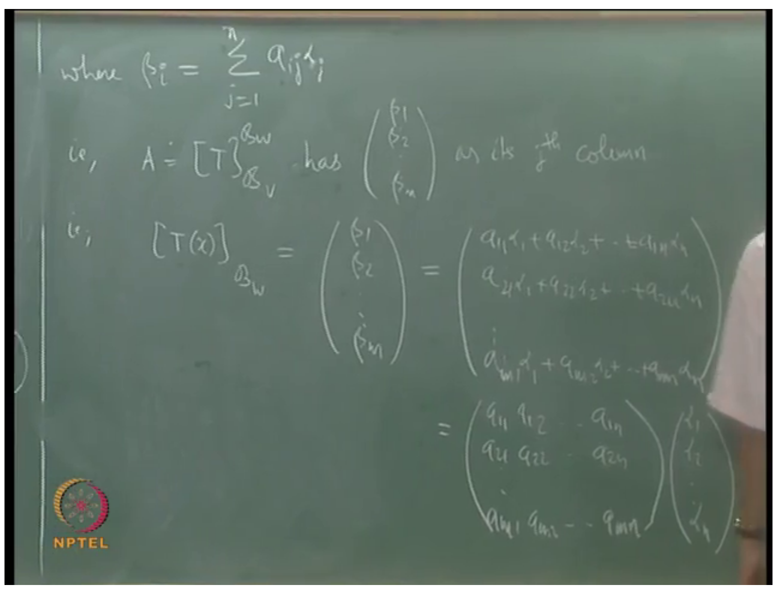

So let me write down there Beta i is submission j equals 1 to n a i j Alpha j that is Beta okay let us go back and see what we have done and consolidate. To summarise, T x has been written as a linear combination of v 1, v 2, et cetera, v n where the coefficients are Beta 1 etc Beta m. T x is a vector in W if I want to know the matrix of T x relative to B w then I must look at the unique representation of T x in terms of the bases vectors that is in terms of vectors in B w that is in front of you. Then this means T x is the matrix of T x has this these entries as the column entries that is T x B w is just this vector okay what I really want is this what I really want is this. Now just plug in these formulas and see that it will unfold, let me write it in full what is Beta 1? i is 1, a 1 1 Alpha 1 + a 1 2 Alpha 2 etc submission is over j, a 1 n Alpha and that is Beta 1.

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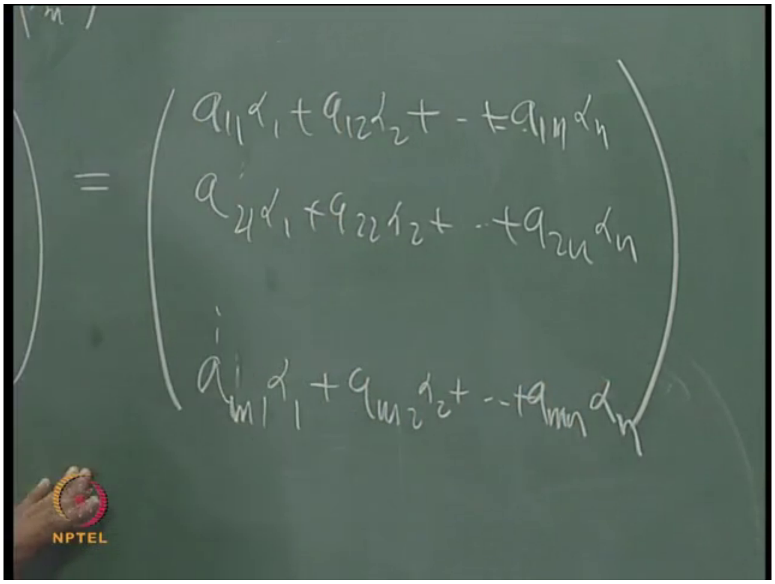

Beta 2; i is 2 a 2 1 Alpha 1 + a 2 2 Alpha 2... a 2 n Alpha n, i equals m... a m 1 Alpha 1 + a m 2 Alpha 2... a n m Alpha m, I have simply written down the expanded forms of Beta 1, Beta 2, etc, Beta m coming from this formula i equals 1 to m.

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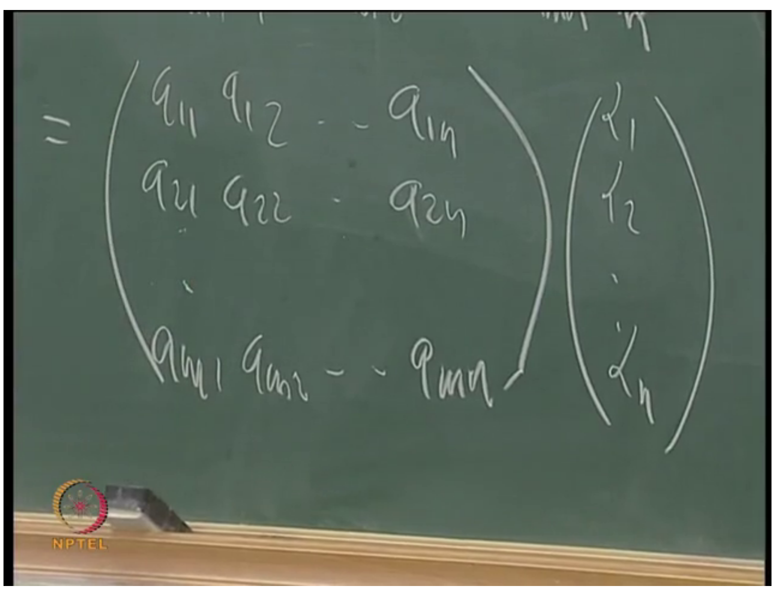

Now what you have on the right is precisely a times x, this is a 1 1 a 1 2 , etc a 1 n, a 2 1, a 2 2, etc, a 2 n, a m 1, a m 2, a m n into Alpha 1 Alpha 2, etc Alpha n okay, but what is this? This is a that is this is this matrix, what is this Alpha 1, Alpha 2, et cetera, Alpha n, we had started with this x d v okay.

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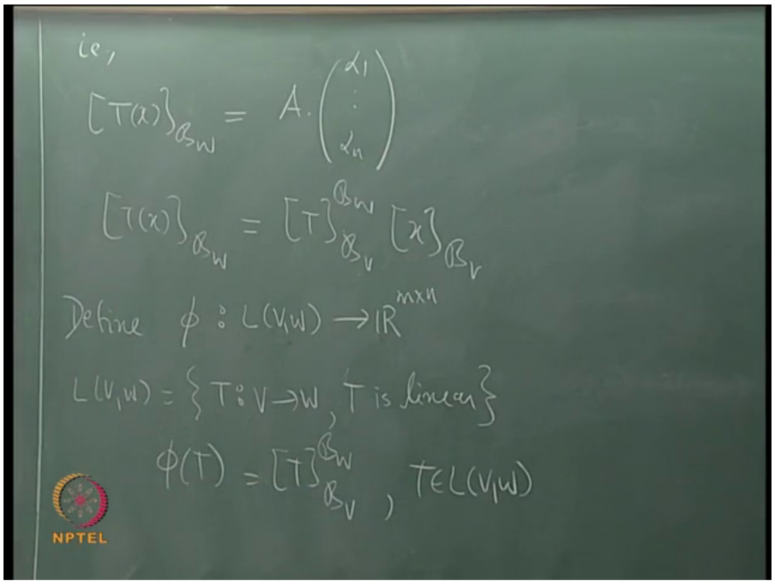

So the right-hand side... Okay this is then the basic relation connecting the matrices of T x the matrix of T and the matrix of x okay. Okay let us look at some of the consequences okay remember to summarise, from a matrix we can define a linear transformation, from a linear transformation with fixed bases we can have a unique matrix, which means there is a one-to-one correspondence, there is a one-to-one correspondence between matrices of linear transformations relative to fixed bases and linear transformation themselves okay. That is why let me define uhh let me define this as a function then we have the following result. Define a function Phi from L V, W, I tell you what L V, W is... to R m cross n, what is L V, W?

L V, W is the set of all T from V into W such that T is linear so I collect linear transformation between from V into W collect all linear transformations that I am calling as L V, W now it is easy to see that this is the vector space it is easy to see that this is a vector space. What are the 2 additions? Vector addition and scalar multiplication; vector addition is addition of 2 linear transformations, how is addition of 2 linear transformations defined point wise? If S and T are linear transformations then S + T of x is s of x + T of x okay so point wise addition then with respect to this operation S + T is a linear transformation from V into W, scalar multiplication Alpha times S how is that defined? Alpha S of x is Alpha times S x this is scalar multiplication, with respect to vector addition that I defined earlier this is a vector space.

This is a vector space over the same field that we started with, we are looking at always the real field case for simplicity so this is a real vector space. This is a real vector space, I would like to actually determine the dimension of this real vector space okay but before I do that let me tell you what this function is; Phi is a function from L V, W to R m cross n. Now we know how to uniquely determine a matrix corresponding to a transformation, all we have to do is fix 2 bases so let us say I have fixed 2 bases; B v for V, B w for W. When I fix these 2 bases I know that there is a unique matrix corresponding to any linear transformation T, any linear transformation T will be an arbitrary element here so the definition of Phi of T will be just the matrix of T relative to the ordered bases B v, B w, T belongs to L V, W.

Okay I am defining a function from the vector space L V, W for the vector space R m cross n, how do I get the numbers m and n? m is the dimension of W, n is the dimension of V, I have fixed 2 bases for these 2 vectors spaces then I know how to determine the matrix of the linear transformation T so can you see then this is well-defined? Phi of T, T is a transformation on the right this is a matrix so that is an element of R m cross n and how is this unique how is this a function, this is function because relative to 2 fixed bases the matrix of a linear transformation is unique so the right-hand side is unique so this is well-defined it has some further properties. It is linear 1st, can you see that Phi is linear? Phi is not just a function, what we will show is that it is linear, it is in fact invertible it is 1 to 1 and onto.

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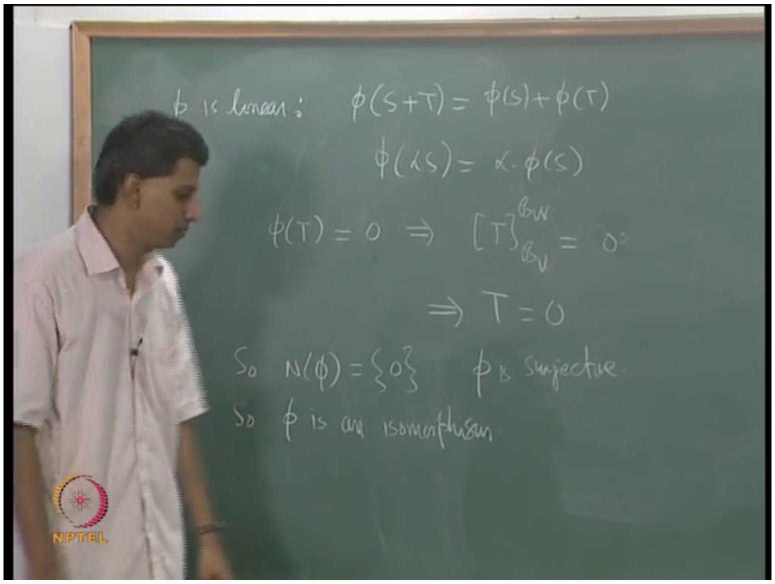

We will also use another property it is a map that preserve products okay, I will I will explain this a little later. Phi is linear, what is the meaning of that? you must verify that it satisfies Phi of S + T equals Phi of S + Phi of T and Phi of Alpha times S is Alpha times Phi of S, these properties follow simply because of the corresponding property for matrix. Okay so please verify these what exactly am I saying, if A and B are 2 matrices then A + B into x is A x + B x. Alpha times A into x is Alpha into A x that will give you these 2 properties. Phi is 1 to 1 and onto so what about that? Phi is Injective and surjective, Phi is linear okay, Phi of T is equal to 0 the 0 matrix implies the matrix of T relative to B v B w it is a 0 matrix same 0 here 0 okay.

The matrix of T relative to be B v B w that is 0 means each entry is 0 especially jth column is 0, if the jth column is 0 what it means is that T u j is a 0 vector, if T u j is a 0 vector how do you write this as a unique linear combination of v 1, v 2, etc, v n? The only way is that these coefficients are 0, which means T u j is 0 but T u j 0 can happen is a bases vector, T u j is 0 for every j, T u 1 is 0, T u 2 is 0, etc, T u n is 0 but we know if this happens then T has to be only the 0 transformation. T u 1 is 0, T u 2 is 0 et cetera, T u n 0 the only transformation that will satisfy this property is the 0 transformation so T is 0 so what have you shown?

We have shown that null space of Phi of T is single term 0, null space of Phi, Phi is the transformation we have assume that it is a 0 transformation. Please remember there is a difference, this is a 0 matrix and this is a 0 transformation, this is the 0 transformation that I have written down. Null space of Phi is 0 which means Phi is Injective, how is Phi surjective? How do you rephrase this? Given a matrix, is there a linear transformation corresponding to this matrix we know that there is (())(38:04) okay so Phi is also... We have already seen that Phi is surjective really, Phi is surjective also and so Phi is an isomorphism, it is a homomorphism which is bijective so it is an isomorphism so Phi is an isomorphism.

See the fact that Phi is surjective needs a little argument but I am going to leave that as an exercise. What you must actually do is take a matrix, define a linear transformation then how are these 2, the linear transformation and the matrix are defined, the linear transformation must be such that the matrix of the linear transformation relative to B v B w is the matrix that I started with okay that is really what we need to verify okay but I am still going to leave that as an exercise so Phi is an isomorphism.

We know that an isomorphism preserves dimensions, so what is the dimension of T L V, W what is the dimension of L V, W? What is the dimensions of R m cross n m into n? Can we give a bases for R m cross n? Think it over... standard bases, the bases consisting of the following matrices, one in the 1st entry all other entry is 0, 1 in the 2nd entry all other entry is 0, do it for all the m and n entries, the dimension of R m cross n is m n so that I mention of L V, W is m n okay, so L V, W is m times n dimensional if V is n dimensional, W is m dimensional.