Linear Algebra
Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology Madras
Lecture no 18
Module no 04
Equality of the Row-rank and the Column-rank II

In the last lecture we were trying to prove this result that the Row rank is equal to the column rank okay. The proof was left incomplete so 1st task for today is to complete that and then I will discuss a notion of the matrix of a linear transformation over finite dimensional spaces and then look at some of how to compute the matrix for instance 1 or 2 examples of linear transformation where I will tell you how to construct the matrix of a linear transformation and then discuss probably some of the elementary preliminary properties and then continue this discussion further. Okay so let me recall that our 1st task is to prove that the Row rank is equal to the column rank I will quickly go through what we have done earlier.

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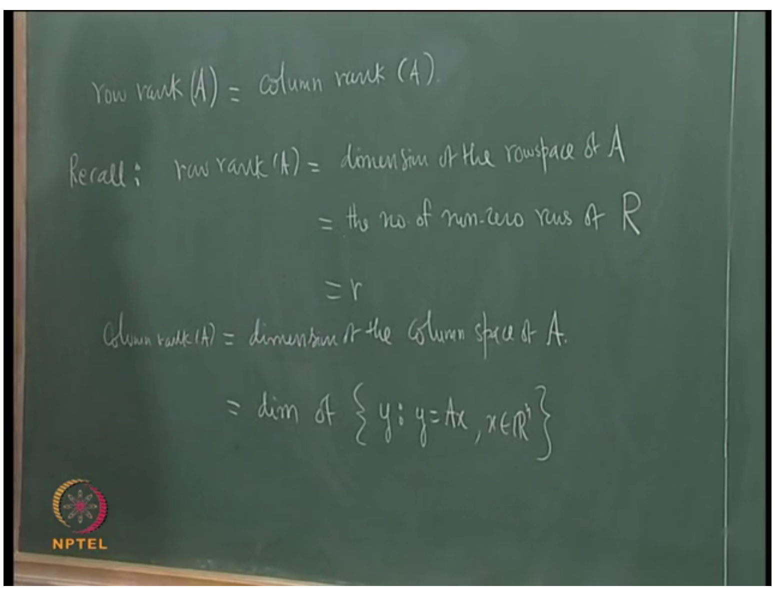

A is an m cross n real matrix and row rank of A is equal to the column rank of A, where what is a row rank, so all these will be recalled quickly. Row rank is equal to the dimension of the row space of A dimension of the Row space of A and if R is the Row reduced echelon matrix row equivalent to A then this is equal to the number of non-zero rows of R, we use the notation small r for that okay. Let me recall again this capital R is a row reduced echelon matrix row equivalent to A then we had seen that the Row rank of A is equal to the number of non-zero rows of the row reduced echelon form of A, what is the column rank of A? Column rank of A is equal to the dimension of the column space of A dimension of the column space of A where what is a column space, it is the subspace spanned by the columns of A.

And we had seen earlier that anything any vector in the column space of A can be written as a linear combination of the columns of A by definition and so if y is in the column space of A then y is equal to A x for some x okay. So let me write, this is the dimension of the subspace y such that set of collect all y such that y = A x for some x in R n, this is the subspace this is the column space of the matrix A so it is the dimension of this space that is called the column rank, the number of nonzero rows of the row reduced echelon form of the matrix A, we have seen that is a Row rank of A.

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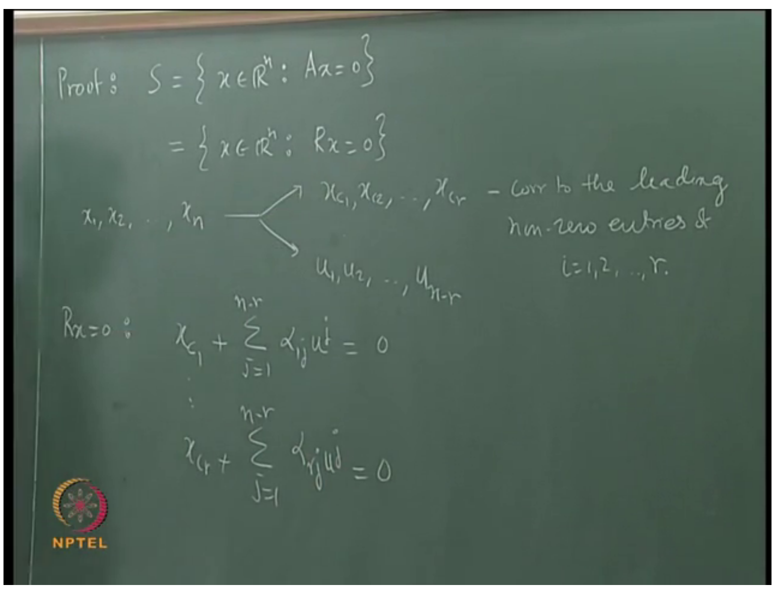

We want to show these 2 are equal and where we started the proof was to look at the system so I am getting into the proof and the 1st few steps we had discussed earlier I am recalling these steps. S is the solution set of the system A x equal to 0, we are using this notation S is a solution set of the system A x equal to 0 this is the subset of R n this is the subspace of R n and what we know is that this is the set of all x in R n such that from what we have discussed when we looked at row reduced echelon form, we know that these 2 sets are the same, the solution set does not change. Now we are again looking at R x equal to 0, there was one interpretation that we have seen earlier uhh R is the number of non-zeros I will keep that.

We have the unknowns x 1, x 2, etc, x n, these are split into 2 types; one type corresponding to the leading nonzero entries of the nonzero rows of R so I have the variables x c1, x c2, etc, x cr okay these are the variables corresponding to the leading nonzero entries of the 1st R rows of capital R that is i equals 1, 2, 3 etc, r that is the n variables have split into 2 categories; one corresponding to the leading nonzero entries. The other ones are the rest of the variables we are calling them u 1, u 2, et cetera, u n - r, the number of variables is n, R have been eliminated here the rest of them are n – r.

Then we had seen earlier that the system R x equal to 0 when expanded using this new notation we will have equations like this; x c1 + submission j equals 1 to n – r some constant let us say Alpha 1 j uhh I am using u, u j this is equal to 0 etc, x cr submission j is also 1 to n - r Alpha r j u j equal to 0 these are the r equations these are r nontrivial equations of the system R x = 0, the rest of the equations give no information, 0 on the left, 0 on the right. I will now adopt slightly different notation, this is just to recall what we had done earlier, the re-labelling of these variables in this manner I will now get back to the original x with the following notation.

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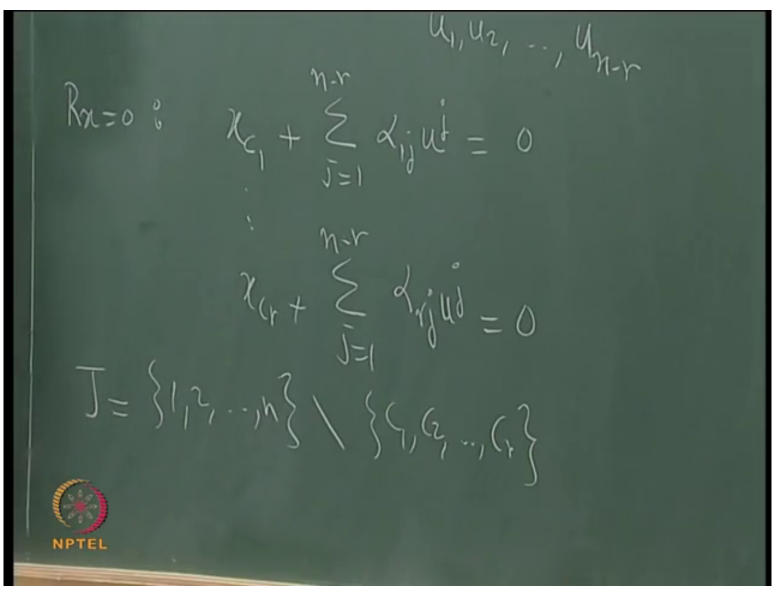

So let me introduce the set j introduce set J, this is 1, 2, 3, etc, n where I have deleted c 1, c 2, et cetera, c r Ideally these numbers c 1, c 2, etc, c r from these n then the cardinality of number of elements in j is n – r

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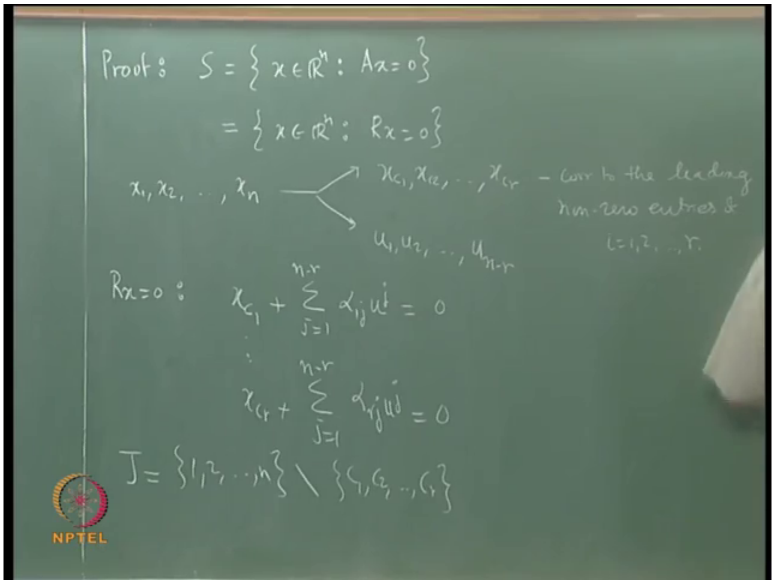

Number of elements in J = n – r, this equation I will rewrite x c1 + submission j element of J Alpha 1 j, I go back to the variable x j. I am sorry this is yeah this is fine right... x c1 is a number so I must use a subscript here so not superscript there so let me go back and correct this...

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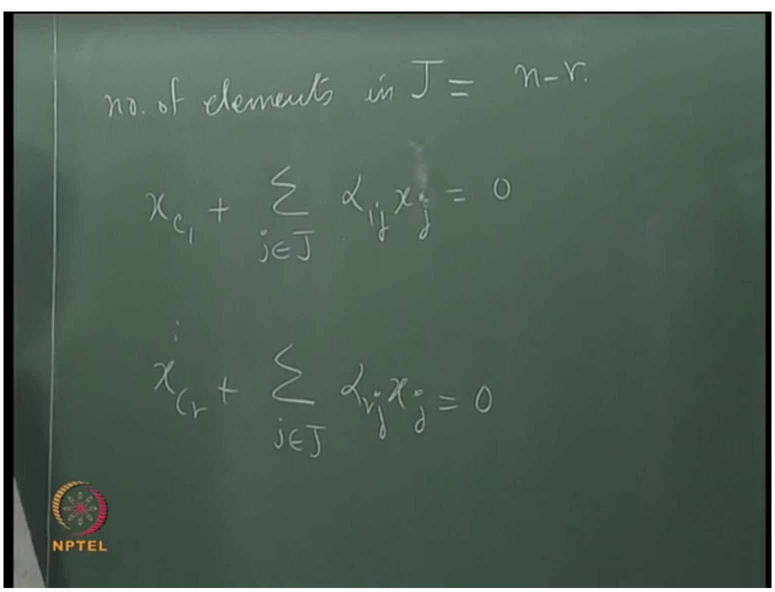

This is Alpha 1 j u j subscript, see these are numbers each other right I have 0 this is a number, this is also a number and this is the number that comes from this u 1, u 2, et cetera u n – r so this is u j. Similarly, these equations will be Alpha 1 j x j equals 0 etc x cr + submission j element of J Alpha r j x j equal to 0, all that I have done is to re-label the variables. u 1, u 2, et cetera, u n – r as x j for j element of J, u 1, u 2, et cetera, u n – r are now x j for j element of J is that okay. If it is in j then it does not respond to c 1, c 2, etc, c r those are written separately. Okay what is the advantage of it, again get back to what we have studied earlier, how do you get the set of all solutions of this system, look at the so-called free variables that is look at the variables that correspond to...

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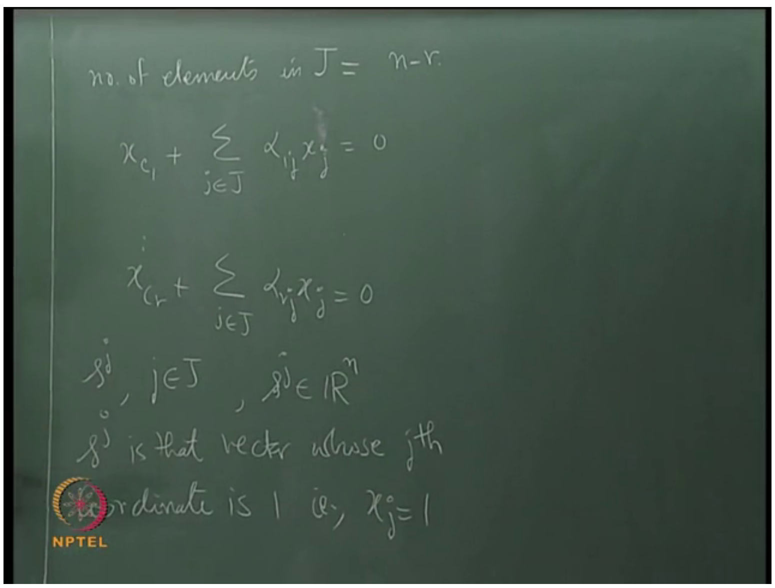

Look at the unknowns that corresponds to those entries in j, look at the entries look at the uhh free variables u 1, u 2, et cetera, u n - r earlier, that the variables x j for j in J. you assign arbitrary values to them, comeback and substitute, the domain x c1, etc x cr that will give you one set of values, take another set of values for x j j in J another set of values, you compute all the solutions using this okay. Among these solutions let us give a new notation for the following variables, how do I I am introducing S j for j in J, I am introducing the vector this is a vector, I will use subscript for real number scalar and superscript for vectors S j j in J each S j let me emphasize belongs to R n to avoid ambiguity. Each S j belongs to R n, how many S j are there? There are n – r S j, what is the definition of S j? S j is that vector which has the value let me write, S j is that vector whose jth coordinate is 1 that is I am looking at one specific assignment for the free variables.

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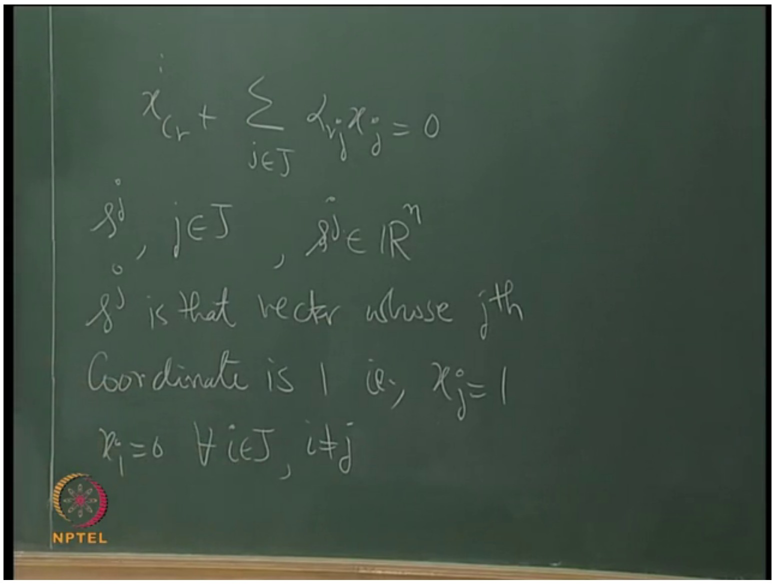

I will take a j in J and then look at the following specific assignment of the free variables, I will take j in J and then put x j equals 1 that is the coordinate corresponding to the 1^{st} j that I have chosen from capital J that is equal to 1, x i equals 0 for all i element of j, i 0 equal to j. I am only looking at the free variables now I am only looking at the free variables, there are n – r free variables u 1, u 2, et cetera u n - r earlier x j j in J now we are the n - r variables, I will look at a assignment now of values for these free variables. Pick a j in J let us say 1 belongs to J then I will write down a vector I will call that S 1 1^{st} coordinate is 1, there are some other elements in J, assign 0 for all those values substitute in this equation, you will determine x c 1, x c 2, etc x cr, give those values fill up that vector that is my S 1 is that clear? okay.

For any particular j in J assign the value x j equal to 1 that is a free variable, J corresponds to free variables, capital J corresponds to indices which are the free variables so I assign the value x j equal to 1, or the other corresponding to J again only those are free, the rest of them these are not free these you have to go back to these equations substitute and then determine this. What I am doing is fixing one of these equations, fix that variable x j equal to 1, assign the others to be 0 and then determine x c 1, et cetera x cr I know some values 1 – 1, e, Pi, etc comeback fill it up I have S j, I do this for each j in J there are n – r S j.

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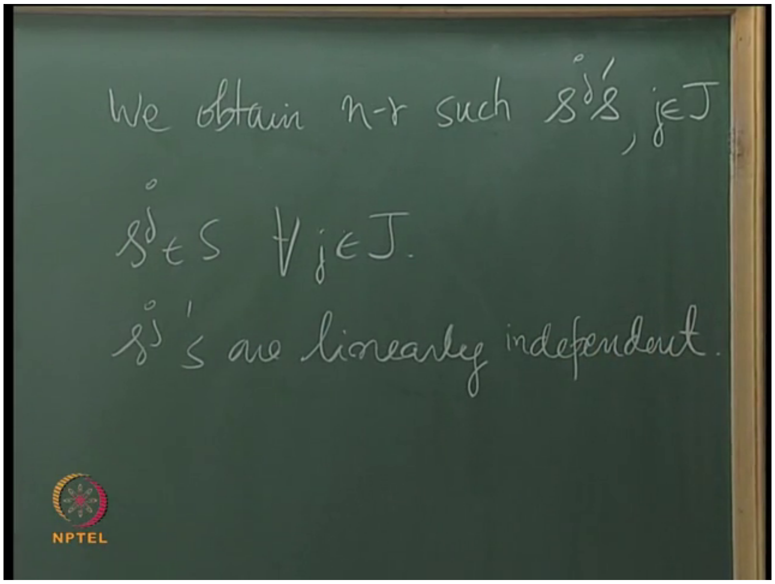

We obtained n - r such S j's okay, each of the S j is a solution of R x equal to 0, each of the SJ is the solution of R x equal to 0 so each S j belongs to S each S j is a solution so each S j belongs to S. Can we say that these vectors are linearly independent? Immediately what is the reason? The reason is very similar to the standard bases vector, vectors being linearly independent. This entry is 1 this entry is 1 for S j let us say I look at s j + 1 that entry will have S j + this... j + 1 entry 1 all other free variables will have the value 0 forget about the values that these variables take. There is a 0 it is like a column it is like a column that is a 1 in 1 entry all other entries are 0, these vectors have to independent okay.

So for one thing these S j's are linearly independent, the argument as I said is very similar to the argument that the standard bases is a base the standard bases vectors are linearly independent so S j's are linearly independent okay. Let me tell you what I am trying to proof, I am trying to proof that the solution set the solution space the solution subspace is of dimension m – r, I want to show that the solution space is of dimensions m – r I want to show that it has a bases consisting of n - r elements, I am explicitly determining such a bases. The 1st step is to prove that these vectors are independent, second step is to prove they are they form a spanning set that I will do next, if I had done this then it follows that the subspace S the subspace S is of dimension m – r, use rank nullity dimensions theorem we will get row rank equals column rank okay.

So the next step is to show that these vectors span any solution that is if I take some x star as a solution some x star that satisfies A x star equal to 0 I must show that this x star is a linear combination of these m – r vectors okay, I am giving the argument you can fill up the details. How do we get a solution of A x = 0, there is only one procedure that we know what is the procedure? There are free variables give some values to the free variables come back and determine these variables that is a solution, this is a general method okay. I have n - r free variables take one of them I give arbitrary values to n - r variable, let us say I take the jth coordinate I look at the entry corresponding to x j that entry is Alpha let us say then I can write that as Alpha times 1.

Then look at the next entry corresponding to j in J, some x j + 1, x j + 2 some entry that is some beta that is beta times 1. I keep doing this for each j in J I look at x j that entry and then I write that as if it is 0 I leave it as it is if it is not 0 it is some constant Alpha then I write it as Alpha times 1, leave it and do this for all x j for j in J. I will have n – r vectors, then you see then that the x star that I started with is a linear combination of these that is immediate almost. It is again very similar to the standard bases, in a standard bases there is only one entry all other entries are 0 this is more or less like the standard bases because of the reason that these values x c 1, et cetera, x cr are not in your control they can take other values these are non-zero values.

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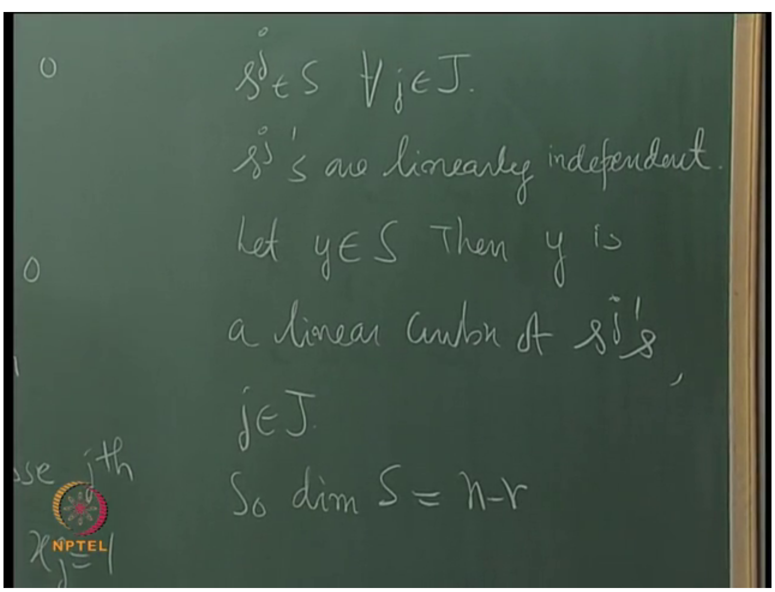

They are not in your control but look at the rest of them, rest of them are free and we have look at a very particular choice in constructing x j this 1, 0, 0, type and so please fill up the details, what follows is that each solution so let me write let y belong to S then it can be seen that y is a linear combination of these S j's for j in J. In argument almost duplicating the imitating the argument for the standard bases can be applied here show that these vectors are not only independent, they span the solution set S so what is the meaning then. So dimension of the subspace S is n - r okay I have given an explicit bases for this subspace, let us go back and use this so let us go back and see what S is.

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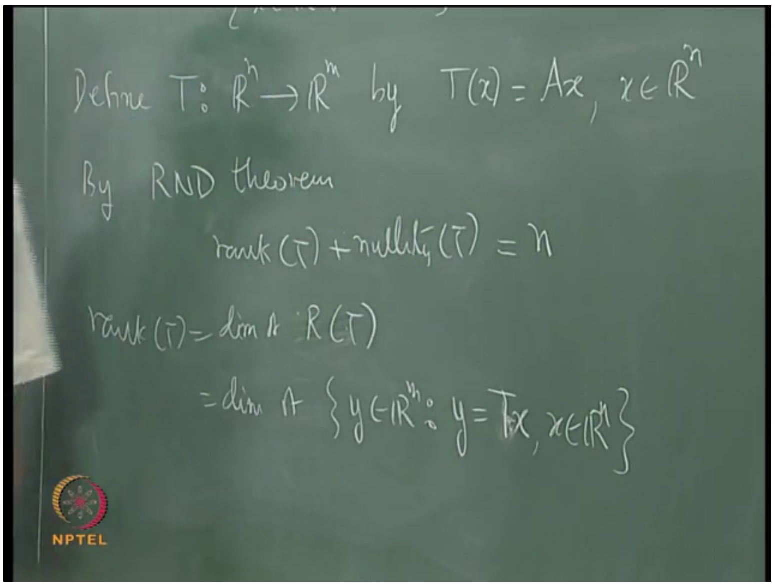

S is set of all... I will now use this set of all x such that A x = 0 okay the dimension of this subspace is n – r. Now let me define linear transformation T from R n to R n by T of x equals A x matrix multiplication x in R n, what is A? The matrix that I started with, A is the matrix that I started with I am defining a linear transformation, this is only to this is why media to apply nullity dimension theorem. Rank nullity dimension theorem is applicable for a linear transformation, I am going to practically apply it to a matrix, to apply it to a matrix I want this system. T x = A x then we know that this T is linear okay, rank nullity dimensions theorem can be applied to the linear map T.

By rank nullity dimension theorem rank of T + nullity of T equals dimensions of the core domain I am sorry domain that is n rank + nullity is the dimension of the domain, core domain could be even infinite dimensional okay. Rank of T, this time in terms of the languages in terms of linear transformations, earlier it was row space column space. Rank T, what is a Rank T? Rank of T is the dimension of the range space okay so rank of T is the dimension of range space of T that is dimension of range space of T. Let us write the complete definition it is a set of all y n R n such that y can be written as A into x for some x in R n, I am sorry y equals T x I am writing down the range of T okay but T x is A x so this is the dimension of the subspace y such that y equals A x but we have encountered the subspace before.

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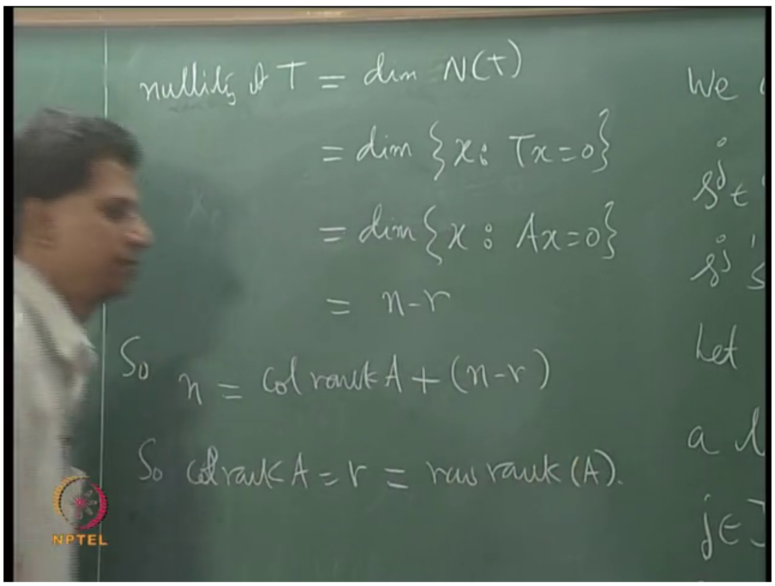

Columns space dimensions of the column space column rank column rank of A so rank of T is a column rank of A. What is nullity of T? Dimensions of the null space of T dimension of N of T that is what dimensions of the subspace of all x such that T x = 0 that is the dimension of all x that satisfies A x = 0 because T x equal to A x, this is what we have determine as n – r. Dimension S is n – r, S is the set of all A x such that A x = 0 so this number is n – r, nullity of T is n – r so go back to this equation; n is equal to rank of T that is column rank of A + rank of T + nullity of T, column rank of A + n – r cancel n, column rank of A equals r what is r now? Number of nonzero rows of capital r that is the row rank of r that is the row rank of A okay that is the row rank of A.

Now this is a proof that involves lots of calculations primarily using rank nullity dimensions theorem but there is a more sophisticated way of proving this using transposes, we will look at this proof quite later okay but this is using all the calculations that you do uhh for row reduced echelon matrix homogeneous equations, non-homogeneous equations, et cetera okay. So this is one of the most important results in linear algebra that is why I have made it a point to prove this completely okay so let us move on to the next topic the matrix of a linear transformation okay.

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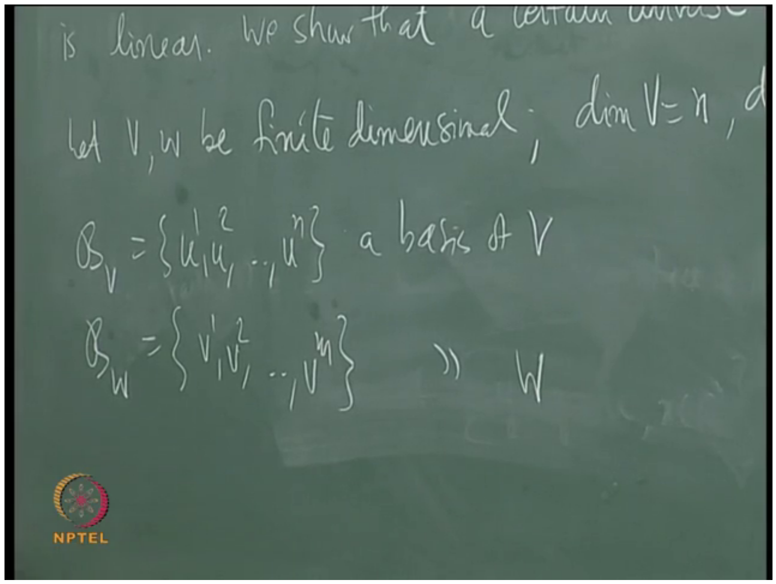

So I want to discuss a notion of the matrix of a linear transformation, this is another fundamental notion uhh I hope you remember the statement that I made some time ago when we discussed examples of linear transformation that something we have done even now. Given a matrix so recall this, given a matrix A with real entries m cross n uhh the mapping T from R n to R m defined by T of x equals A x this is linear. Given a matrix there is a natural linear transformation that can be associated with this matrix and I also make a statement there is a certain converse which is true. Now what is this converse, I will make this statement of the converse precise and also show how to construct how to proof this statement in a constructive manner.

The converse statement is that given a linear transformation there is a matrix associated with this linear transformation which behaves in precisely this manner, given a linear transformation between finite dimensional spaces there is precisely one matrix corresponding to bases such that the matrix will do what you have here so this is more or like defining equation of a linear transformation between finite dimensional spaces okay. So that is the statement that we are going to proof so how do we go about it? I have still not made the statement precise so will simply say a certain converse is true.

Okay, so let us go back and look at finite dimensional vector spaces so I have let V, W I have 2 finite dimensional vector spaces over the same field let us say the field of real numbers let us say dimension of V is n and dimensions of W is m. Let me also write down 2 bases explicitly, let me use the following notation script B v will be bases for V, let us say the entries are u 1, u 2, et cetera, u n, this is a bases of V. B w will be bases for W, let me call it v 1, v 2, etc v m bases of W. So I start with 2 given bases then given any vector x and v it can be written as a linear combination of these vectors.

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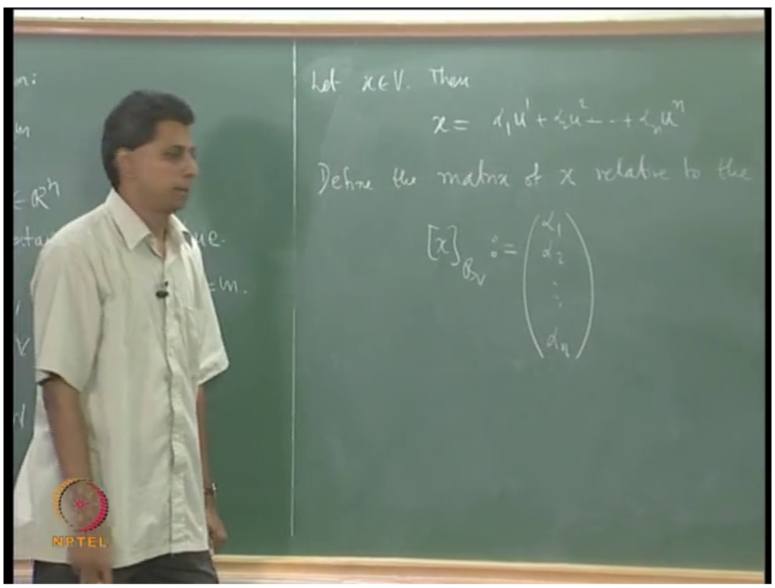

Let us say Alpha 1 u 1 + Alpha 2 u 2 etc + Alpha n u n. Alpha 1, Alpha 2, Alpha n are real numbers, they are unique for the x that I start with, these real numbers are unique for the x that I start with. What I will do then is define the matrix of x matrix of a vector matrix of x relative to the bases that I started with relative to the bases B v, define the matrix of x relative to the bases B v by the notation for me will be x B v with this parenthesis. This matrix if it is a vector it is a column, so what is the column? It is Alpha 1, Alpha 2, et cetera, Alpha n is equal to this.

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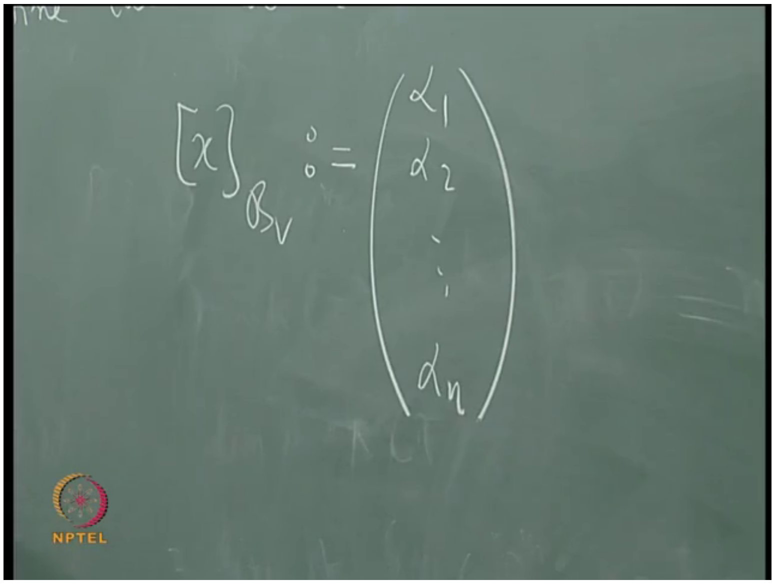

This is actually a function, in order for this to be a function the right-hand side must be unique, whenever x is unique but we know that it is 0 because the representation is given in terms of bases. So this is then the definition of the matrix of a vector relative to a fixed bases okay let us observe 1 thing immediately, I take the same bases B v, instead of u 1, u 2, et cetera u n I take u 2, u 1, etc u n then this matrix will change this will be Alpha 2, Alpha 1, et cetera okay, so there is really an ordered bases that I am dealing with okay. you must deal with ordered bases but here after when I write u 1, u 2 etc u n then it follows that u 1 is the 1^{st} vector in that bases, u 2 is the 2^{nd} vector et cetera u n, this order I should always remember when I write down the matrix of a vector okay. So the notion of the ordered bases needs to be uhh introduced but I will just skip it over.

Ordered bases means a bases whose vectors form a finite sequence, sequence means that is the 1^{st} element of the sequence, 2^{nd} element of the sequence, et cetera so it is just an ordered set of vectors which also forms a bases, why is it important? When you write down the matrix it is important because you say there is a 1^{st} coordinate, there is a 2^{nd} coordinate, et cetera. Alpha 1 the 1^{st} coordinate, now I am writing vector in an abstract finite dimensional vector space using numbers, I am writing down the I am giving representation to a vector in an abstract finite dimensional space using real numbers Alpha 1, etc, Alpha n by this so there is a coordinate, 1^{st} coordinate of the vector x, 2^{nd} coordinate of the vector, etc.

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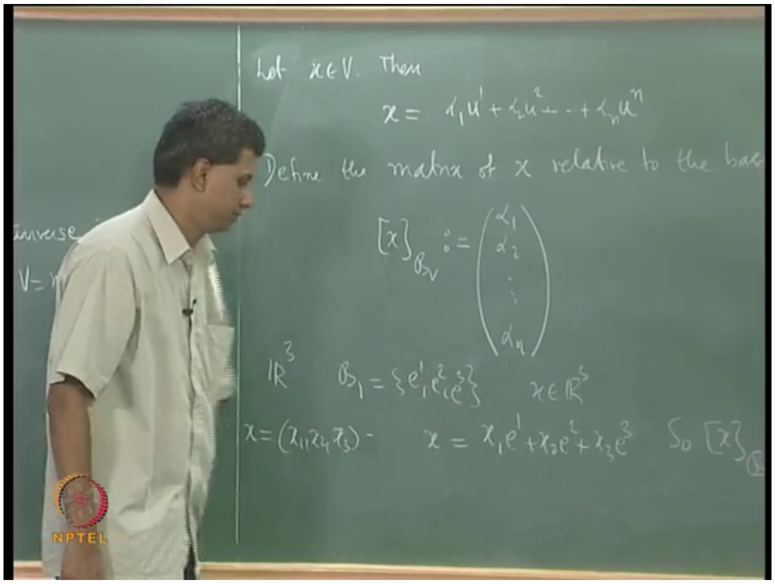

In order to make since that is the 1st coordinate, 2nd coordinate, etc, you need to have a fixed bases where the elements are taken in only that order okay that is an ordered bases. Let us make one simple uhh calculation in elementary example just to consolidate, let us take R 3 for instance for R3 there is a standard bases let me call that B 1, the standard bases e 1, e 2, e 3 given the standard bases in this order this is always implicit the order is implicit, what is the matrix of x relative to B 1. Given any x in R 3 what is the matrix of x relative to this? This is the simplest bases you can write down the matrix immediately but let us do it from the scratch. This x can be written as okay I will start with the following.

x is the vector which has 3 coordinates so I have x = x 1, x 2, x 3, I intentionally writing this as a row vector then this x can be written as x 1 e 1 + x 2 e 2 + x 3 e 3 this representation is unique and so what is the matrix of x relative to the standard bases, I am calling that B 1 that is the column vector according to the definition x 1, x 2, x 3 okay it is as easy as just taking a row and putting it this order. This is the simplest bases suppose I have another bases let us make calculation corresponding to that.

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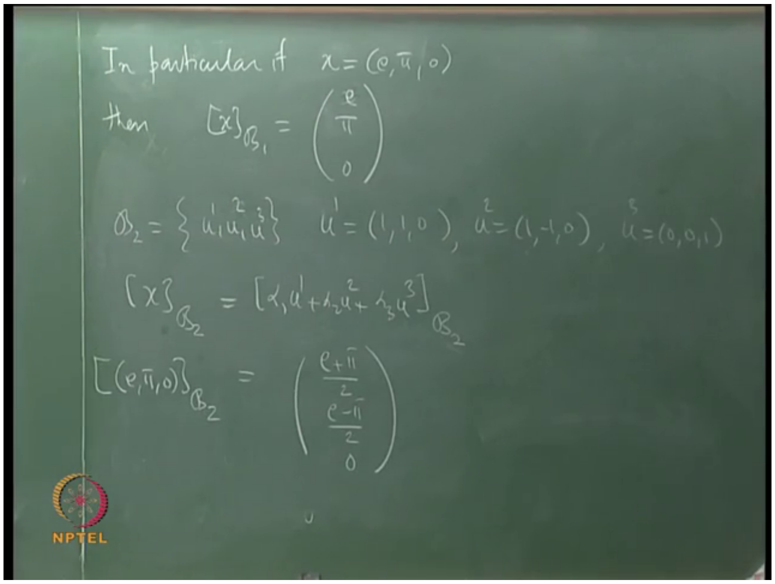

In particular if let us say x is e, Pi, 0 then the matrix of x relative to B 1 is e, Pi, 0 okay. Let us say I have another bases B 2 consisting of... This time let me call them u 1, u 2, u 3, where u 1 let me say the vector 1, 1, 0, u 2 is the vector 1, - 1, 0, u 3 is the 3rd standard bases 0, 0, 1. I have taken another bases just to illustrate that the matrix corresponding to these bases will be different from the original 1 corresponding to the standard bases. I want to calculate x relative to this B 2 okay, instead of doing the general vector let me take this particular vector and do the calculations here. I want to determine these numbers, let me now call them Alpha 1 u 1, + Alpha 2 u 2 + Alpha 3 u 3, 1st I must write this, I have just written x and then I need to determine the matrix of this whole thing relative to B 2.

Please verify the calculations, it will turn out to be see this is 1, 1, 1, -1 so it will turn out to be the following uhh for this x for this x that I started e, Pi, 0, you can verify that it is e + Pi by 2, e – Pi by 2 0 this is the column vector. This is the matrix of the vector e, Pi, 0 corresponding to the 2nd bases which is obviously different from the matrix of the vector corresponding to the 1st bases standard bases. So when I change the vector rather when I change the bases, the representation of the vectors will obviously change okay, but there is a relationship between them we will be able to demonstrate that there is a relationship between them okay.