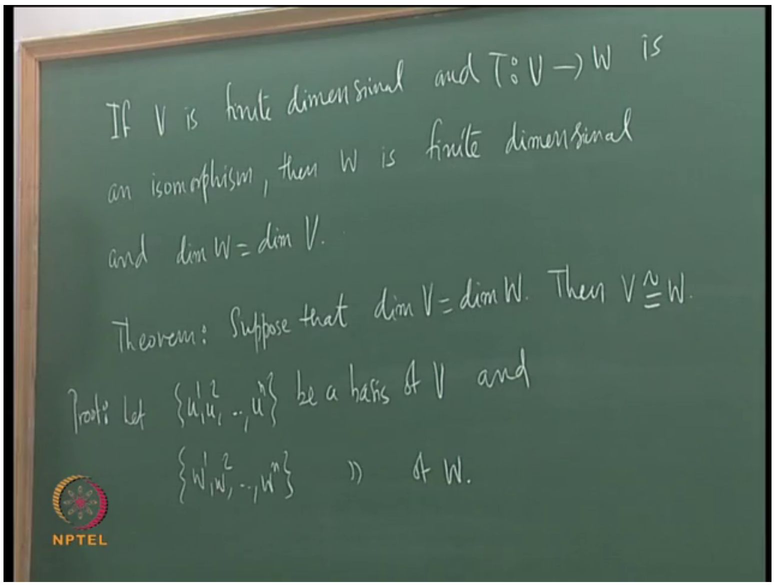
Linear Algebra

Professor K. C. Sivakumar
Department of Mathematics
Indian Institute of Technology Madras

Lecture no 17
Module no 04

Isomorphic Vector Spaces, Equality of the Row-rank and the Column-rank I

(Refer Slide Time: 0:36)

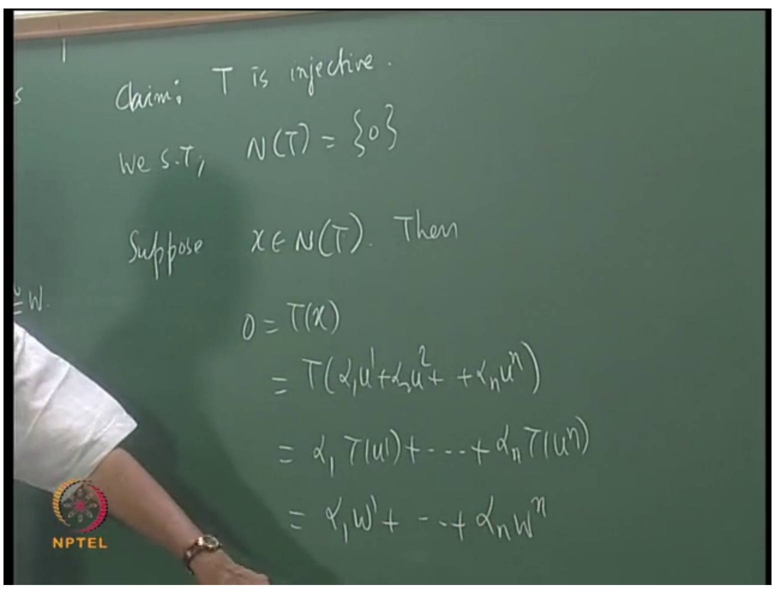


We have seen yesterday that if V is finite dimensional vector space and T from V to W is an isomorphism, then W is also finite dimensional in fact we have dimensions of W equals dimensions of V this result we have seen yesterday okay. Let us look at the converse of this result that is if we have 2 finite dimensional vector spaces whose dimensions are the same, are the isomorphic okay, the answer is yes that is what I will do today the proof of this result and also some consequences of the rank nullity dimension theorem, there is a little unfinished business. Okay so this is what we will 1st proof today, I have V and W finite dimensional vector spaces with the same dimensions then V is isomorphic to W then V is isomorphic to W that is what we will prove now.

You will see that this is the proof is natural, let us take 2 bases so proof of this theorem let us take u 1, u 2, et cetera u n to be bases, this is bases of V. I know that dimension of W is also n so I will enumerate the bases for the space W, they are bases of W the numbers are the same the number of elements in these 2 bases this number is the same just because dimension V is dimension W. Now what we know is that there is a linear transformation uhh that maps each u i to the corresponding W i and this linear transformation is unique okay. The only thing that we need to do is to verify that this linear transformation is invertible Injective and surjective. See we want to show V is isomorphic to W so we must show that this T is invertible okay.

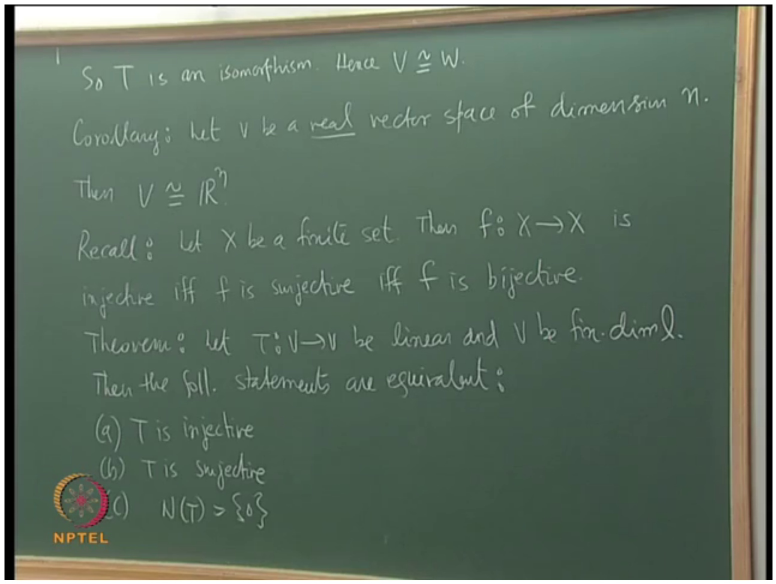
So let us define define the mapping T from V into W by T of u I equals W I, we know that there is one such linear transformation we also know that this transformation is unique, we must show that this transformation is invertible from the rank nullity dimensions theorem it is enough if we show it is Injective, the dimensions are the same so it must be surjective also so let sure that it is Injective.

(Refer Slide Time: 4:20)



To show that it is Injective we will show that null space of T is single term 0, we have seen that this is an equivalent condition so let us look at an arbitrary element in the null space of T then T x = 0 and this x belongs to V, and V has u 1, u 2, et cetera u n as a bases so this is T of some linear combination Alpha 1 u 1 + Alpha 2 u 2 et cetera Alpha n u n. T is linear, Alpha 1 T of u1 etc + Alpha n T of u n but T of u 1 is W 1, etc T of u n is W n so this is Alpha 1 W 1 etc Alpha n W n so I have this linear combination Alpha 1 W1 et cetera Alpha n W n equal to 0, I also know that vectors W n they form a bases for W so in particular they are independent. So it means Alpha 1 equals Alpha 2 equals Alpha n equals 0 but what is x? x is Alpha 1 u 1 etc Alpha n u n so x is the 0 vector, the definition of x is given here this is x and so T x is equal to 0 in place x equal to 0, T is a linear transformation so T is Injective the rank nullity dimension theorem T is surjective so T is invertible so T is an isomorphism okay.

(Refer Slide Time: 6:36)



So T is an isomorphism hence V is isomorphic to W okay, so the dimensions coincide then they must be isomorphic vector spaces. There is an easy corollary to this result, let V be real vector space of dimension n, let V be real vector space of dimension n then V is isomorphic to R n, V be a real vector space of dimension n then V is isomorphic to R n. We can similarly show that V is a complex vector space of dimension n then V is isomorphic to C n okay. Remember the dimensions of R n over R is n that is what I mean here, real vector space R n is the vector space over R. Look at the vector space C n, C n over C also has dimension n but C n over R that is also a vector space that has dimension to n because any complex number is an (())(8:22) pair of real numbers okay. To represent a complex number you need 2 real numbers so if it is a complex vector space of dimension n then it is isomorphic to C n alright, if you want to write it as R n then it is R 2 n that you need to write okay.

Okay so this is to summarise about isomorphism between vector spaces, as I said I want to go back to this rank nullity dimensions theorem look at some consequences. Okay uhh please recall this result that you must have learned till now when you studied functions. Suppose x is a finite set then we know that a function F from x to itself we know that a function from x to itself is... This function is Injective this 1 channel is Injective 1 1 if and only if F is surjective if and only if

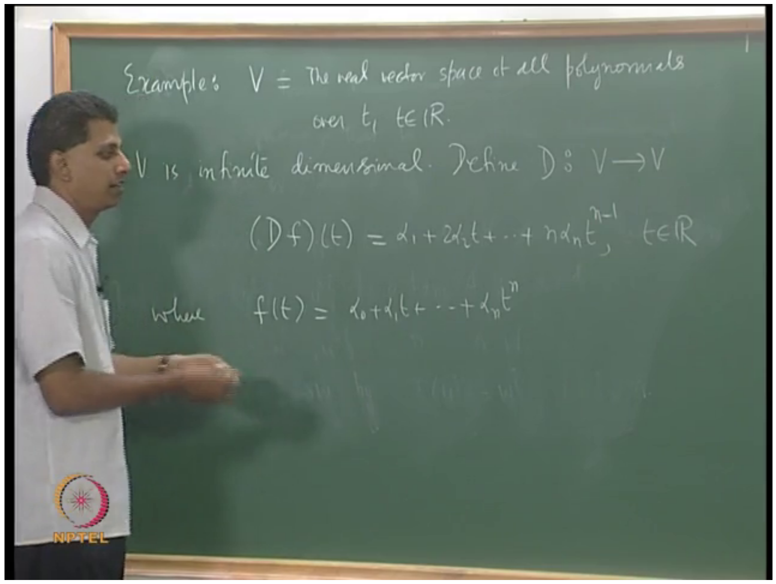
F is by Injective okay I hope you have seen this result over finite set. Over infinite set this is not true, over finite sets their function is Injective if and only if it is surjective if and only if it is by Injective so again over finite sets it is enough if you verify that it is either surjective or Injective then it will follow that it is by Injective it is invertible.

A similar result holds and consequences of rank nullity dimensions theorem for a linear transformation over finite dimensional vector spaces, so this is really a particular case of the rank nullity dimensions clear which means I will not prove difficult but I want you to compare the following results with the result that I have written down just now okay this result is the consequences of rank nullity dimension theorem and the proof is left is an exercise. Let T from V to V be linear and V be the finite dimension; T is linear, V is finite dimensions, T is operator on V, T is linear transformation from v to itself that is an operator then the following are equivalent. Then the following statements on T are equivalent; 1^{st} statement is T is Injective, 2^{nd} statement is T is surjective, 3^{rd} statement for instance I could include null space of T equals single term 0, I could include one more statement range of T is whole of V but these are equivalent statements.

Easy consequences of rank nullity dimensions theorem for instance look at a implies b or a if and only if b, T is Injective if and only if null space of T is single term 0 which from the rank nullity dimensions theorem tells us that rank of T is n but rank of T is the dimension of the range space of T it is a subspace of V, V has dimension n, this subspace has dimension n so this subspace must be equal to the entire V and so range of T equals V so T is surjective et cetera okay. So there is nothing new in this result, it is only a particular case of rank nullity dimensions theorem where instead of W I have taken Vand is also allows us to compare with the result that I have stated here. For a function over finite set injectivity is equivalent to subjectivity equivalent to bijectivity.

But what happens in the infinite dimensions case, in the infinite dimension case this result is not true okay that is the reason why there is this restriction so let us look at an example. I want to give an example of linear operator say which is surjective but not Injective so look at the following.

(Refer Slide Time: 13:42)

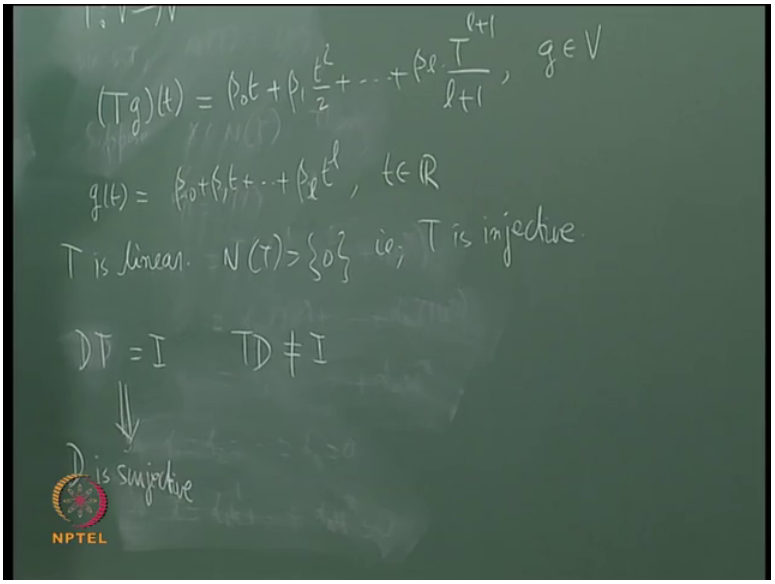


Consider V as the vector space the real vector space of all polynomials vector space of all polynomials whose coefficients are real numbers defined over the variable E, E is a real variable. Now this is an infinite dimension vector space for instance this is a subspace of C 01 the space of continuous function which we have shown is infinite dimensions of course being a subspace of an infinite dimensions space does not mean this must also be infinite dimensions but then we know that 1 t, t square, et cetera, they are linearly independent so this is for infinite dimensions, V is infinite dimensional the same idea that we used to prove C 01 is infinite dimensional applies here. Let us define the differentiation operator, define D from V to V, any polynomial is infinitely many times differentiable so define D from V to V by D f of t to be... What is the form of F depending on that D f of t will be defined.

So let me say D f of t where f of t now f is in V so f is a polynomial, f of t is let us say Alpha 0 + Alpha 1 t et cetera + Alpha n. Then D of t is defined as Alpha 1 + the next term is Alpha 2 t square that is 2 Alpha 2 t + etc + n Alpha n t to the n - 1, this is the definition of D this is called differentiation operator. Takes a polynomial, compute its derivative that is the operator D then since differentiation operator is linear transformation this D is linear okay you can apply it to each term this D is linear let me also define... By the way uhh is D Injective? What is the null space of D?

Student: Set of all constants.

(Refer Slide Time: 16:39)

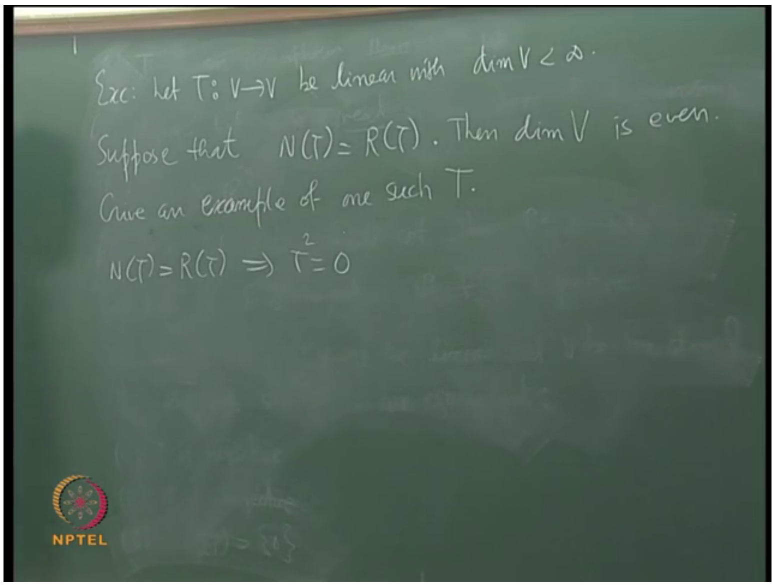


Null space of D is set of all constants... Constant polynomials okay so D is not Injective okay so let us now define another operator, uhh this is so-called indefinite integral operator I will do that. I am going to define another operator I will call that T from V to V is defined by let us take T g of t, what is the form of g? If g of t is let us say beta 0 + beta 1 t etc beta L t to the L then I will do this so-called indefinite integration that is integral of this, which is beta 0 t + Beta 1 T square by 2 etc + beta L T to the n + 1 by L + 1 so this is what I will define for operator T, this is so-called indefinite integral okay this is like integral 0 to t g of t D t. This is again linear, this T is linear okay what about range of T okay what about null space of T?

T is Injective okay T is Injective, let us look at the relationship between D and T, what about T D what do you expect T D to be? It is differentiate and integrate okay, let us start with D T what is D T? D T is identity okay please check this D T is identity, what about T D? T D for example for constants if you apply you will see that it is 0 so T D cannot be identity T D cannot be identity, so what is the moral of the story? moral of the story is... See this D is not Injective but it has right inverse, D is not Injective the operator D that we have defined is not Injective, D has a right inverse, from this it can be shown that D is surjective, I am going to leave this as an exercise from this it can be shown that D is surjective okay.

What about T, T is Injective but you can verify that T is not surjective because of this okay that is also an exercise for you T is not surjective, T is Injective. Let me say this right away implies T is not surjective okay so I have given an example of an operator over an infinite dimensional space, one operator which is Injective but not surjective the other operator which is surjective but not Injective okay, differentiation operator surjective but not Injective, the indefinite integral operator is Injective but not surjective okay. So please check these calculations and so this result is not for infinite dimensional spaces.

(Refer Slide Time: 22:26)



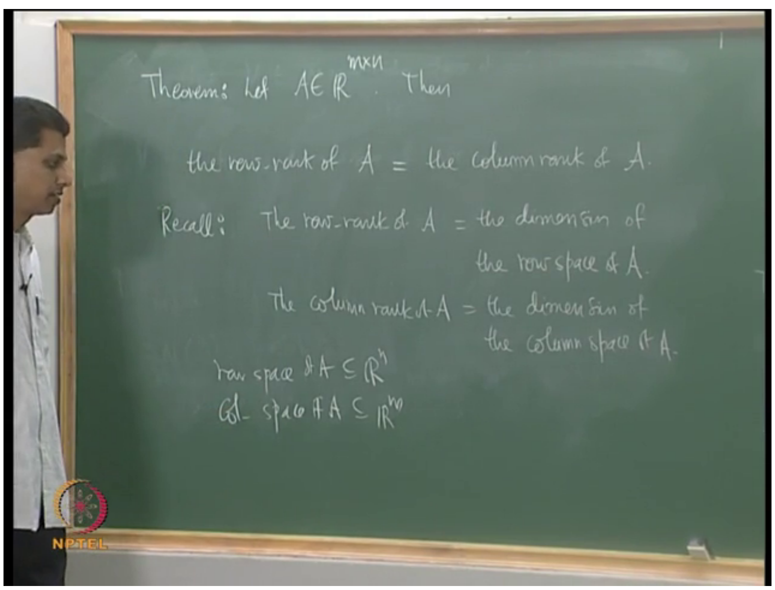
There are other consequences in particular let me give this little exercise, using rank nullity dimension theorem one could solve this problem. Let T from V to V be linear with V finite dimensional, suppose that suppose that null space of T = range space of T, V is finite dimensional, what is the conclusion that you could draw on the dimension of V?

Student: (())(22:47) dimensions

It is even... Then dimension of V is even it is an even positive integer, okay not that is an easy consequence of rank nullity dimensions but what I want you to do is to give an example of such a linear transformation. Given example of one such transformation T okay now in order to sort out the 2nd problem I will give one hint; T is an operator that satisfies the condition null space of T is a range of T okay then 1st try to show that T square equal to 0. Null space of T equals range of T so that this implies T square is 0, T square is a 0 operator. Construct an operator that satisfy this condition, to construct an operator that satisfies this condition to construct 2 by 2 for example, let us construct T over R 2, T from R 2 to R 2 remember these spaces must be the same otherwise null space of T and range of T, null space of T is a subspace of V, range of T is a subspace of W.

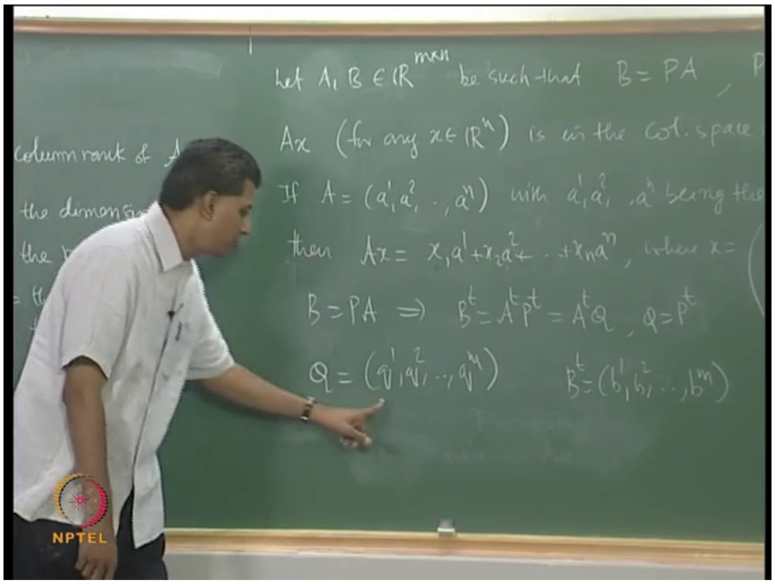
So when we talk about equality I can to that only when that domain is equal to the core domain. Take T from R2 to R2, you want to construct T such that T square is 0, start with the matrix a that satisfies the condition a square equal to 0, use that matrix a and then construct the linear transformation, T x = A x so that I am going to leave you uhh leave with you as an exercise but what you can try is, is the converse true that is the related question. If T square is equal to 0 then does it follow that range of T is equal to null space of T that is another question so construct an example T that satisfies this, verify if the converse is true okay. One final application of the rank nullity dimensions is to show that the row rank of a matrix is the column rank of that matrix, this statement was made long time ago when we discussed elementary row operations row equivalence, et cetera so we would like to prove the following theorem.

(Refer Slide Time: 25:40)



We would like to show that the row rank of A equal to column rank of A okay, before that what is a row rank what is a column rank? So before I prove so let us let us recall these definitions, the row rank of a matrix is defined to be the dimension of the row space of A the row rank is the dimension of the row space of A, the column rank is a dimension of the column space of A okay. As we have observed the row rank the row space rather is a space of R n, the column space is a subspace of R m is that clear? Each row has n coordinates, each column has m coordinates so the column space is a subspace of R m and the row space is a subspace of R n okay. So it is an interesting and important result that these spaces may lie in different these subspaces may lie in different places but their ranks are the same their dimensions are the same.

(Refer Slide Time: 28:14)

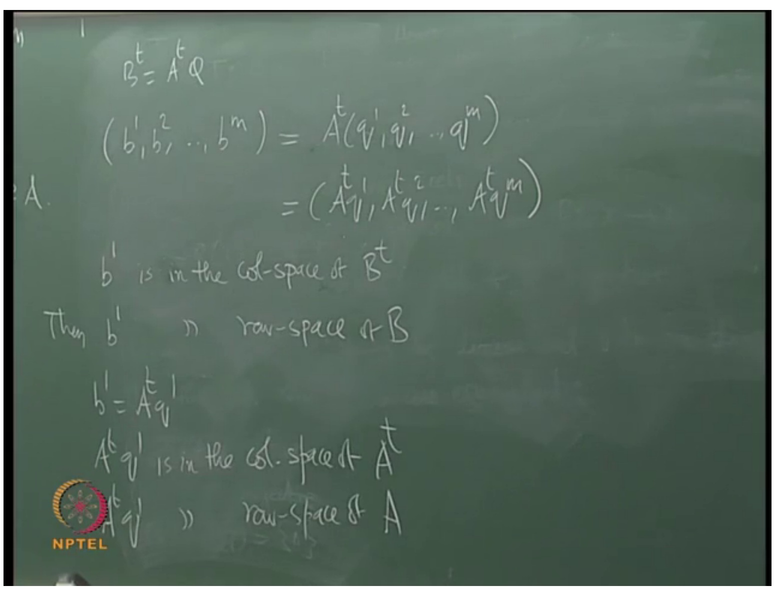


There are other consequences of rank nullity dimension theorem but we need a little work before settling this identity, so let us recall what we did for row (())(28:01) let me 1st prove this result. R start with 2 matrices is of the same order, which are related by the equation 2 matrices A and B related by this equation B = P times A where P is square matrix of order m so this is n cross m, m cross n, the product is m cross n, B is n cross n. What I want to do is to conclude that if B is equal to P A then the row space of B is contained in the row space of A, I want to conclude that row space of B is contained within the row space of A. To prove this I will involve what we had seen earlier that anything... So I am going towards the proof of this.

Look at A x for any x, A is in m cross n, look at A x for any x this is in the column space of A this we have seen before, in particular I wrote this A x as okay I want to conclude that A x in the column space of A, why is that so? you can see that if I write A as a 1, a 2, et cetera a n these are the n columns of A with a1, a 2 etc a n being the columns of A. If A is equal to this then we can verify that A x is x1 a1 + x2 a2 etc + x n a n where I have used, x is the column vector; x1, x2, et cetera x n okay now this we have observed before. See what you have on the right is a linear combination of the columns of A what I am saying is that this is precisely A x where A is a matrix that we started with and x is this column vector okay that is to reinforce A x in the column space of A.

Okay and letters go back to this equation B equal to P times A; now B equals P times A after taking transposes using the fact that transpose satisfies the reverse order law, this gives me B transpose as A transpose P transpose, B transpose is A transpose P transpose, let me call this as A transpose Q so Q is P transpose let me write Q, see P is m cross m so Q is also m cross m, Q has m columns m rows so let me write Q as q1, q2, etc q n again these are the columns of Q just as how I wrote down the columns a 1, etc a n for A, these are the columns of Q. Let me write down the columns of B transpose; I will call that b1, b 2, etc. B transpose is b1, b2, etc , B is of order m cross n so there are m columns, B transpose will have m columns so b1, b2, et cetera, b n, B is m by n so B transpose is n by m in particular number of columns of B transpose is m, the columns of B transpose I am denoting them by b1, b2, etc b n okay, please be clear about the notation here. I am using a 1 etc a n for A, q1, etc, q n for Q but b1, etc, b n for B transpose okay for simplicity.

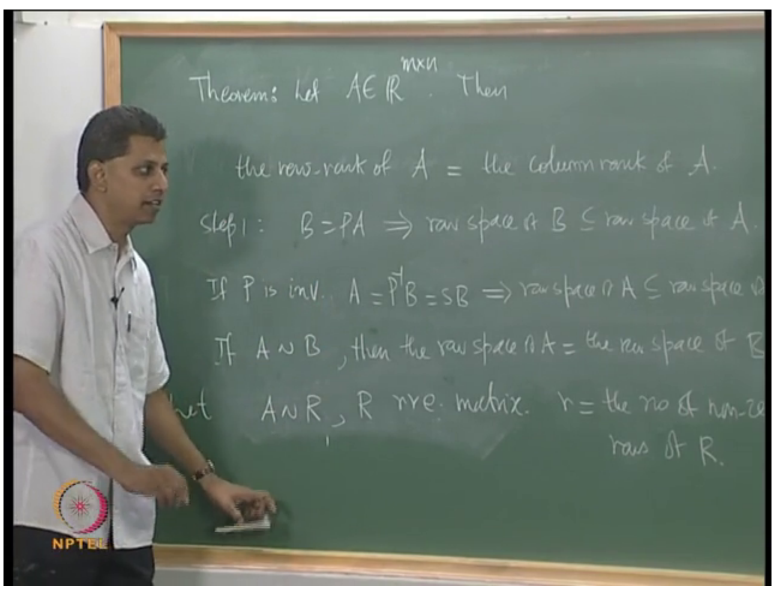
(Refer Slide Time: 32:59)



Okay now look at this equation once again; B transpose is A transpose Q. B transpose equals A transpose Q gives me b1, b2, etc b n equals A transpose q1, q2, et cetera q m and this is something we have seen before you can push the matrix inside, this is A transpose q1, A transpose q2, etc, A transpose q n okay that is A into q1, q2, etc, q n A transpose into q1, q2 then this matrix can be pushed inside. This can be verified by matrix multiplication I remember having told you this before. In particular look at b1; b1 is the 1st column of B transpose, b1 is in the column space of B transpose then b1 transpose is in the rows okay then b1 is in the row space of b, column space of B transpose row space of B. See in fact you will observe that b1 is a the 1st column of B transpose so b1 is the 1st row of B in particular it is in the row space of B okay.

Look at what we have on the other side; A transpose B but b1 is A transpose q1 okay look at A transpose q1 in its own right, a matrix times a vector that is column vector of A transpose. Is it clear that A transpose q1 is let me say in the let me say it is the first column of let me rewrite... say I want to make use of this fact that A x is in the column space of A. A transpose q1 is in the column space of A transpose, A transpose q1 is in the column space of A transpose so it is in the row space of A. A transpose q1 is in the row space of A, it is in the column space of A transpose so it is in the row space of A, it is a linear combination of columns of A transpose, which is the linear combination of the rows of A, so what have we proved?

(Refer Slide Time: 36:38)



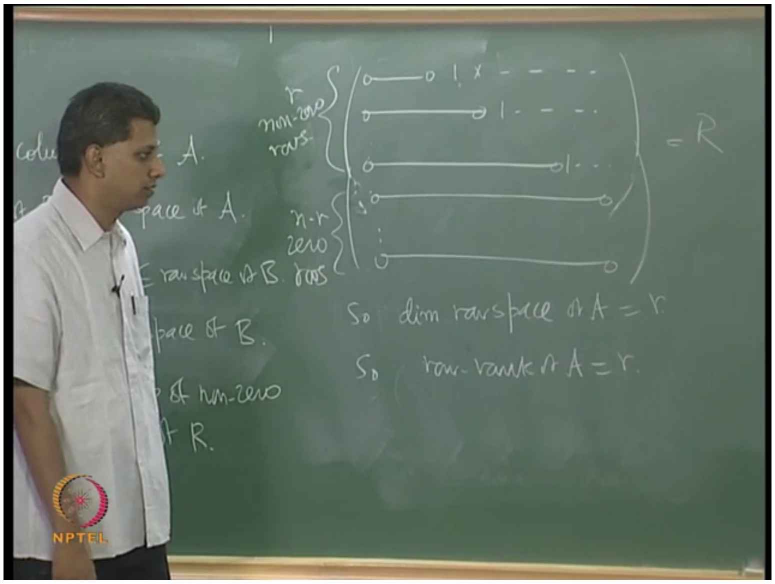
On the one hand it is b1 is in the row space of B on the other hand it is in the row space of A, so if B is equal to PA we have shown that that the rows of B are linear combinations of the rows of A that is the first statement okay let me write I want to prove this statement, what we 1st observed is that if B equals P A then the row space of A sorry the row space of B is contained in the row space of A, if B is equal to PA then the row space of B is contained in the row space of A that is what we have proved just now. Suppose P is invertible then I can P multiply by P inverse. If P is invertible... By pre-multiplying by P inverse I get I write A as P inverse B okay so what I have done is, you can call this S times B, I have written A as S times B.

Apply the same idea, if A is S times B then the row space of A is contained in the row space of B, the argument we have given just now can be applied here in this case so this implies row space of B sorry row space of A is contained in row space of B okay if this matrix P is invertible then I get this result, which means if B equals P A with Be invertible then the row spaces of A and B are the same. In particular in particular row equivalent matrices have the same row space because of A and B are row equivalent then A can be written as P times B where P is the product of elementary matrices. So row equivalent matrices have the same row space if A is row equivalent to B then the row space of A equals the row space of B that is because A can be written as P times B for some invertible matrix P.

Look at the row okay now I want to determine a bases for the row space of A look at the row reduce take alone form R of the matrix A, R is a row reduced take alone matrix row equivalent to A as before, R is a row reduced take alone matrix row equivalent to A then the first let us take R to be the number of non-zero rows of R, small R equals the number of non-zero rows of capital R okay, okay what is the dimension of the row space of A? Can we say it is R for the following reason? What is the row space of R? Row space of R is the subspace consisting of the linear combination of the rows of A.

Now R first R rows are non-zero the rest n – R are 0 so the n – R 0 rows do not contribute anything to the row spce, it is only the contribution that comes from the first R nonzero rows of R so the row space of capital R is spanned by these R vectors, are these R vectors linearly independent? We need to verify that but that is easy, and argument similar to the standard bases can be given okay. Instead of just mentioning it let me write down in the form of R and it will be clear from the 1^{st} R nonzero rows that is nonzero rows are linearly independent in fact.

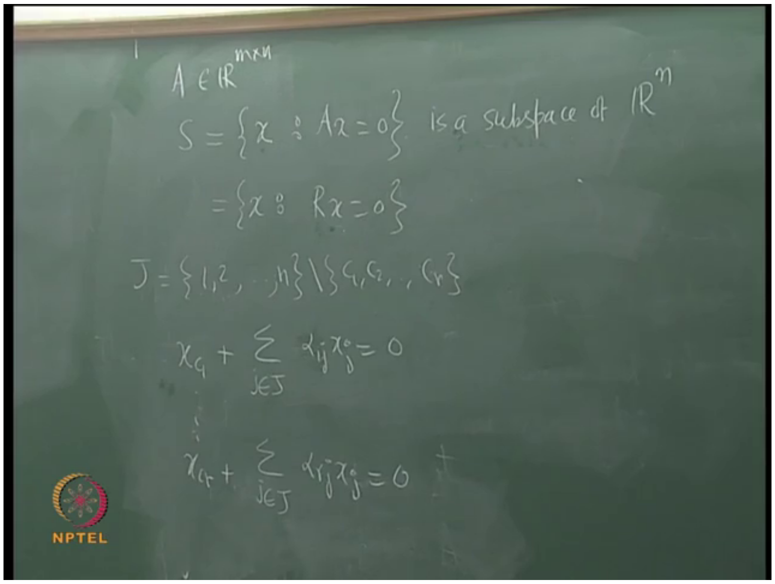
(Refer Slide Time: 40:55)



Remember this is how we used to write, we have certain 0 then there is a 1 and some entries here then some Zeros here, 1 appears to the right of this some other entries et cetera, finally I have lots of zeros here this and then I have the 0 rows that is I have this corresponds to the n – R 0 rows. I am writing down R, this is my matrix R these are then R nonzero rows and it is clear that this behaves somewhat like the standard bases vectors so you take a linear combination Alpha 1 times this vector + Alpha 2 times this vector, et cetera, Alpha R times this vector equate that to 0 then right away the 1st equation gives you Alpha 1 0, 2nd equation gives you Alpha 2 0 they do not lie along the same column they are in different columns that is if this is in C1, this is in C2 et cetera this is in C R then we C1 less than C2 less than C3 et cetera less than C R.

So it is clear that these R nonzero rows the 1st R nonzero rows of R are independent and that they span the row space of R and so that dimension of row space of A is equal to R, dimensions of row space of A equals R where R is the number of nonzero rows of the row reduced echelon form of A that is row rank of A that is equal to R, row rank of A is equal to R. We need to show that the column rank of A is also R okay, for the column rank we need to do a little more so let me see how much I can cover.

(Refer Slide Time: 43:07)



Now for the column rank I want to start all over again; consider A in R m cross n that I started with earlier, look at the system A x = 0, collect all the solutions call that as S, S is the solution set of the homogeneous system of equation A x equal to 0 okay. Remember that the kernel of A whole thing is we have defined kernel of a linear transformation what is the kernel of a matrix, through this matrix you can define a linear transformation and then the kernel of that linear transformation is a kernel of this matrix okay. But in any case what I want to say is this s is a subspace, this is a subspace of R n set of all x in R n so it is a sub space of R n okay, I would like to calculate the dimension of the subspace it cannot exceed n I know, I want to calculate the dimension of the subspace I want to conclude that dimensions of subspace is m – R.

Then it will follow that the column rank is R, which is the same as the row rank of A okay, so I want to conclude that dimensions of subspace is m – R. To calculate dimensions I also observe that R is row reduced echelon form of A then the solution set of R x equal to 0 and A x equal to 0 or the same. I will use this to calculate the dimensions of S, I will use the row reduced echelon form R to calculate dimensions of S. Let us go back and write down these equations and analyze this once again, let me in this case use this notation, J in the subset of all integers 1, 2, 3, etc, n, which do not have a C1, C2, et cetera, difference C1, C2, et cetera, C r. C1 is the parliament which the leading nonzero entry of the 1st row appears, C2 is the column in which the leading nonzero entry of the 2nd row appears, et cetera okay.

I remove these integers from the integer 1, 2, 3, et cetera , n then I can write down the eye can expand this, we have done this before but I will do it using this notation you will see it is the same. As before xc 1, x c2, etc, x cr corresponds to those variables out of x1, x2, et cetera, x n that corresponds to these columns okay then x c1 + we write submission J equals m - R but this time city submission J over J. Let me say I have C1 no C1 I cannot use let us say Alpha 1 J x J equals 0 et cetera x cr + submission J element of J Alpha R J x J equal to 0, probably I will stop here and then continue next time. Okay remember I want to show that the dimension of this space is m - R that would show that the column space of A is R dimensional which means that column rank of A is R same as row rank of A okay.