Linear Algebra
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Lecture no 16
Module no 04

The Rank-Nullity-Dimension Theorem, Isomorphisms between Vector Spaces

Let us discuss some more properties of linear transformation, uhh the first important result for linear transformation whose domain is a finite dimensional vector space is called the Rank-nullity dimension theorem so let me prove that first.

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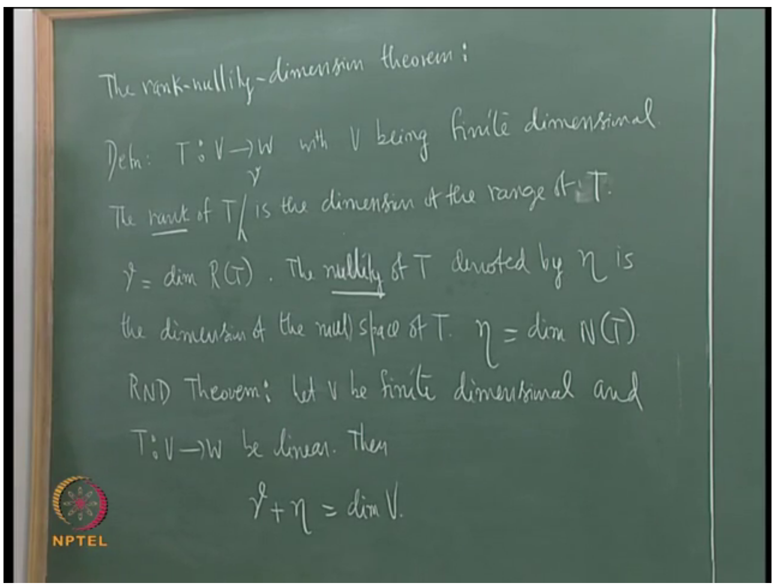

The Rank-nullity dimension theorem okay, first what is the rank, what is nullity? Let T be a linear transformation from V into W, I will assume that V is finite dimension, the rank of T is the dimensions of the range of T range of T. Let us use R for this, then almost like Gamma, this is dimensions of range of T that is a rank, the nullity of T let us denote it by Eta is the dimensions of the null space of T that is let us say Eta is dimensions of n of T, so I define these two numbers nonnegative integers, Gamma and Eta are nonnegative integers. Dimensions of the range of T is the rank, dimensions of the null space of T is nullity okay so these are the numbers. There is a dimensions of domain space V, I have assumed v to be finite dimensional so this theorem relates these three numbers.

The Rank-nullity dimension theorem, I will call it the RND theorem; V is finite dimensional, T be a linear map from V into W, no conditions is on W which means W could be infinite dimensions but V is finite dimensional, T is linear okay then this theorem says that rank + nullity equals dimension of the domain space okay. Now you see that dimensions of W does not come into this equation that is the reason why there is no condition on W, W could be infinite dimension okay. So let us prove this and then look at some of the consequences of this result.

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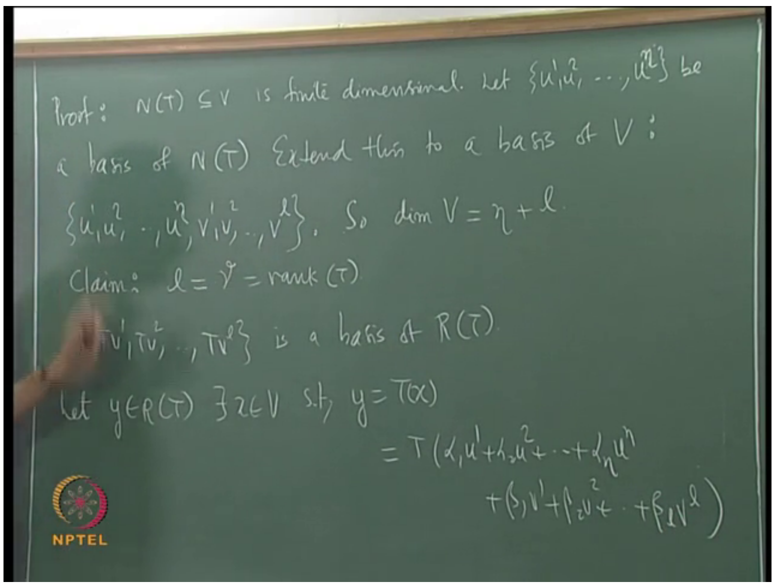

Let me first discuss the proof, I want to make use of the fact that any linearly independent subset of the vector space v can be extended to the bases of v okay. So let me start with null space of T, what I know is that null space of T being a subset of V must be finite dimensional. Null space of T is finite dimensions so I will take up bases consisting of finite elements let us call u 1, u 2, et cetera u Eta u 1, u 2, et cetera u Eta be bases of null space of T, I know that null space of T is Eta dimensions so there are Eta vectors here, this is the bases of null space of T, this is linearly independent this can be extended to the bases of V.

Extend this to a bases of V that is I will have u 1, u 2, et cetera u Eta then let me called the other vectors v 1, v 2, et cetera, v L this is the bases of way, and so what this means is that dimensions of V must be Eta + L okay dimensions of V is Eta + L, where Eta is the dimensions of the null space of T that is the nullity so we need to only show that L is equal to rank of T, we need to only show that L is equal to rank of T okay. So the claim is... Is that clear? So dimension of V is Eta +

L because of this being a bases there are Eta + L vectors, Eta we started with this is the bases for null space of T so Eta is the nullity of T, we need to only show that L is equal to rank of T that is Gamma dimensions of the range space of T, so we will exhibit we must exhibit the bases of range of T consisting of precisely L vectors okay.

It is it is probably not unnatural to take these vectors, take all these vectors, look at the action of these vectors under T, the action of T on these vectors will give you 0 because they are null space, look at these other vectors and probably they should be a bases. In fact what we will show is that you look at T v 1, T v 2, etc T V L, we will show that this is the bases of range of T okay, suppose we show that this is the bases of a range of T then it follows that there are L vectors here then it follows the rank of T as L and so this equation holds okay so that is what we will do. We will show that these vectors are linearly independent and they span the range of T, let us first dispose of this spanning thing. We must show that this is the bases so we must show that it is linearly independent and the spanning set.

So let us take y in the range of T and then show that y is a linear combination of these vectors okay, y is in range of T by definition then there exists x in v such that y is equal to T x there exist x in V, now this is the bases for v and so I can write this y as T of x that is T of this is this x in V it is a linear combination of these vectors so I have something like Alpha 1 u 1 + Alpha 2 u 2, et cetera + Alpha Eta u Eta + let us say Beta 1 v 1, + Beta 2 v 2 + etc + Beta L v L that is my x linear combination of these vectors this is the bases.

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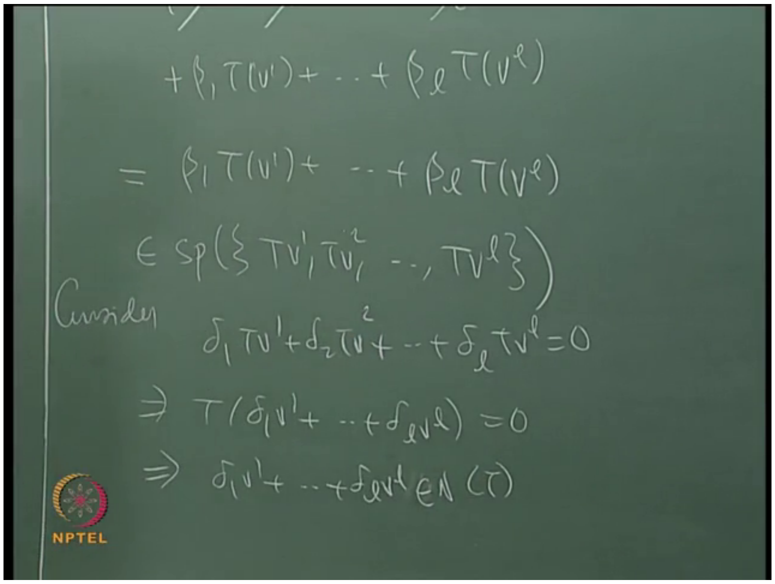

T is linear so this can be rewritten as Alpha 1 T of u 1 + Alpha 2 T u 2 etc Alpha Eta T of u Eta + Beta 1 T v 1 et cetera Beta L TV L okay, but observe that u 1, u 2, etc, u Eta they are in null space of T they are in fact linearly independent they are in null space of T so these terms are 0 so this is just Beta 1 T of v 1 et cetera, Beta L T V L that is this belongs to span of the vectors T v 1, T v 2, et cetera, T V L linear combination of these vectors because these are 0. And so for one thing this is a spanning set, T v 1, T v 2, etc, T V L is a spanning set it spans the range of T.

Next linear independence so let us consider a linear let us consider a linear combination of T v 1, T v 2, et cetera, T V L, equate that to 0 so that each coefficient is 0 so let us consider what shall I say, Delta 1 T v 1 + Delta 2 T v 2 + etc + Delta L T V L suppose this is 0, I must show that each of these scalars is 0, this use the fact T is linear so this means T of Delta 1 v 1, etc Delta L V L is 0, this means this vector Delta 1 v 1, etc Delta L V L that belongs to null space of T, null space of T is spanned by u 1, u 2, et cetera and so that is a linear combination of... yeah so this is a linear combination of those... T of L this vector is 0, x belongs to null space T if T of x is 0 so this vector must be in the null space of T that is what I written. Null space of T has u 1, u 2, et cetera u Eta as a bases so let me just go to that line.

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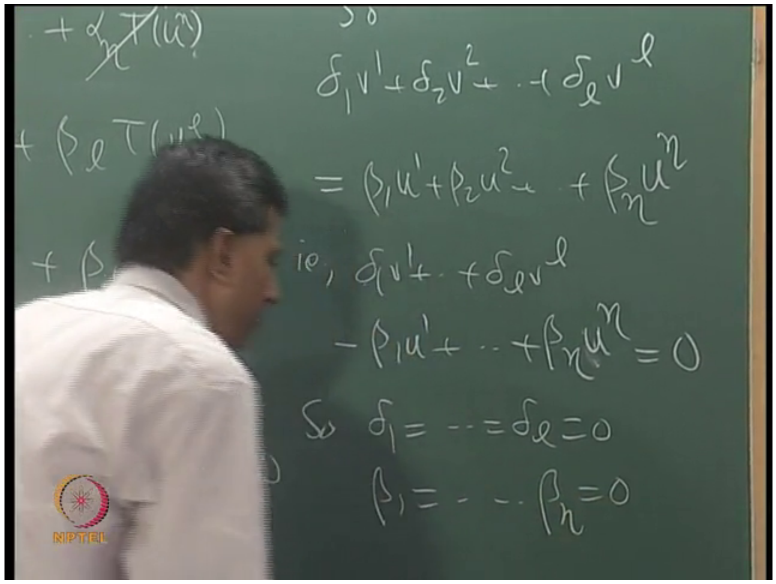

So Delta 1 v 1 + Delta 2 v 2 + etc + Delta L v L must be a linear combination of u 1, u 2, et cetera u Eta because null space of T is spanned by these vectors. So let us say I have Beta 1 u 1 + Beta 2 u 2, et cetera, Beta Eta u Eta, is that okay? Bring this to the left-hand side that is Delta 1 v 1 et cetera + Delta L v L – Beta 1 u 1 + et cetera Beta Eta v u Eta, this is 0 but u 1, u 2, et cetera u Eta, v 1, v 2, etc v L they are linearly independent so this means each of these scalars Delta 1 etc equals Delta L is 0, Beta 1 et cetera Beta Eta is 0. Go back and see what you have here, Delta 1 et cetera Delta L they are 0 so this is a 0 vector I am sorry you have the proof right away, Delta 1 etc Delta L equal to 0 I started with this combination. I started with Delta 1 T v 1+ Delta 2 T v 2 et cetera Delta L T VL equal to 0 I have shown that these scalars are 0 so T v 1, T v 2, et cetera they are linearly independent so T v 1 etc TV L this is a linearly independent subset hence the theorem.

Our claim is that L is a dimension of range of T, 1^{st} we have shown that these vectors T v 1, T v 2, et cetera they form a spanning set then we have shown that they are linearly independent that is Rank–nullity theorem okay, let us look at some consequences.

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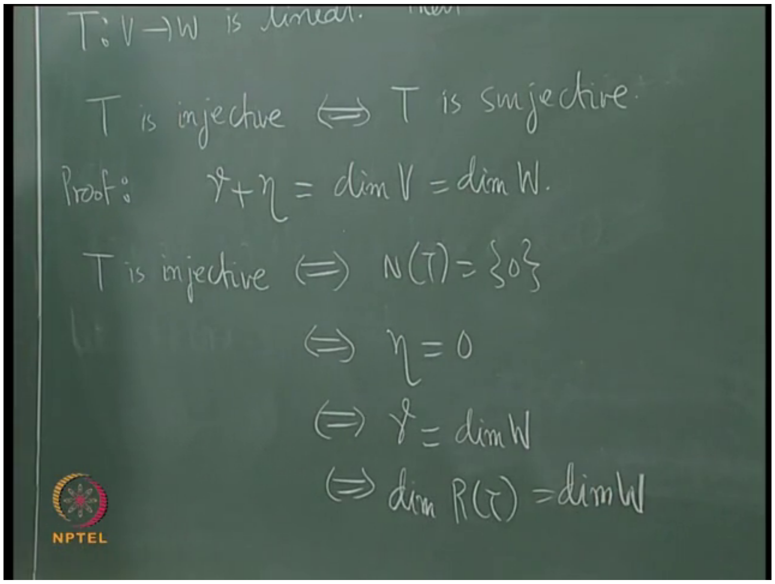

One of the consequences of the Rank-nullity dimensions theorem is the following; I will state this is as 1st corollary; V is finite dimensional in fact I will take this time dimensions W equals dimensions V, T from V to W is linear then we have the following. T is Injective if and only if T is surjective, if the dimensions of the domain and the core domain vector spaces are the same then a linear map is Injective if and only if it is surjective that is now you go back to the last example of the last lecture. The last example of the last lecture from R 2 to R 2, T of x = x 1 + x 2, x 1 – x 2, we verified that it is 1 to 1 as well as onto, if we had known this theorem it would have been enough to verify only one of those. Dimension of core domain and the dimensions of the domain there are the same so it is enough to verify one of those.

Okay, how do we prove this? This is the consequence of the Rank-nullity dimension theorem, rank of T + nullity of T is the dimension of the domain space which is also the dimension of the core domain space okay. Suppose T is Injective we had shown last time that the null space of T is single term 0 in fact they are equal, T is Injective if and only if null space of T is single term 0. Null space of T is single term 0 if and only if nullity of T is 0 because we had defined the dimension of 0 space to be 0 so null space of T is single term 0 if and only if nullity that is Eta is 0. Eta is 0 if and only if Gamma is dimension W but what is Gamma? Gamma is the dimension of the range of T that is this happens if and only if dimensions range of T is dimensions W.

Range of T is a subspace of W, if it has the same dimension as W then it must be equal to W so this happens only if and only if range of T equals W, this is same as saying that T is surjective okay so T is Injective if and only if T is surjective okay, this is one of the consequences. Suppose I have relationship... Dimensions we dimensions look at the finite dimensional case when both V and W are finite dimensional, these 2 are integers one can compare these 2 integers, I will state this then as next corollary.

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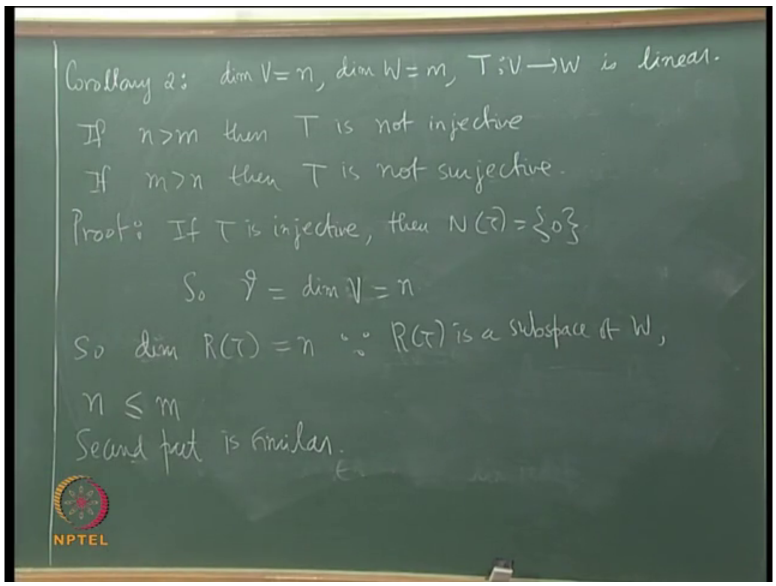

Again V and W are finite dimensions, this time as you dimension V to be n, dimension W to be m, T from v into W is linear then we have the following. If n is greater than m, this equality case has been discussed earlier let us discuss the case when one of them is greater than the other, each is greater than the other in 2 cases; n greater than m is one case, m greater than n is the other case. If n is greater than m then the conclusion is T is not injective if n is greater than m then T cannot be Injective. If m is greater than n then T cannot be surjective okay that will completely exhaust the 3 cases when V and W are both finite dimensions okay proof by contradiction okay.

If T is Injective 1^{st} part if T is Injective then we know that null space of T is single term 0 and so nullity is 0 and so the Rank + Nullity dimension theorem says that rank is equal to dimensions of W that is dimensions V, which is n dimension V is n but Gamma is dimensions of the range space. What this means is that range space of T is dimensions of the range space of T is equal to n but then range space is subspace of W so this n cannot exceed m. Since range of T is a subspace of W and dimension W is m, this n cannot exceed m, m is a dimensions of W, n is a dimensions of V.

1st we have used dimensions v is n so Gamma is n, Gamma is the dimension of the range space, range space being a subspace of a finite dimensional vector space cannot exceed the dimension of that subspace so this n cannot exceed m so that is a contradiction that is if T is Injective then n is less than or equal to m means if n is strictly greater than m then T cannot be Injective that is the 1st statement, the 2nd one is similar I am going to leave that as an exercise 2nd part is similar 2nd part is similar again prove by contradiction. Okay so these are 2 important consequences of the fundamental result on the dimensions of the subspaces; null space and the range space of a finite dimensional vector space.

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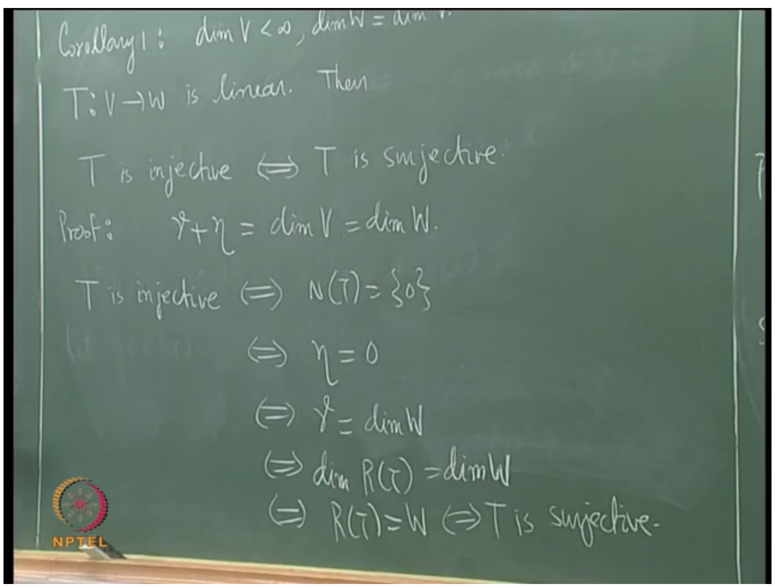

Let us look at linear transformations that are both Injective and surjective a little closer. If a function as Injective and surjective then it is invertible if function is both 1 1 and onto then it is by bijective it is invertible and the inverse function is also 1 1 and onto okay. If F is bijective then F inverse exists and F inverse is bijective okay. The question that we would like to ask here is T is Injective and surjective linear then I know T inverse is Injective and surjective. The question is, is T inverse linear? The answer is yes so let us prove that let us first prove that the inverse of a linear transformation is also a linear transformation and then look at some consequences for vector spaces if there exists an invertible linear transformation between them.

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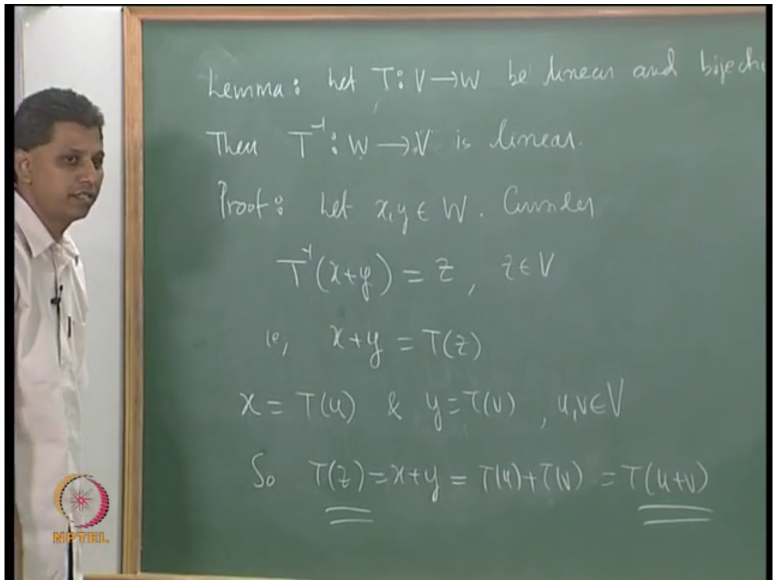

So let me 1st prove this result and state this as a lambda, let T from V into W be linear and bijective by which I mean Injective and surjective, 1 to 1 and onto then we know that T inverse exists as a function and we know that T inverse must be a function from W to V the claim is that this is linear, T inverse is linear, so let us 1st prove this and then look at some of the consequences of this result. Okay I want to show T inverse is linear I have to verify these 2 equations that T inverse is additive and then T inverse Alpha x is Alpha T inverse x. So proof let us start with 2 vectors x, y in W this time and consider T inverse of x + y I must show that this is equal to T inverse x + T inverse y for one thing, let us call this as vector Z this vector Z belongs to V and I remember that x and y come from W.

This means T is invertible so T inverse is invertible, T inverse is T so I am operating by T, T of T inverse x + y is T of Z but T of T inverse is composition, T circle T inverse is identity transformation so x + y equals T Z x . Now look at x and y, they come from W, x and y come from W and I know that T from v to W is bijective so each of these must have a pre-image. x is equal to T of u and y equals T of v for u, v in capital V in fact this must be unique you can verify that, if T is bijective then the pre-images must also be unique. Okay, in any case we have u and v from V such that x is T u, y is T v go back substitute use injectivity of T that is let me now write T Z first, T Z is x + y, x is T u, y is T v so this is T of u + T of v, T is linear, T of u + v so I have T Z equals T of u + v, T is Injective, T x equals T y implies x equals y so Z equals u + v.

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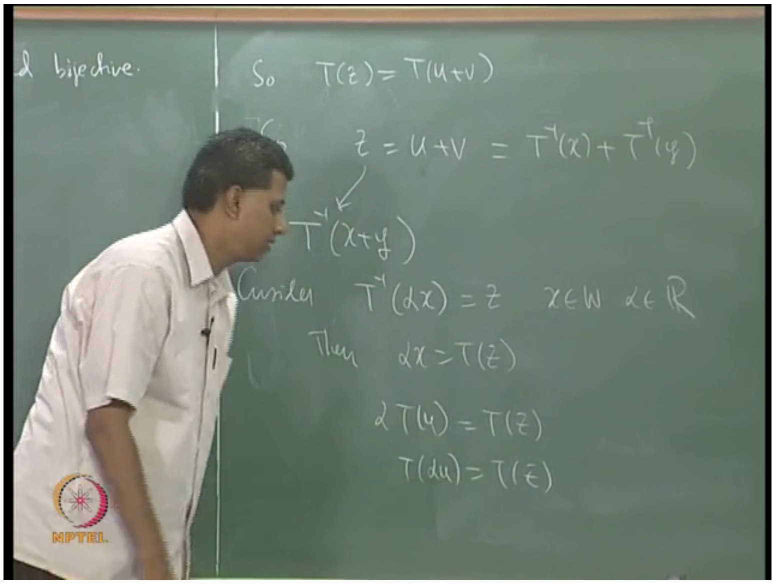

T u is x, T v is y so this is T inverse x + T inverse y but that is on the other hand is T inverse x + y so that is equal to T inverse x + T inverse T so T inverse is additive. The 2nd part is similar let me go through that quickly; consider again T inverse of Alpha x I will call that Z then Alpha x is T of Z, x is in W and so you can write this is Alpha T of u I am using the same u Alpha T of u equals T of Z, T is linear so this goes in. T of Alpha uu equals T of Z, T is Injective, Alpha u equals Z.

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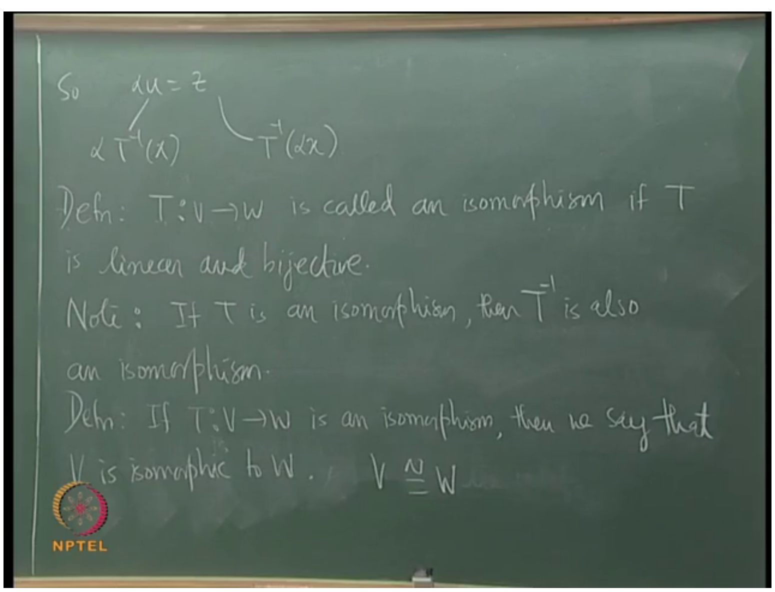

On the one hand Alpha u is Alpha T inverse x, on the other hand Z is the inverse Alpha x so T inverse is linear if you take this, if T inverse exists and if T is linear then then T inverse is linear no dimensions here this is true for any linear transformation but in any 2 vectors spaces. Okay now let us look at what an invertible linear transformation does on finite dimensional vector spaces. So we need the following definition uhh T from V to W is called an isomorphism if T is linear and bijective. Isomorphism means same structure morphism is structure same structure it preserves the structure, it is called isomorphism that is a special name for a linear transformation which is also bijective. What we have seen just now is that if T is an isomorphism then T inverse is also an isomorphism okay.

Note we have just shown that T inverse is linear, T inverse is bijective is known anyway so if T is an isomorphism then T inverse is also an isomorphism. What does an isomorphism do to finite dimensional vectors spaces? Before that I know the notion isomorphic vector spaces. If T from V to W is an isomorphism then we say that V is isomorphic to W, if there is an isomorphism from V into W then we say that V is isomorphic to W and then use the following notation to denote that there is an isomorphism from V into W, we will use this notation on the left I have V on the right I have W this is the symbol for isomorphism, V is isomorphic to W. If V is isomorphic to W can I conclude that W is isomorphic to V? Why?

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If T is an isomorphism then T inverse is an isomorphism from W to V and so now we can say that V and W are isomorphic, it is not just V is isomorphic to W we can in this case say V and W are isomorphic. Also if V is isomorphic to W, W is isomorphic to Z can I conclude V is isomorphic to Z? yes because composition of bijective maps is bijective, composition of linear maps is linear I have not proved this before by tips not difficult to prove. Composition of linear maps is linear and inverse of the composition there is a reverse order law similar to the matrix inverse T 1 circle T 2 inverse is T 2 inverse Circle T 1 okay and so and what is the trivial isomorphism from vector space to itself? Identity.

Identity is linear bijective, inverse is linear bijective, so V is isomorphic to itself, V is isomorphic to W implies W is isomorphic to V. V isomorphic to W, W isomorphic to Z implies V isomorphic to Z so this is an equivalence relation. This partitions the set of all vector spaces into equivalence classes, the classes are the property that if you take 2 vectors spaces from 2 different classes they cannot be isomorphic, if you take 2 vectors spaces that are isomorphic they belong to the same class okay. What this also does an isomorphism also does is to split finite dimensional vector spaces according to their dimensions that is the important thing. Isomorphic classes corresponds to precisely the dimensions of the vector space that is if 2 vector spaces are not isomorphic I know they are finite dimensional, they cannot be of the same dimension.

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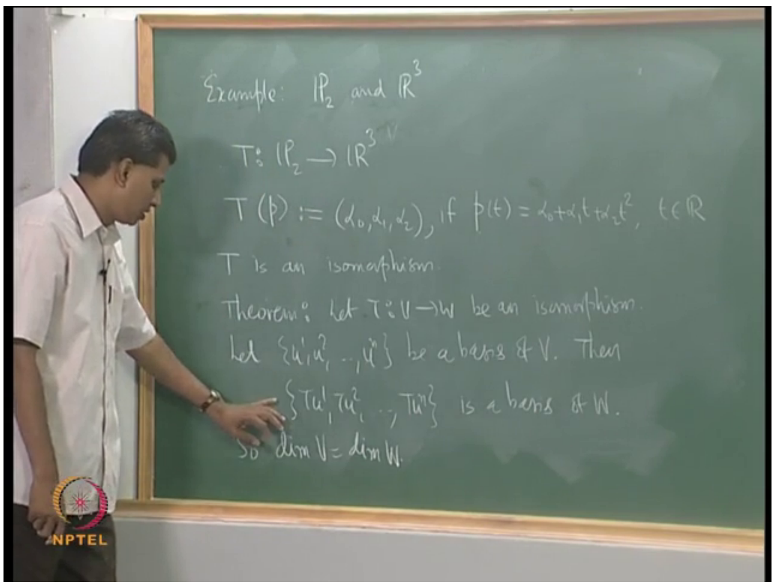

On the other hand if 2 vector spaces have the same dimension then they belong to the same isomorphic class that is there is an isomorphism between them okay this is what we will prove in a short while from now, so let me 1st give an example of an isomorphism and then proceed okay. Now you will notice that after me writing down the isomorphism that it is really trivial to have written this down so I want to look at this example. Let us consider P 2 and R 3, P 2 is V, R 3 is W, P 2 is a space of all polynomials with real coefficients degree less than or equal to 2, dimension of P 2 is 3, R 3 is of dimension 3, I want to give an isomorphism between them. Let me define T from P 2 to R 3 by T of a polynomial p, I must have on the right-hand side a vector with 3 coordinates on the right-hand side I must have a vector with 3 coordinates, can you give a natural vector on the right? Just take the coefficients.

What I know is that p is a polynomial so p of T is Alpha 0 times 1 + Alpha 1 times T + Alpha 2 times T square, p is a real variable, Alpha 0, Alpha 1, Alpha 2 are from R again this is a real polynomial. Define T of p to be 3-dimensional vector Alpha 0, Alpha 1, Alpha 2 then probably I am going to leave this as an exercise; T is linear, T is 1 1 it is enough then T is onto because the dimensions are the same so this is an isomorphism, T inverse is an isomorphism from R 3 to P 2 but maybe we can take that as an exercise again what is the inverse of this transformation okay. Let me just state that T is an isomorphism, what I have exhibited here this is an example of an isomorphism, T is an isomorphism.

Okay you see that this is almost natural, how to associate in isomorphism between vector spaces of the same dimension okay, we will try to imitate this in a general case but before that I want to proof this result. I have T from V into W and isomorphism, T from V to W be an isomorphism. I am assuming that V is finite dimensional that is like u 1, u 2, et cetera u n be a bases of V, so I am assuming that V is finite dimensional, I have exhibited a bases given that T is an isomorphism what can be shown is that so what is your guess? From this can I get a bases for W? Look at T u 1, T u 2, etc, T u n, this is a bases of the vector space W, as a consequence dimensions of V is equal to dimensional W okay, this is what I said that if you have an isomorphism between finite dimensional vector spaces then the vector spaces must have the same dimension.

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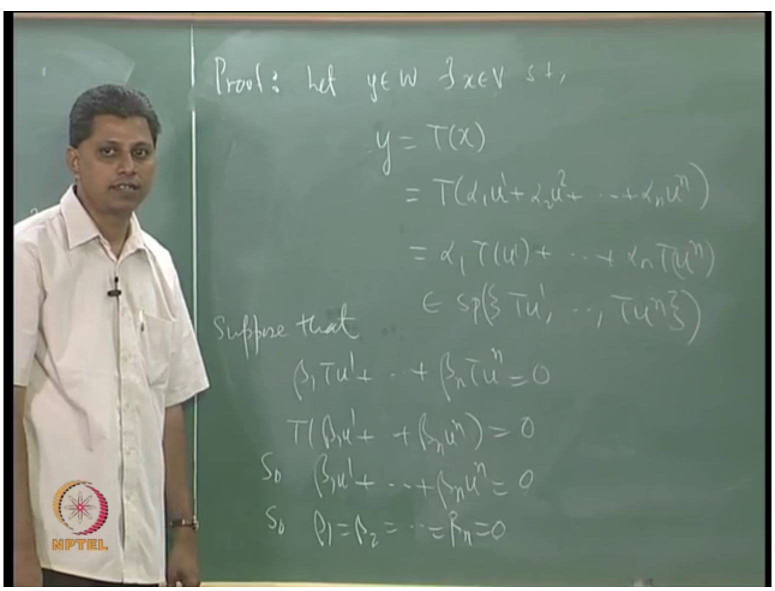

We will prove the converse also, if 2 vector spaces have the same dimension then there is an isomorphism very similar to this particular example, let me 1st proof this result, I want to show that this is the bases of W spanning set linear and (())(38:35). Let y belong to W, T is an isomorphism so T is bijective, T is surjective, there exist x in V such that y is T of x, x is in V this is a bases of V. I can write this is T of Alpha 1 u1 + Alpha 2 u 2 et cetera + Alpha n u n, T is linear Alpha 1 T u 1 etc, Alpha n T u n. I have written this is a linear combination of T u 1, et cetera; T u 1, T u 2, etc, T u 1 is a spanning set okay nothing much in this just that T is onto that is what we have used. For linear impedance we will prove we will use injectivity of T is linear independence.

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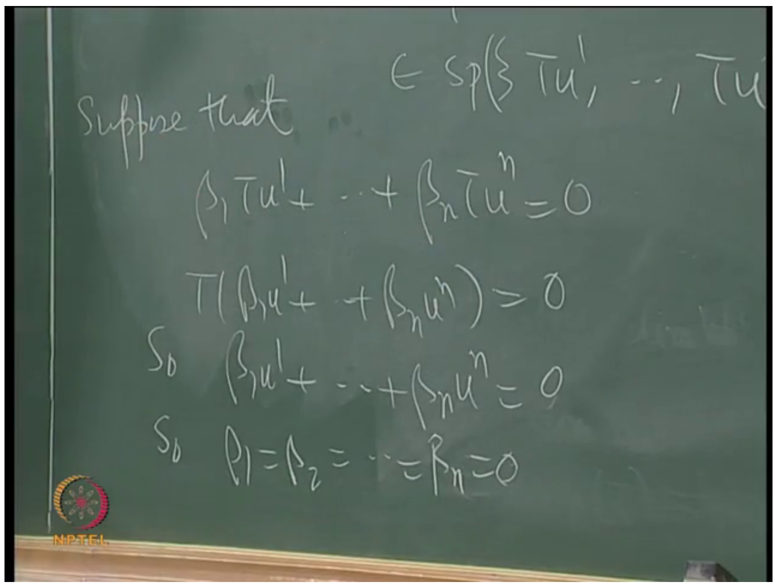

Suppose that suppose that a linear combination of these vectors is 0, let us take Beta 1 T u 1, etc Beta n T u n to be 0. This means T of Beta 1 u 1 etc + Beta n u n equal to 0, this means this vector inside Beta 1 u 1 etc Beta n u n belongs to null space of T but T is Injective, null space of T is single term 0 so this must be the 0 vector. Beta 1 u 1, etc + Beta n u n is the 0 vector but remember that u 1, u 2, etc, u n they form a bases for V so these vectors are linearly independent so Beta 1 , Beta 2, et cetera they must be 0 each of these scalars is equal to 0. So remember I started with Beta 1 T u 1 etc Beta n T u n I have shown each of the scalars is 0 so it follows that T u 1 etc T u n is linearly independent.

We have shown already it is a base it is a spanning set so it is a bases, it is a bases of W and they are n in number, the number of elements in this bases is n so it follows that dimensions W is n that is same as dimension of V. In the next lecture I will prove the converse and also I will consider other examples.