Linear Algebra

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Lecture no 15
Module no 04

The Null Space and the Range Space of a Linear Transformation

Let us continue our discussion on Linear Transformation, we discussed 1 property the other day us look at some more properties, some examples then the notion of null space of linear transmission range space of a linear transmission, some examples where we will calculate these subspaces and then probably today the Rank nullity theorem okay. Let me recall the result that we discussed in the last lecture.

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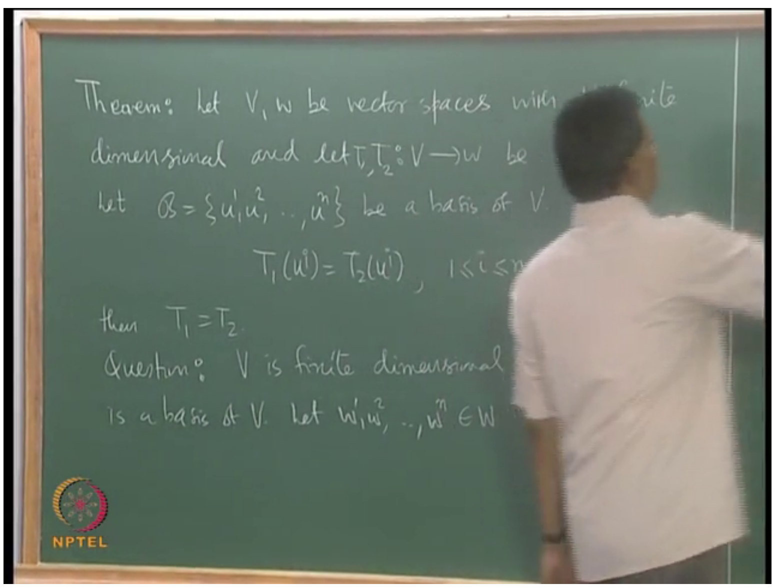

What essentially proved I will rewrite it, Vector V, w be vector spaces over the same field with V finite dimension and let T from V into w be a linear transformation simply call it linear T from V to w is linear. The 1st result that I discussed is about uniqueness so let us say there are 2 vectors to transformation T1, T2 okay linear maps. Let script B = u 1, u 2, et cetera u n, I assume that this is the bases script B be a bases of V, V is finite dimension so there is a finite bases for V. If T1 of u i = T2 of u i for all the i lying between 1 and 10 then we have shown last time that T1 and T2 are the same maps okay this was the 1st property, this theorem was proved in the last lecture.

Let us look at certain other results this question was asked last time again, V is finite dimensional V is finite dimensional, let us say B as above is a bases of V, let w1, w2, etc, w n belong to the W, W need not be finite dimensional so there is no condition on these w, they can be even the same vectors they can all be the 0 vectors.

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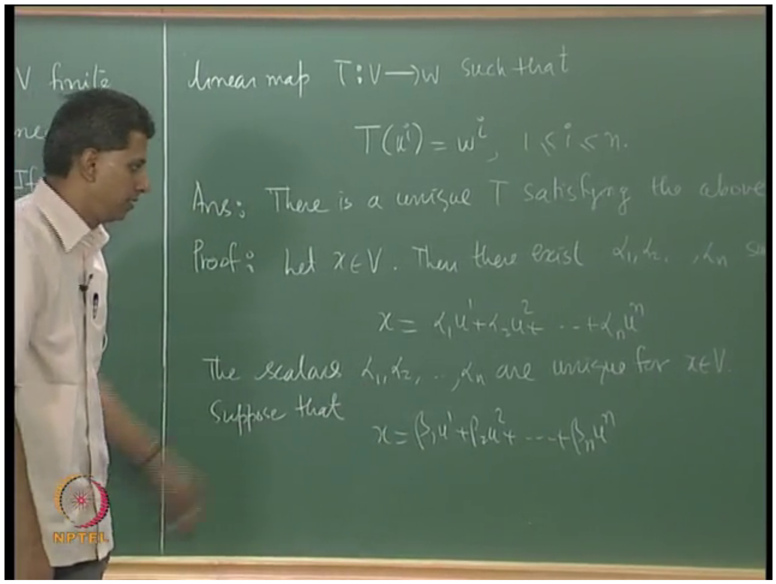

A question that we addressed last time was, is there a linear map a linear map T from V into W such that such that T of u i equals w i discussion was after the last time, let us answer in the affirmative okay we will prove that given any any bases B given any vectors and number the same as this, they can repeat given any set of n vectors there is a unique linear transformation that satisfy this condition okay, so the answer is there is a unique T satisfying the above there is a unique T satisfying the above that is what we will show okay so this is another theorem so let us prove this theorem.

Let us take a general x vector x and V, V has script B as the bases these are the vectors so I have a linear combination, then there exists scalars Alpha 1, Alpha 2, et cetera Alpha n such that such that the x can be written as Alpha 1 u 1 + Alpha 2 u to + etc + Alpha n u n this comes from the definition of a base. A base is just spanning set so any vector A and B it is a linear combination of the bases vectors, but what is also important to observe is that these numbers are unique for this vector x. The scalars Alpha 1, Alpha 2, et cetera, Alpha n are unique for the x that we started with, the scalars Alpha 1, Alpha 2, etc, Alpha n are unique for the vector x that we started with.

What is the meaning of this statement? If there exists beta 1, beta 2, et cetera such that x is beta 1 u 1 + etc + beta n u n then Alpha i equals beta i for all i okay.

Let us prove that again quickly again, so I am going to demonstrate that the statement is true that is quick. Suppose that x is also beta 1 u 1 + beta 2 u 2 et cetera + beta n u n so I have another if possible let there be another representation for x in terms of the bases vector u 1 u 2 etc u n.

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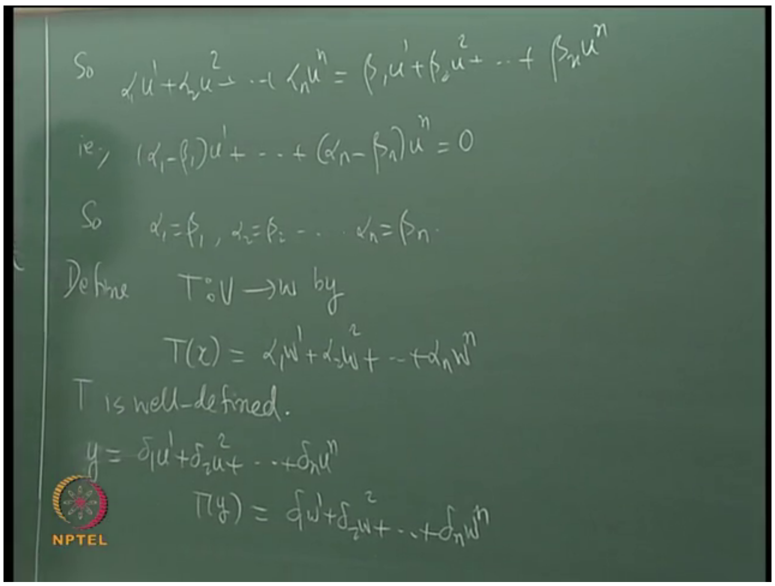

Then make use of these 2 equations to write Alpha 1 u 1 + Alpha 2 u 2 et cetera + Alpha n u n = beta 1 u 1 + beta 2 u 2 et cetera + beta n u n. I must have this which I can rewrite this is same as saying Alpha 1 – Beta 1 u 1 + etc Alpha n – Beta n u n = 0 but invoke the factor u 1, u 2, etc u n are linearly independent, this means Alpha 1 = Beta 1, Alpha 2 equals beta 2, et cetera Alpha n equals beta n, so the presentation of any vector in terms of a bases that representation must be unique. I fixed the bases, for a fixed bases there may be several other bases, for another bases I have another representation that is a different matter, for this bases there is a unique representation. Okay, what is the need for proving this uniqueness? We will use this uniqueness to define a mapping thing.

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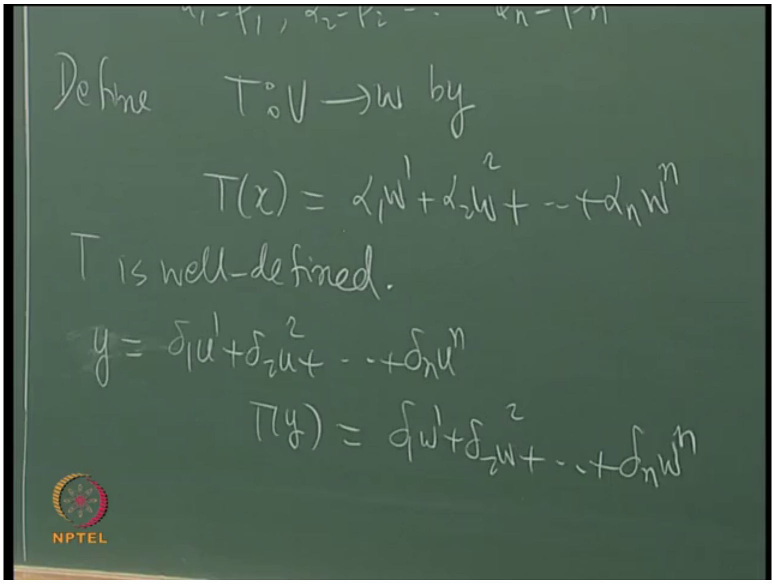

Define T from V into W by T of x to be... I use these scalars Alpha 1, Alpha 2, et cetera for this x and then use vectors W1, W2, etc do T of x will be for me Alpha 1 W 1 + Alpha 2 W 2 etc + Alpha n W n where remember these Alpha are chosen from the representation for x in terms of the bases that we started with that is unique so this is a well-defined map. The reason why we proved uniqueness is to show that this is a well-defined map, well-definedness here means if x = y then F x = F y, if x = y then T x = T y, I am going to leave that as an exercise, then T is well-defined that is an exercise. What is the meaning of this?

The image for x cannot be 2 different elements that is well-defined, T of x cannot be to defend vectors, the well-definedness come from the unique representation for a vector in terms of a bases. Okay so t is well-defined I am going to leave as an exercise, we need to verify that this T is linear and that T satisfies the required equations, the required equations are these. We show that T is linear and that T satisfies these equations okay. Okay is linear 1st let us let us take a representation, x I have taken as before let me take a representation for y, y is let us say Delta 1 u 1+ Delta 2 u 2 + etc Delta n u n, I am taking y as another vector I want to show T x + y is T x + T y then if this is my y then the definition of T y as above will be Delta 1 W 1 + Delta 2 W 2, et cetera + Delta n W n okay.

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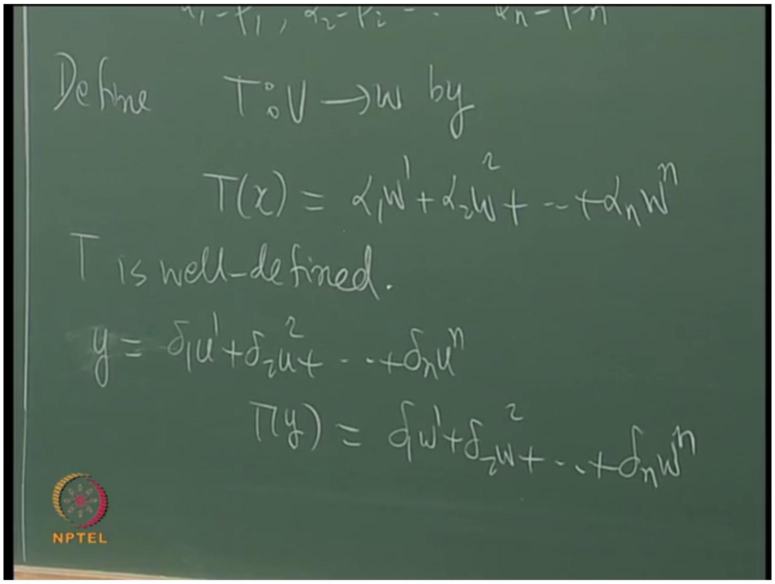

Remember you must go back whenever you want to write T of something you must go back to the representation of y in terms of the bases vectors and then use the linear combination of these scalars along with W n etc W n okay, we must show that T of x + y is T x + T y. So consider x + y first, x + y this happens in the vector space linear combination I can add the coefficients Alpha 1 + Delta 1 u 1 etc Alpha n + Delta n u n this is x + y, this representation is unique and so T of x + y I can now write down that is this scaler Alpha 1 + Delta 1 into W1 + et cetera Alpha n + Delta n W n.

This happens in the vector space W, Alpha 1 W 1 + Alpha 2 W 2 et cetera Alpha n W n collecting the 1st term from each parenthesis + correct the second term Delta 1 W 1 Delta 2 W 2 et cetera Delta n W n and then go back and see that this is precisely T x, this is T y so this is T of x + T of y so we have shown that the T is added to... T of x + y is T x + T y. T of Alpha x, Alpha is a fixed scalar again I must know the presentation for Alpha x, the representation for Alpha x will be Alpha Alpha 1u 1 + Alpha Alpha 2 u 2 et cetera + Alpha Alpha n u n and so T of Alpha x will be Alpha Alpha 1 W 1 + Alpha Alpha 2 W 2 et cetera Alpha Alpha n W n this comes from the unique representation of Alpha x.

This is Alpha Times Alpha 1 W 1 + Alpha 2 W 2 et cetera Alpha n W n but that is precisely T x so T of Alpha x is Alpha T x from Alpha and so T is linear okay. So T is linear, does it satisfy these n equations, I must verify that T satisfies these equations okay.

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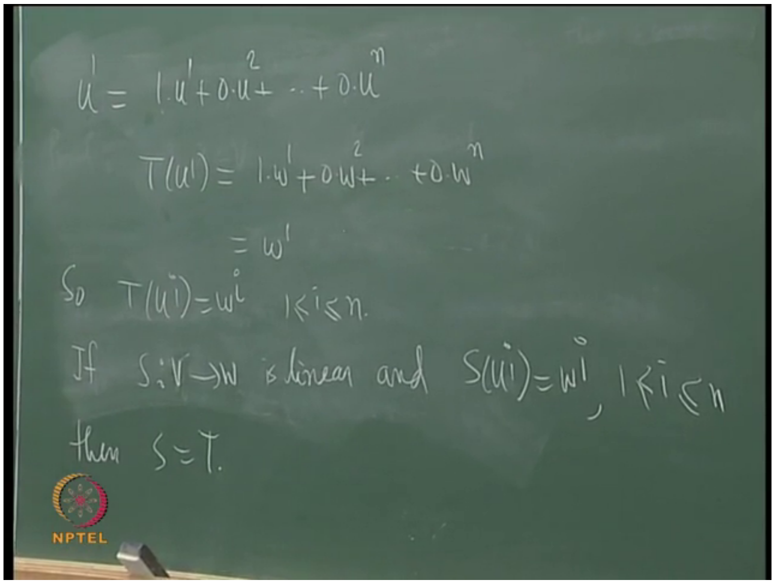

Okay but look u 1 for example say I want to know T of u 1, T of u 2, et cetera T of u n but in order to know u 1 I must know the unique representation of u 1 in terms of the bases vectors, but what is the unique representation for u 1? 1 times u 1 + 0 u 2 et cetera + 0 uN, for one thing this is a representation for another I know it is unique. Now by my definition T of u 1 is this scalar into W1 + this scalar into W 2 et cetera this scalar into W n so this scaler into W1 + 0 W 2 0 W n that is this is just W 1. So I have T of u 1 equals W 1 being satisfied, T of u 1 equals W1 the 1st equation has been satisfied but what I have done for u 1 can be done for any other vector u 2, u3, et cetera.

For u 2 the representation is 0 times u 1 + 1 times u 2, et cetera so use that to conclude that this equation T of u i equals W i these n equations are also satisfied okay. The fact that this T is unique is similar to the previous theorem proof of the previous theorem, the fact that T is unique is similar to the proof of the previous theorem that is you want to show T is unique, take ... I will not prove it I will just give us sketch, take another linear transformation I will call that S, if S from T into W is linear and S of u i equals W i, see the claim is there is a unique linear transformation T that satisfies these equations. Suppose there is another transformation S such that S of u i satisfying these n equations S of u i equals w i... yes S is from V to W satisfying these equations then it is easy to see that S = T okay, so this last part is similar to the proof of the last part of the previous theorem.

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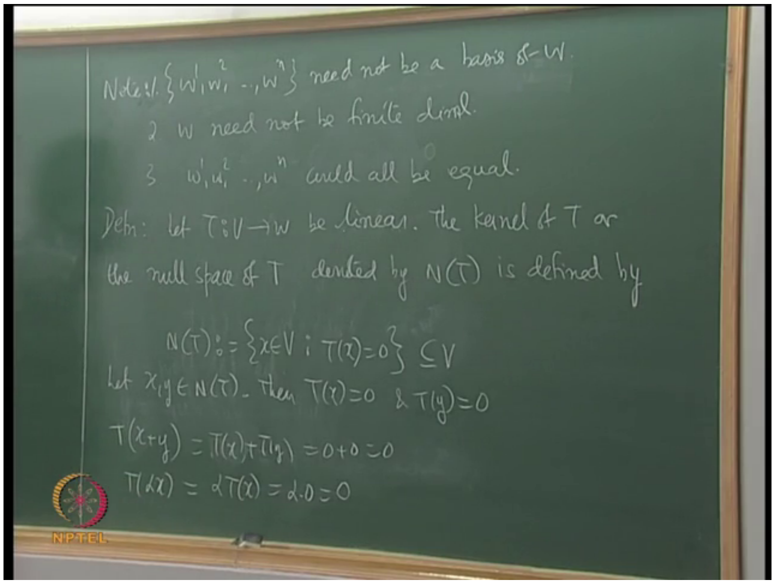

I am going to skip this okay so let me emphasize what I said earlier when I wrote down this Theorem. Note W1, W2, et cetera, W n, this need not form a bases of W 1st observation these need not be a bases of W in fact, W need not be finite dimensional W need not be finite dimensional in fact I can say that W1, W2, W n could all be equal could all be equal and the worst case could be 0 also. Okay, there is absolutely no condition on W n etc W n that we need in this Theorem... just arbitrary vectors. Okay so these are some of the elementary properties 2 elementary properties really in connection with existence uniqueness okay. Let us look at certain subspaces associated with linear transformation.

Let me 1st define the null space of a linear transformation, I have T from V into W as a linear map, the kernel of T or the null space of T, kernel word comes from group theory for instance, null space is typical vector space notion, null space of T I will call it I use this notation N of T denoted by this is defined as follows; the kernel of T or the null space of T notation is N of T, what is the definition? N of T is the set of all x and V such that T of x equals 0 set of all x and V such that the of x equals 0. Since this is this x is in V this is a subset of V okay so let us observe that this is contained in V so this is a subset of V, now this can be shown to be a subspace of V okay let us I will prove that quickly this null space is a subspace of V.

Let us take we want to show that a subset is a subspace then show that it is close with respect to addition and scalar multiplication so let us take 2 vectors, let x, y belong to null space of T then T of x = 0 and T of y is 0, I must show that T + y belongs to null space of T so start with T of x + y, I must show that T of x + y is 0 but T is linear so T of x + y is T of x + R of y that is 0 + 0 that is 0 and so x + y belongs to null space of T. This is with respect to addition, with respect to scaler much easier; T of Alpha x is Alpha T of x that is Alpha into 0 that is 0, so we have shown that x + y belongs to null space of T, Alpha x belongs to null space of T so this null space is in fact a subspace that is why it is called a null space. So null space is a subspace of V, there is another important subspace associated with linear transformation T.

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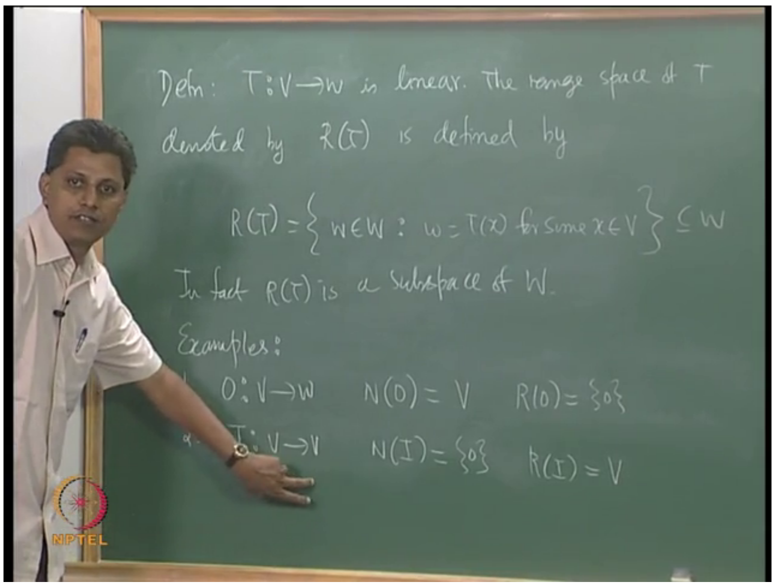

Some information about linear transformation T can be derived by looking at the null space okay. As I mentioned there is another subspace associated with linear transformation T which also has some information about T that is called the range space of T, T is linear, the range of T or the range space of T; I will use R of T for that denoted by R of T is defined by R of T is the set of all W in W such that W equals T of x for some x into V that is I collect all those vectors W and W now, this is a subset of W this is a subset of W, what is the property of subset? This subset has a property that for every element for every vector of this subset there is at least one pre-image in V, for every W in range of T there exist some x and V such that W equals T of x, every vector in W has a pre-image corresponding to T this is called the range space any of those that this is the subspace of W the other vector space.

Now I am going to leave this as an exercise for you to prove that range of T is a subspace, range of T is a subspace of W this is an exercise for you. Let us now look at some examples and then determine the range space, null space to consolidate these notions, let us dispose off the trivial cases, we want to look at examples that dispose of the trivial case the 1st one is the 0 map 0 from V to W the 0 map, what is the null space of the 0 map, null space of 0 is the whole of V that is because 0 operating on x, 0 for all x and V so null space of 0 is whole of V, what is the range of 0? Range of 0 is single term 0 this 0 coming from the vector space W, range of 0 single term 0.

Example 2; identity linear transformation V to itself identity linear transformation on V, what is the null space of I? Single term 0, what is the range of I? It works complimentary to the 0 operator, range of I is V okay this is like complimentary of 0 operator. Identity from V to V, 4 identity mapping W must be equal to V so range of I is V okay so these are the trivial examples let us look at other examples.

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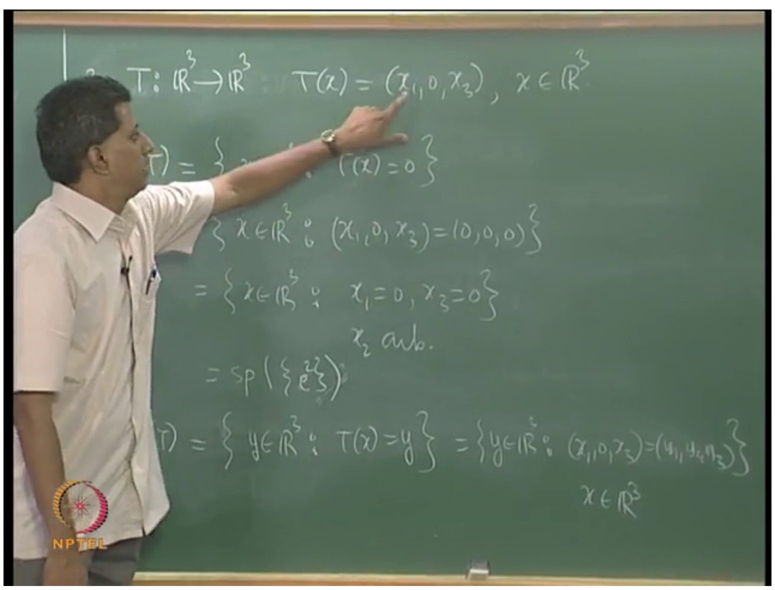

Example 3; let us look at the projections T from let us say R 3 to R 3 defined by T of x will equals x1, 0, x 3, projection onto the so-called x-Z plane, the 2nd coordinate is 0 we have verified that it is a linear transformation, what is the null space of T? Null space of T is a set of all x in R 3 such that T of x is 0, this is a set of all x such that T of x equals to 0, x1, 0, x 3 this must be the three-dimensional zero vector, what is the condition that these equations impose on the unknowns x1, x2, x3 that is what we must observe, this is the set of all x and R3 such that x1 is

0, x3 is 0, it doesn't impose any condition on x2 so it is x1 equal to 0, x equals zero, x2 is arbitrary let me just emphasise x2 is arbitrary.

Can you also give a bases for null space of T now? Do you agree that this is span of E1, E 3 sorry just E 2, span of E2, any multiple of E2 must be in null space of T and anything in null space T is a multiple of E2 because first and third coordinates are 0 okay that is null space of T what about range of T? Range of T is the set of all this time I will use y and R3 such that T of x equals y, set of all y in R3 such that the of x equals to y, x belongs to R3 for some x in R3. Let me write on this side, this is the set of all y in R3 such that x 1,0, x3 equals y1, y2, y3, some x in R3, I have just used the definition of T of x. T of x is x1, 0, x3 equals y1, y2, y3. Now you see that this really imposes a condition on y2; y1 and y3 arbitrary so y2 is 0 that is the only condition okay so can I write okay.

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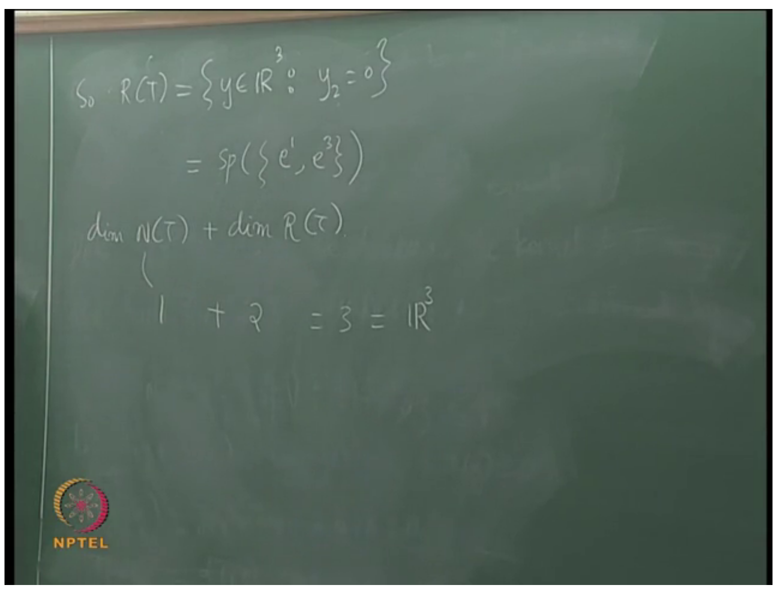

So range of T for this example is the set of all y in R3 such that y2 is 0. Okay now can you give a bases similar to what we did for null space, can you give bases for range of T? Span of y2 is 0 so look at e 1 and e 3, both these vectors y2 is 0 e 1 and e 3 and you can verify that these two vectors satisfy this condition. Now in this example you observe that the dimensions of null space of T + dimensions of the range of T, the dimension of null space of T null space is one dimension that is 1, dimension of a range of R is two-dimensional 2, this is equal to 3 that is the dimension of the domain space of T, T is from R3 to R3. The domain space is three-dimensional forget about this, the domain space is three-dimensional we observe that dimension of null space + dimension of a range of T is the dimension of the domain space, this is part of a general result the Rank–nullity theorem.

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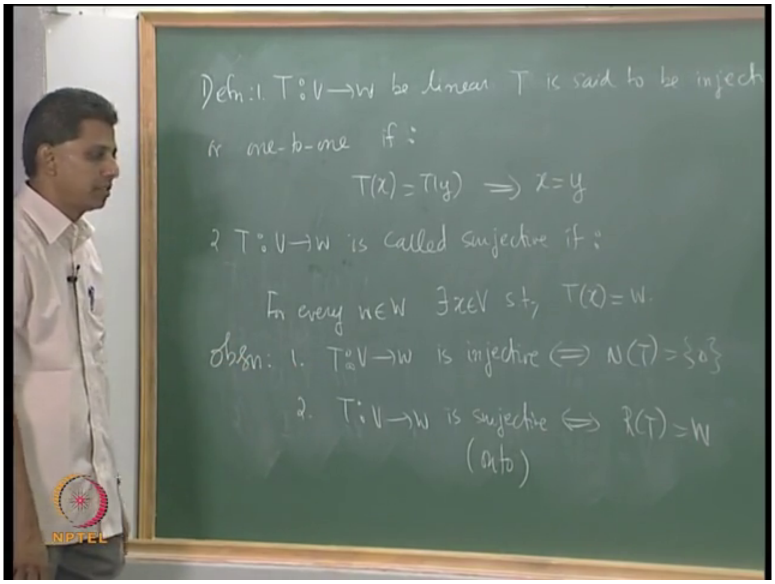

Okay let us look at two more examples, before I discuss these two numerical examples let me also give you the notion of injecting and surjective linear transformation. Injective and surjective linear transformation let me give the definition, a linear map T from V to W, T said to be interactive or one-to-one if as a function it is injected that is if T of x equals T of y implies x is equal to y, distinct elements have distinct images, x 0 = y implies T x0 equal to T y that is same as saying Tx equal to Ty implies x is equal to y, distinct elements have distinct images that is injectivity the notion of injectivity of a linear transformation. Subjectivity I will call this first part, second part T from V to W is called its linear.

I will not emphasise that again all the objects in this course will be linear transformation, T from v to W is called surjective if for every W vector W in W there exists at least one vector x and V is that T of x equals W; T is said to be surjective if every element in the core domain has a pre-image, every element in the court domain has a pre-image there exist x as W = T of x, there exists at least one x touch that W equals T of x this is subjectivity of a linear transformation. Before I look at those two or three examples let me also get the connection between this notion and the subspaces null space T and range of T, let me first make the following observation.

Observation one; T is linear, T from V to W is injective if and only if null space of T is 0 subspace, T is injective if and only if null space of T is a 0 subspace. Observation 2; T from V to W is surjective if and only if the range of T is the whole space W T is surjective if and only if range of T is a whole space W. Of these two observations the second one is straightforward that comes from the definition; T must be an onto map for example just as injective T is 1 to 1, surjective T is onto so in some cases I will use this notion onto, we will refer to T as an onto map if it is surjective okay. So from the definition of onto map it is clear range of T is W, let us quickly proof that first statement is true; T is injective if and only if null space of T is single term 0.

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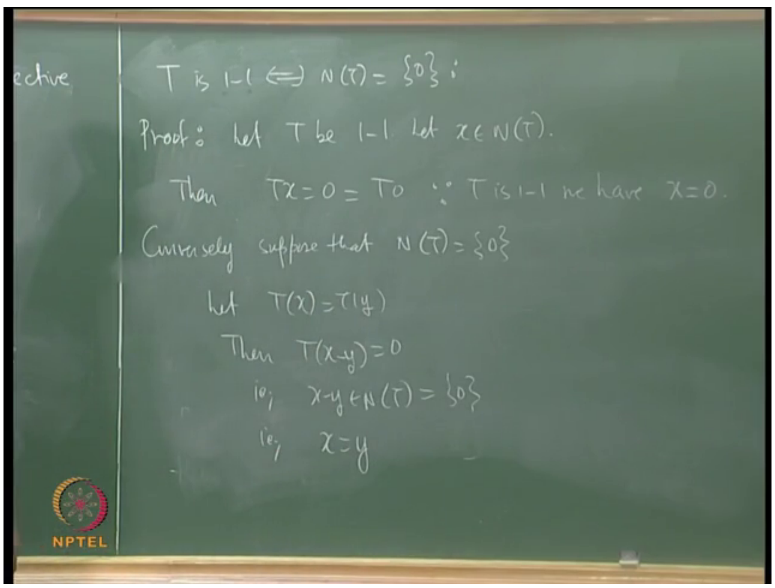

T is one is 1-1 if and only if null space of T is single term 0, proof of this let us take this implies this let T be 1-1; I will use this notation for 1-1 injectivity let T be 1-1 I must show that null space of T a single term 0 let us take x in null space of T I must show that this x is 0 okay then I have T x equal to 0 but T is a linear transformation so T of 0 is 0 we know this property there basic property so T x equals T 0 but this is like T x equal to TY. Since T is 1-1, it follows that x = 0 okay, 0 belongs to null space of T there is no problem null space T is a sub space so it has got at least zero vector, but what happens in this case when T is injective is that 0 is the only vector in the null space of T that is what we have shown, we have started with an arbitrary x in null space of T we have shown that x is 0, so this proves one way if T is 1-1 the null space of T is 0.

Converse conversely suppose that null space of T is single term zero we must show that T is giving injective. Let us start with T of x equals T of y we must show that x is equal to y that is injectivity then T of x – T of y = 0, T is linear, T of x – y equal to 0 which means x - y belongs to the null space of T, which I know has only 0 vector and so what I have shown is that x - y is a 0 vector so x = y so I started with T x = T y, I have shown that x = y so T is injective so our linear transformation is 1-1 if and only if the null space is single term 0 is onto if and only if the range space is entire core domain W okay. Let us now look at three examples, an example of a linear transformation which is 1-1 but not onto, another example of a linear transformation which is onto but not 1-1, third example which is both 1-1 and onto.

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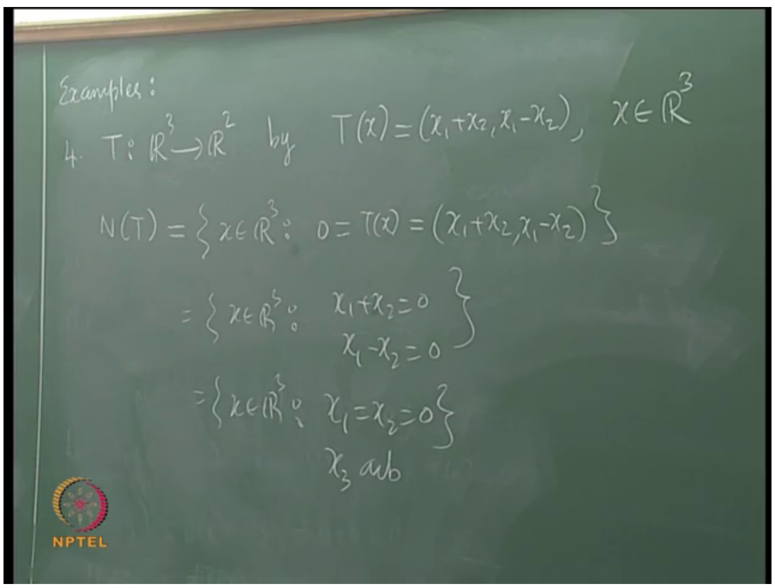

So I want to continue with examples, this is probably example 4, let me define T from R3 to R2 by T of x R3 there are 3 coordinates R 2, let us say x 1 + x 2, x 1 – x 2 for each x in R 3 let me define T of x in this manner. Then this is a linear transformation that does not need any proof, we have seen several times, T is linear, what is null space of T let us calculate that. Null space of T is a set of all x in R 3 such that T of x is 0; x1 + x2, x1 – x 2 that is x1 + x2 is 0, x1 – x2 is zero, what can we conclude from these 2 equations? x1 = x2 = 0 okay. The set of all x in R3 such that x1 equal to x2 equal to 0. Is T 1-1, is T 1-1 map? It is not because x3 is arbitrary so let me emphasise x3 is arbitrary so the null space is actually one dimensional it is spanned by e 3.

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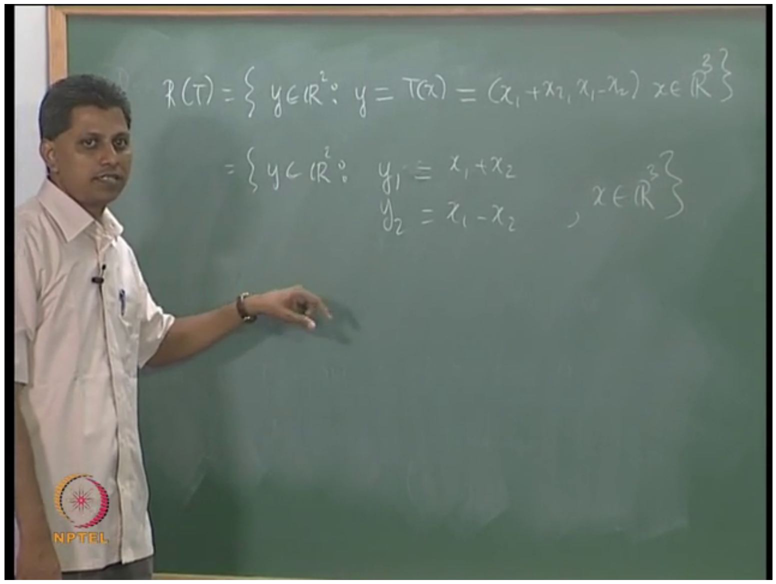

What about range of T? What is range of T, okay let us calculate. See this see this T is not 1-1 because null space consists of a nonzero vector at least one nonzero vector, range of T set of all y in R 2 such that y equals T of x, again it is x1 + x2, x1 – x2, x is in R3 that is my range space. This is the set of all y in R2 such that y 1 equals x1 + x2, y 2 equals x1 – x2 for x in R3, the question is... What is the condition that these equations impose on y in R 2 if at all there is a condition, we want to determine range of this linear transformation T, the question that arises is, do these equations impose any condition on the left-hand side numbers y1 and y2? The answer is no, let me give an argument for that.

Let us take uhh this is a subspace of R2, let us take the 2 bases standard bases vectors for R2 1-0 0-1. Suppose I show that the standard bases vectors have pre-images, does it follow that any vector will have preimages, does it then follow that range is a whole space? It would then follow that t is onto okay so the question is, look at these equations this is my question, do there exist x1, x2 Of course x3 such that such that this time x1 + x2 right so let me write like this uhh; x1 + x2 equals 1, x1 – x2 equals zero, I want to answer this question, I want to ask a similar question, do there exists numbers such that x1 + x2 is zero, x1 – x2 is 1 this is one question, this is another question, is it clear that these two systems are solutions?

Look at the first case, x1 equal to x2 equal to half in fact that is a unique solution; second equation is trivially satisfied half – half there is no x3, x3 can be taken to be arbitrary, second set of equations second system x1 equals x2 equals; x1 is half, x2 is – half, x1 is half, x2 is – half, this is zero, this is one so both these systems have a solution but what is the advantage? This is the vector e 1, this gives rise to e 1 that is 1 0, this gives rise to e 2 the second standard bases vector 0 1, these two vectors can be... These two vectors can be written as a linear combination of certain vectors in R3 that is what this means. See, these equations do not include x3 so extreme is arbitrary really, x1 and x2 only satisfy certain condition, x3 can be taken to be arbitrary so I can write so the summary is the following.

All that I'm saying is e 1, e 2 belong to range of T, what we have seen just now is that e 1, E2 belong to range of T. Since E1, E2... okay what this means is that range of T is 2 dimensional but range of T is a space of R2 which is two-dimensional so the spaces must be the same so range of T is the whole of the space W that is R2 in this example. Okay so this is the example of or linear transformation which is not 1-1 but onto, range of T is the whole of R2 I have shown this by showing that I have picked probably the simplest bases of R2 and then shown that any bases vector in R2 can be written... any bases vector of R2 has an image in R3. T of x equals e 1 we have solved, T x equal to E2 we have solved so these bases vectors have pre-images in R3 and so range is the entire space so T is onto so this is an example of a linear transformation which is not injective but surjective.

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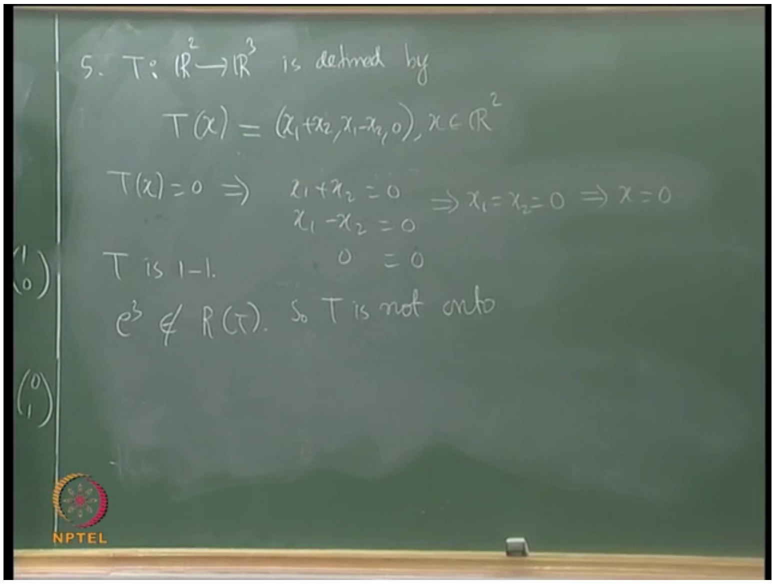

See this x come from R2, I will write the same coordinates x1 + x2, x1 – x2, 0. T is from R2 to 3 this time okay then T is linear that is easy to see, my claim is T is injective, let us prove it quickly let us start with T x = 0 I want to show that x is equal to 0. T x = 0 implies x1 + x2 equal to 0, x1 – x2 equal to 0, 0 equal 0 the row reduced echelon form the last row is 0, this is not the row reduced echelon form but you can see that this has 2 nonzero rows and the last row is zero. Now look at these two equations; these two imply immediately x1 equal x2 = 0 that is the same as saying the vector x is 0 so I have started with T x = 0, I have shown that x must be 0 so T is injective T is injective, T is not surjective can you give me one reason?

T is not surjective which means I must exhibit a vector in R3 which does not have a pre-image, which one? 0 0 1. e 3 does not belong to range of T because e 3 has a coordinate 1 but if it is in the range of T then the third coordinate must be 0 so e3 does not belong to range of T so T is not surjective so T is not onto okay, this is an example of linear transformation which is injective but not surjective.

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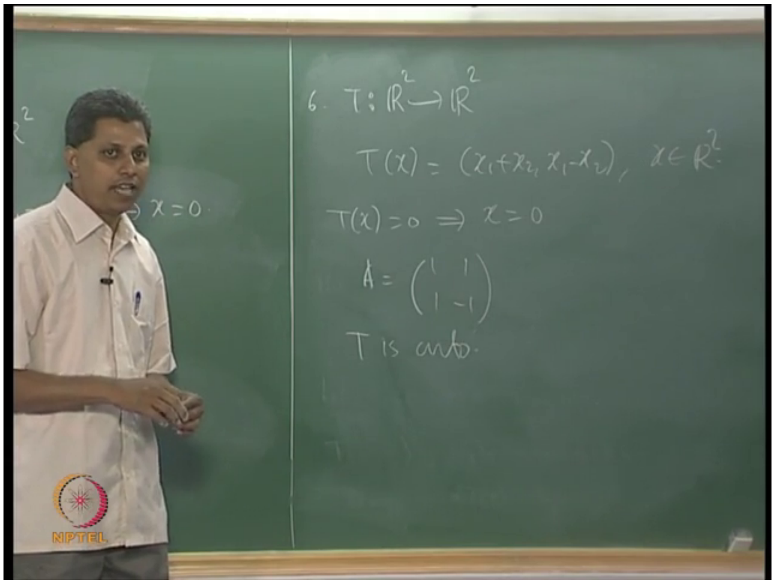

One final example before I conclude, T from R2 to R2 is defined by T of x equals x1 + x2, x1 – x2 okay this linear transformation has a property that it is both injective and surjective. Injective T of x = 0 implies x1 + x2 is 0, x1 – x2 is 0 as before, x is 0. Now subjectivity again one can show that the right-hand side vector if I take e 1 as a right-hand side vector then the system x1 + x2 = 1, x1 - x2 equal to 0 has a solution, change the right-hand side vector to e 2, x1 + x2 equal to 0, x1 - x2 equal to 1 has solution that is one way of looking at it. The other way of solving this problem is to link this with a particular metrics, we are actually solving the system of linear equations, link this to a particular metrics that metrics come from the definition of T.

Look at the metrics whose entries are 1 1 1 -1 okay, now what we have shown is that this matrix has a property, we have shown T as 1-1 which is same as saying this matrix has a property that the system A x = 0 has 0 as the only solution. Then from an equivalence condition that we have proved before it follows that A x = B has a solution for all right-hand side vectors that is the reason why A x = e 1 as well as A x = e 2 have solutions. But if A x equal to e 1, A x = e 2 have solutions it means T is onto so please fill up the details, this T is onto also, link this with what we have learned before okay. The second argument to show that T is onto relies basically on if you go back to that example you will realise it relies basically on solving 2 systems of linear equations, 2 systems of linear equation A x = e 1, A x = e 2; e 1, e 2 are the standard bases vectors in R2.

So I would like to ask, thus A x = B have a solution for all B? If I know the answer, if I know S is the answer for this question then e 1, e 2, any number e n they will have a solution. So my question is to find out whether A x = B has solution for any B, I'm saying in this example the answer is yes because we have shown that T is 1-1 which is same as saying that the homogeneous equation A x = 0 has zero as the only solution, the homogeneous equation A x = 0 where A is the metrics which comes naturally from the transformation T.

This A x = 0 equal to 0 has zero as the only solution, we know that this is square matrix, homogeneous system has 0 as the only solution then we know that this matrix is invertible which is same as saying that the system A x = B for any right-hand side vector B has a solution, which is what we wanted. So A x equal to e 1 has a solution, A x = e 2 has a solution, I'm not interested knowing the solution I only want to know that the solution exists, A x = e 1, A x = e 2 both are solutions so standard bases vector e 1 and e 2 have been written as a linear combination of certain vectors in R2, they are in range of T that is really what we want to conclude, e 1, e 2 both are in range of T, it follows T is onto as before so this is an example of a linear transformation both 1-1 and onto, so let me stop here.