Linear Algebra
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Lecture 14
Linear Transformations

See one of the central notions in linear algebra is that of a linear transformation this is also central not only in linear algebra but the entire mathematics. Today I will discuss the notion of linear transformation give several examples, now these examples will justify the statement that I made just now you will see that linear transformations arise in differential equations in integral calculus in matrices for transformations between vector spaces etcetera. So let us first look at the notion of linear transformation look at several examples and probably towards the end of the class we will look at some properties some simple properties some not so simple properties and then we will be able to compare how a linear transformation behaves with the general function between vector spaces, okay.

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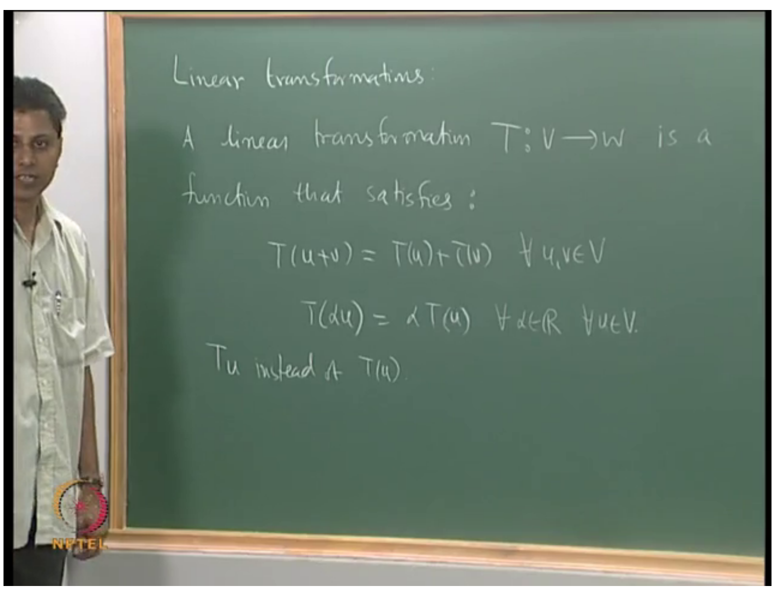

So let us begin with the notion of a linear transformation between two vector spaces, if I have two vector spaces let us call T as a linear transformation, what this means is that V and W are vector spaces T is a function to begin with so I am just emphasizing it is function that satisfies the following two conditions T of u plus v equals T of u plus T of v for all u, v in V and T of alpha u equals alpha T of u this is true for all alpha in the underlying field which we are assuming is the real field and for all u in V, okay.

So a linear transformation is a function linear transformation between two vector spaces is the function between those two vectors spaces that must satisfy these two conditions, let us observe that the right hand side vector is in W, okay. What is inside? u plus v that is in v T of that is in W and the formula for T of u plus v s given by this the formula for T of alpha u is given by this right hand side, okay. Sometimes we write Tu instead of T of u just a notational convenience.

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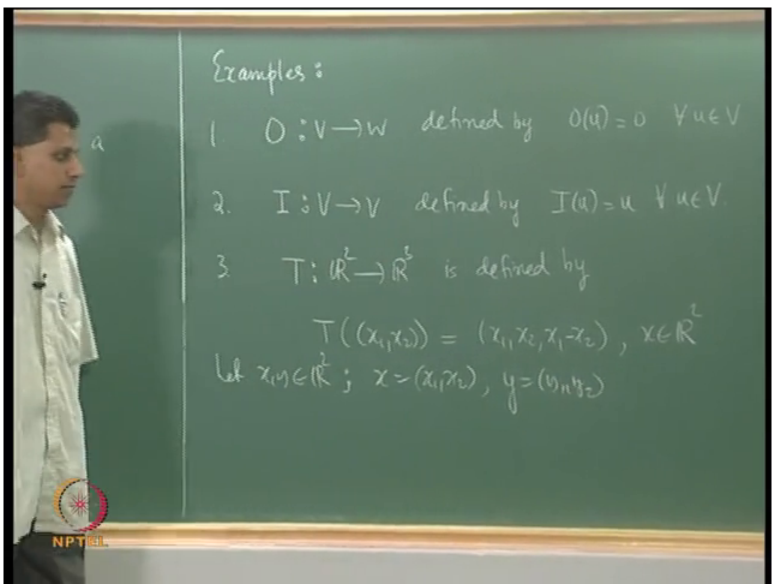

So let us now look at examples let us dispose off the trivial examples, first one you look at the map which I will denote by 0, 0 from V to W defined by 0 of u is 0 for all u in V, see the left hand side 0 or (())(4:01) is a 0 transformation the right hand side 0 is a 0 vector in W, okay trivially this is a linear transformation because it satisfies these two conditions this is called the 0 transformation, example 2 I will call it I I from V to v, you know what it is the identity transformation defined by I of u equals u for all u in V. Now you see that the right hand side u is the same as the left hand side u that you started with so you need W to be equal to V for identity transformation W is equal to V.

So these two are trivial examples of linear transformations, let us know look at one non-trivial example and then probably two examples from geometry motivated by notions from geometry.

So let me first look at a non-trivial linear transformation let us say T from R2 to R3 T from R2 to R3 is defined by T of something, okay so T of x1, x2 a typical element in R2 is x1, x2 T of that element let us say the right hand side is x1, x2 x1 minus x2 and I need to remember that x comes from R2 T of x1, x2 is the first component of the x1 second component x2 third component x1 minus x2.

So what you observe is that this x1, x2 belongs to R2 this belongs to R3 this is linear let me first take this as the first example and then verify these two conditions that this is a linear transformation we need to verify first condition so let us take x, y in R2 and the notation that I will use is x equals x1, x2 y equals y1, y2 I must verify that T of x plus y equals T x plus T y first, okay.

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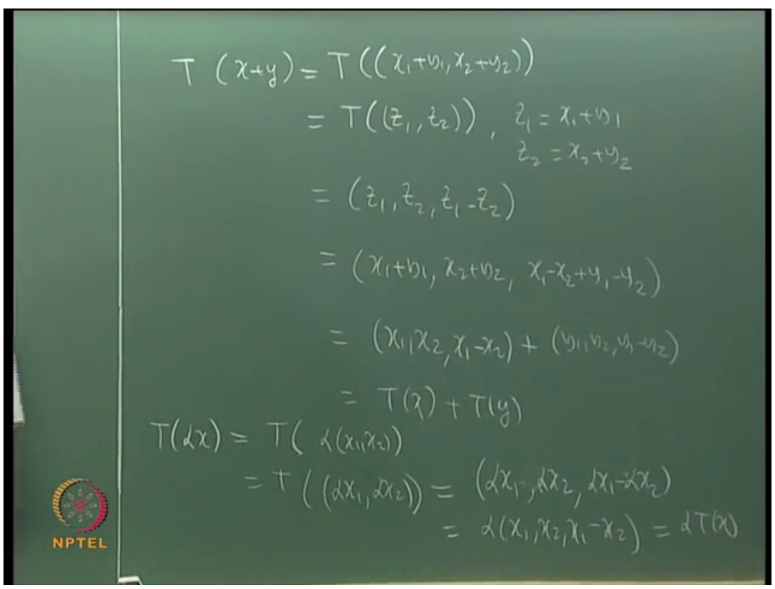

Look at T of x plus y this is T of the vector x plus y I know is coordinate wise x1 plus y1 x2 plus y2 I will call this as T of z1 comma z2 where z1 is x1 plus y1 z2 is x2 plus y2. So I have T of two coordinates I have the formula there so this is z1, z2, z1 minus z2 that is a definition T of x1, x2 is x1, x2 then x1 minus x2 so T of z2 z1, z2 z1 minus z2 z1 is x1 plus y1 z2 is x2 plus y2 z1 minus z2 is I will write it as x1 minus x2 plus y1 minus y2, I can write this as x1, x2 x1 minus x2 plus y1, y2 y1 minus y2 this is addition in R3 this is operation plus in R3 the first term now is T x the second term is T y, okay so T x plus y is T x plus T y we can also verify T of alpha x equal to alpha T x let us do that quickly T of alpha x is T of alpha into x1, x2 that is T of alpha x1 comma alpha x2 which by definition is alpha x1 minus sorry alpha x1, alpha x2 alpha x1 minus alpha x2 which let me write as alpha times x1, x2, x1 minus x2 which is alpha T x, okay.

So this is a simple verification that T is a linear transformation, okay the T that defined that is defined here is a linear transformation.

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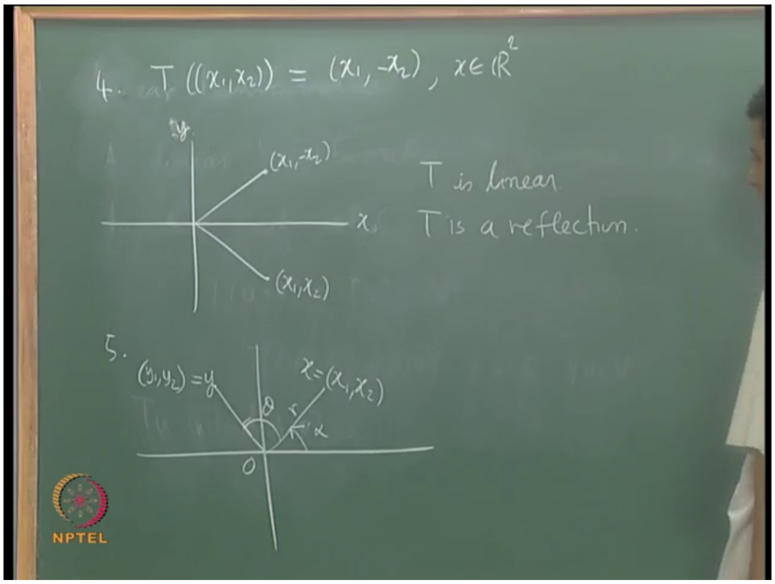

Let us look at other examples let me now look at two examples coming from geometry look at the transformation that sends a vector in R2 on the plain I have a transformation which sends a vector X to a vector obtained by reflecting with respect to the horizontal axis, that is example 4 you tell me if this is what I am talking about T of x1, x2 is x1 minus x2 this is the transformation which takes the vector x1, x2 to x1 minus x2 so it is reflection with respect to the horizontal axis, okay if you think of R2 as a horizontal and a vertical then this is what it does, say x1, x2 then if x1, x2 is here let us say x, y and I will call this a general point x1, x2 then it must go to this point which is a reflection of this this is x1 minus x2 I could have written x1 minus x2 here x1, x2 here but does not matter reflection this is an example of a linear transformation I am not going to verify that it satisfies those two defining equations this T is linear this comes from geometry reflection.

Example 5 rotation, okay let me call it T is a reflection a reflection with respect to some axis I have taken the horizontal axis, 5 is rotation rotation let us first derive the formula and then get the transformation from that formula rotation means the following I am again in R2 I have a vector here at a distance R from the origin and I will call that x1, x2 this makes an angle so this is R for me this makes an angle let us say alpha with the horizontal axis positive x axis I am rotating this when you rotate this length does not change the distance from the origin does not change by rotation.

So let us say I have something like this here this is my y and I write y as y1 comma y2 the rotation is by an angle theta the vector x has been transformed to the vector y by an angle theta, can I write down a formula for a transformation that sends x to the vector y, okay now you know this horizontal vertical components if you want you can use Pythagoras theorem a, okay.

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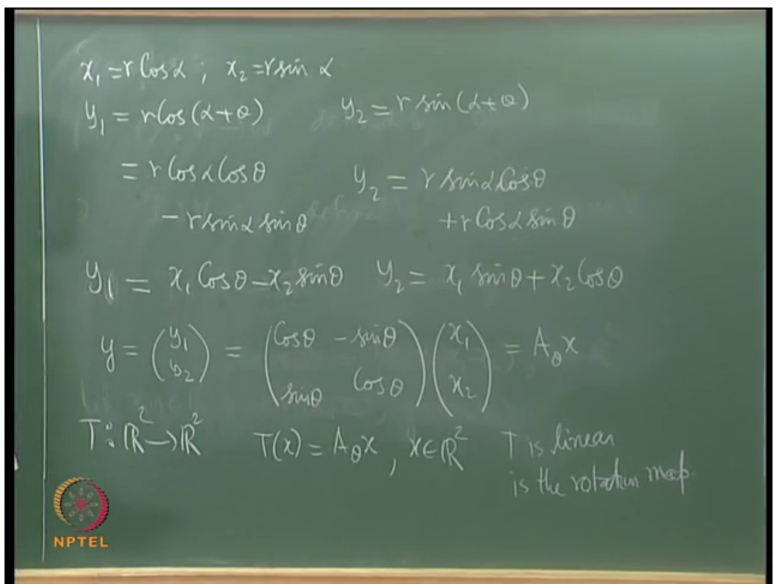

What I have is x1 is the horizontal component that is cos alpha x2 is sin alpha, yes there is an R coming otherwise we can write a similar formula for y, y1 is r cos alpha plus theta and y2 has a similar formula let me write down this, it is okay let me write y2 similar y2 is r sin alpha plus theta so this is r cos a cos b r cos alpha cos theta minus r sin alpha sin theta, y2 for me is r sin alpha sin theta sin a cos b plus cos alpha sin theta, okay. So this is r cos alpha go back to this this is x1 cos theta minus x2 sin theta that is my y2, y1 rather y2 is on the other hand r sin alpha that is x2 so let me write this first r cos alpha x1 sin theta plus x2 cos theta x2 is r sin alpha into cos theta so I have these two expressions for y1 and y2 again horizontal vertical components.

Then let me write y as a column vector this time let me write y as a column vector then this y I know is y1 is x1 cos alpha minus x2 x1 cos theta minus x2 sin theta so let me write just the coefficients cos theta minus sin theta this into x1 comma x2 I will write that also as a column vector, so I am now writing a matrix equation something like Ax equal to b, b equal to y equal to Ax y2 is what I need to write next x1 sin theta this is cos theta so I have written y equals A theta x this matrix the entry is depend on theta so that is A theta x I have written y as A theta x so this is the transformation formula if you give me x I substitute into this I get y of course I must know theta as, okay I must know the angle of rotation.

Now look at A theta let me now use this A theta to define T from R2 to R2 by the formula T of x equals A theta x I am defining T of x that is a transformation is A theta x rotation rotates x to y, then use matrix multiplication to conclude that this is a linear transformation, okay this t is linear T is linear and it is the rotation map T is the rotation map or the rotation transformation, okay. So these two examples come from geometry, let us also look at some other examples coming really from geometry but this time we may have to look at higher dimensions that is fifth example.

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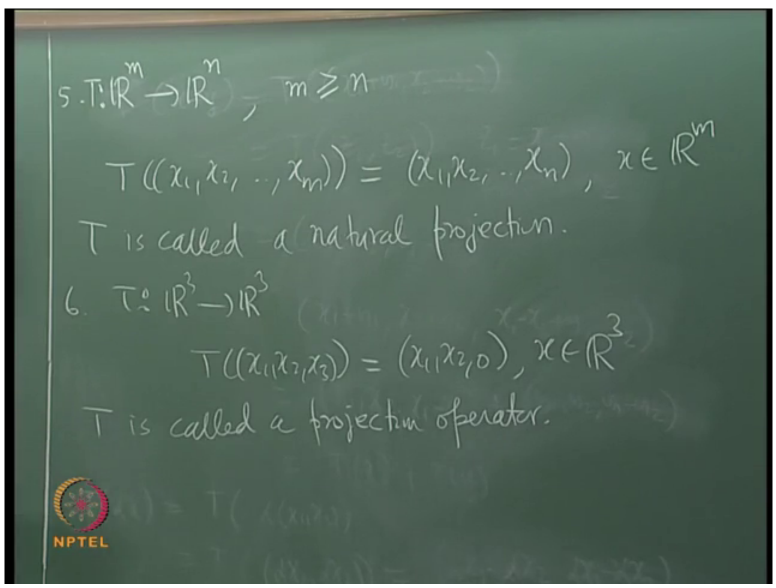

Let me look at T from Rm to Rn let me take the case when m is greater than or equal to n I will define T x1, x2, etcetera, xm m is greater than or equal to n that is n is less than or equal to m so the number of coordinates on the this on this right hand side I have a vector which has less than m coordinates there is a natural definition x1, x2, etcetera, xn, okay x is in Rn.

See what I have dropped is what I have done is to drop the coordinates from n plus 1 to m, okay this is called a natural projection T is called a natural projection on Rm, now you can verify that this T is linear these are some of the simplest examples of linear transformations this T is linear. Let us look at the usual projections, projections that we encounter in engineering drawing for instance T from R3 to R3 defined by T of x1, x2, x3 is let me say it is x1, x2, 0 these will be called projection operators T is called a projection operator on R3 we will reserve the word operator when the vector space is V and V are the V and W are the same if V is equal to W then linear transformation will be in particular called a linear operator.

So this is an operator it is called the projection operator you see that any point on the plain rather any point in three space is dropped on to the horizontal plain the plain let us say x, y plain any point in the x, y, z plain the z coordinate is 0 so we are looking at the projection of any point in three space on the so called x, y plain that is the projection operator, this is just one of those examples I have another several other examples for instance T of x1, x2, x3 could be x1, 0, x3 or x1, 0, 0 on the x axis etcetera all these are called projection operators.

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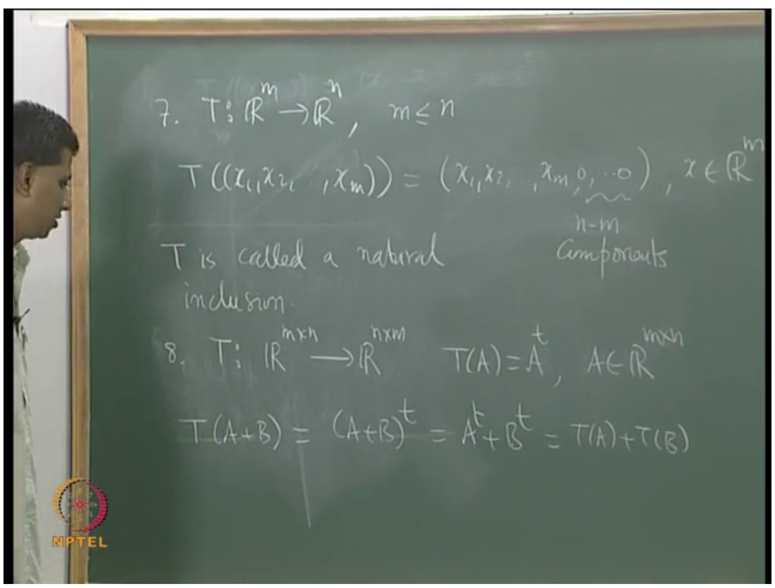

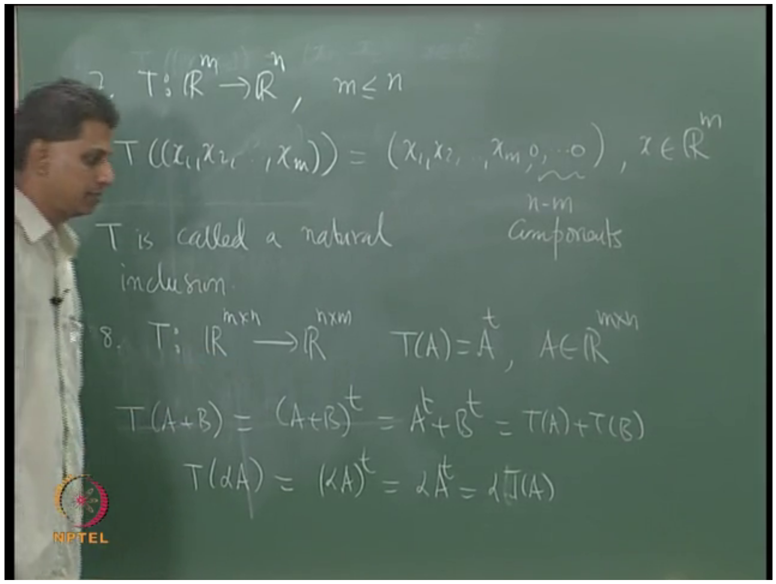

Let us look at other example let us take this time T from Rm to Rn with m less than n it can be equal also the definition is as follows this will be x1, x2, etcetera, xm if it is strictly less than n the other coordinates are taken to be 0. So these are n minus m components m is less than or equal to n so this may not be there then it will reduce to the natural projections but otherwise there are certain coordinates which are 0.

Now this is not an operator this is what is called as natural inclusion, again T is linear it is called a natural inclusion in particular this allows us to think of Rm as sitting inside Rn if n is greater than m you can thin k of Rm as sitting inside Rm, so this is natural inclusion this is another example of a linear transformation. Let us look at other examples let us take one from the space of matrices let us say T from R m cross n the set of all the vector space real vector space of m by n matrices to the real vector space of n by m matrices defined by T of A equals A transpose this A transpose has been defined earlier if A is equal to Aij then A transpose is Aji and so if A is m cross n then A transpose is n cross m this is linear that is because we need to verify T of x plus y equals Tx plus Ty.

Let us look at T of A plus B, T of A plus B by definition is the transpose of A plus B but the transpose can be verified to satisfy this formula A transpose plus B transpose, okay this is easy consequence of addition A transpose plus B transpose A transpose is T of A B transpose is T of B so this is additive T of A plus B is T of A plus T of B T of alpha times A is T of its alpha A transpose alpha A is multiplying alpha to each component of each term of the matrix A, so that can be taken outside it is alpha A transpose.

Remember its real case if it is a complex case you must take alpha bar outside alpha A transpose that is alpha T of A, so T is linear we have verified. So this one comes from transformation between vector spaces of matrices.

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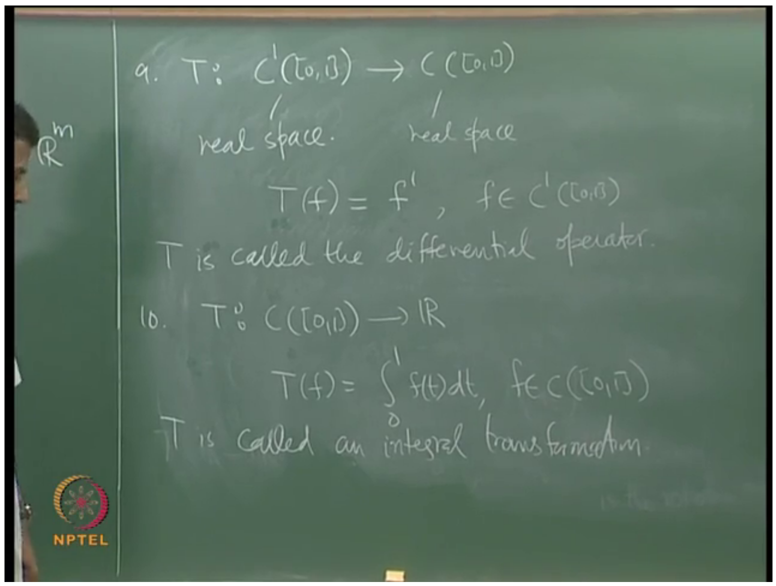

Let us look at example from differential calculus let us take T from C prime 0, 1 I usually say this is a complex valued space of complex valued continuous functions on the interval 0, 1, C 0, 1 C prime 0, 1 space of complex valued continuous functions on 0, 1 with the property that the first derivative is continuous, I will consider this as a real space T from C 1, 0, 1 to C 0, 1 again real space the mapping is T of a function f it is f prime its derivative derivative function T of f is f prime the derivative function T of sin theta is cos theta f comes from C1 0, 1 first of all this is well defined because T of f is f prime f if f is C1 0, 1 does f prime belong to C 0, 1 that is the case because of the definition of C1 0, 1 f prime is the first derivative that must be continuous.

So this f prime belongs to this so this is well defined there is a function to verify that this is linear comes from differential calculus d by dt of f plus g is df by dt plus dg by dt, d by dt of alpha times f is alpha df by dt. So this t is linear this is called the differential operator this T is called the differential operator, okay see it is not just a superficial connection to differential calculus what we will see is that later when maybe in the next lecture we will discuss the notion of the rain space of a linear transformation, null space of a linear transformation there you will see that the null space of a linear transformation when T is a differential operator especially coming from constant coefficient that is it is a differential operator with constant coefficients then the null space is precisely the set span by the solutions which are called so called complementary functions of the differential equation, okay.

So this connection is not just superficial, okay this will be made clear later. So this is called the differential operator coming from differential calculus, one from integral calculus and probably I will stop this list the last example is 10 let us say I have T from C 0, 1 again for the sake of simplicity I will take this to be a real space of continuous functions on 0, 1 to R this time the domain vector space is 1 dimensional it is just R T defined by T of f is integral 0 to 1 f of t dt T of f is integral 0 to 1 ft dt the Riemann integration.

We know that this is well defined again because from integral calculus we know that every continuous function is Riemann integrable. So the right hand side is well defined and you can verify easily that this is linear transformation that is for two functions f and g that are continuous integral 0 to 1 f of f plus f of T plus g of T dt is equal to integral 0 to 1 ft dt plus integral 0 to 1 gt dt T of that is T of f plus g equals Tf plus Tg T of alpha f is alpha times 0 to 1 ft dt so that is alpha f alpha t of f rather.

So this is linear this is called an integral operator I will simply say integration integral transformation this is again a linear transformation, okay.

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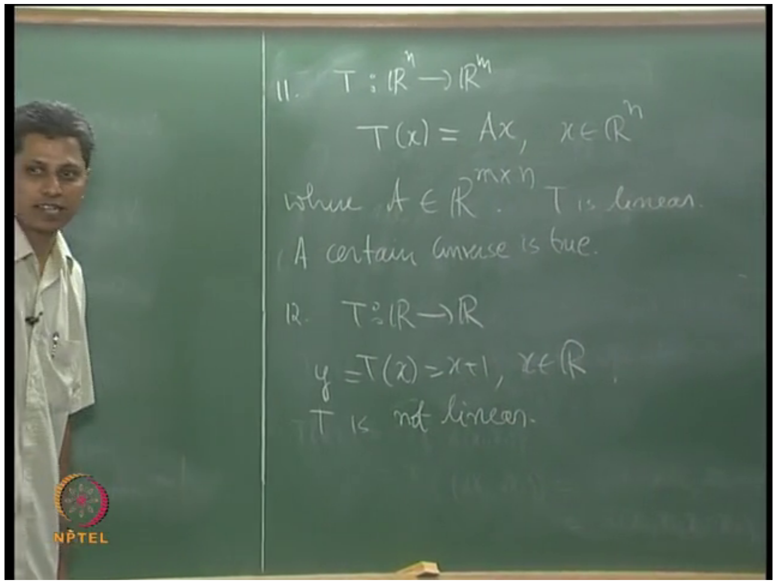

Probably one final example which sort of summarizes several of the previous examples not all of them, I will state that as example 11 let me say T is from Rn to Rm is defined by, so I have T of x equals Ax where I am given an m cross n real matrix A I am given an m cross n real matrix A through this matrix I am defining a transformation this transformation is between Rn and Rm it is from Rn to Rm defined by this equation Tx equal to Ax.

This is matrix multiplication you see that if A is m cross n then Ax is 1 cross m cross n m cross 1, okay that is a vector is Rm so this is well defined most of the examples that we have discussed previously 0 transformation, identity transformation, the second example, the third example reflection, the fourth example rotation, natural projection, natural inclusion, projection operator all these are particular cases of this for different choices of A this T is linear which follows by matrix multiplication this is linear, okay this sort of summarizes all those examples, now what is also true which is the most interesting part of linear algebra is that a certain converse is true that is if I have a linear transformation between finite dimensional vector spaces then there is a matrix which has a property that the transformation T satisfies this equation for that matrix, okay if I have a linear transformation between finite dimensional vector spaces then there is a matrix A we can construct a matrix A such that this holds for the linear transformation T that we started with, okay.

So let me just say that a certain converse is true and this is this holds for finite dimensional vector spaces, okay. So this list should probably convince you that linear transformations are indeed important objects before I proceed to the certain simple properties let me also consider this notion of what is linear sometimes is not really the linearity that we would like to have as illustrated here that is I want to give example 12 which is not really an example let us look at T from R to R defined by T of x equals x plus 1, okay T of x is x plus 1 the translation.

Now we can plot this on R2 that is you can call this as y, then I have y equals x plus 1, now this is a straight line not passing through the origin, okay you can verify that this T is not linear you can verify that this T is not linear in spite of the fact that intuitively in R2 y equals x plus 1 is a line, okay. So if you have a formula representing a line in R2 this does not necessarily correspond to a linear transformation this is just a simple point I wanted to illustrate, okay. In any transformation that transforms a line to a line is not necessarily a linear transformation is what I wanted to emphasize, okay so this is not a linear transformation you can verify by simple examples that this T is not linear, so anything that looks like linear is not necessarily linear, on the other hand if it is a straight line passing through a origin then this will be a linear transformation, okay.

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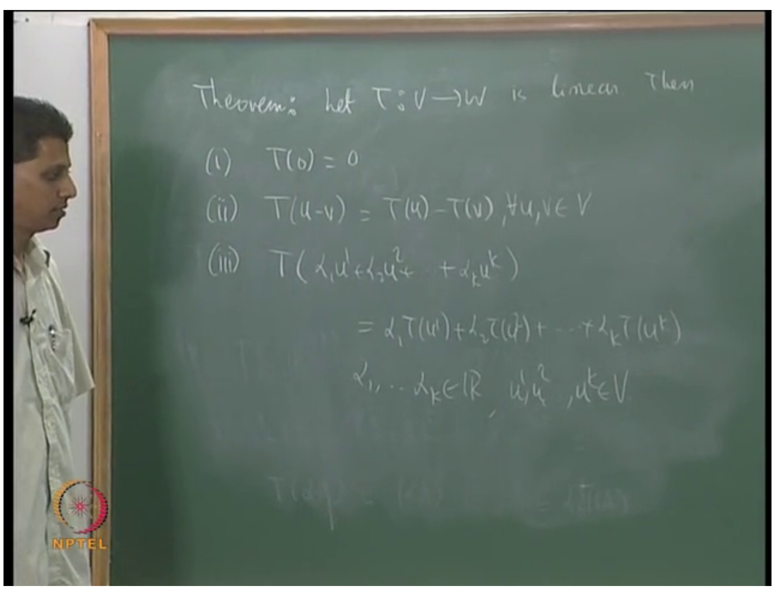

Let us now look at some simple properties, okay first property is the following so let me write down this theorem T from V to W T from V to W is linear then I have the following properties, property 1 0 must be map to 0 this is the first property, for instance you could use this property in that last example T of x equal to x plus 1 T of 0 is not 0 so that is not linear T of equal T0 equals 0 property 2, we know that T of u plus v is Tu plus Tv this holds for u minus v also T of u minus v is T of u minus T of v and property 3 T of u plus v equals Tu plus Tv this can be extended to a finite sum T of alpha 1 u1 plus alpha 2 u2 etcetera let us say alpha k uk this is equal to alpha 1 T of u1 plus alpha 2 T of u2 plus etcetera alpha k Tuk this additivity property that is condition 1 that a linear transformation must satisfy can be extended to finitely many terms in fact linear combinations, that is here these coefficients alpha k are in R u1, u2, etcetera, uk they come from V, let us quickly verify that these properties hold.

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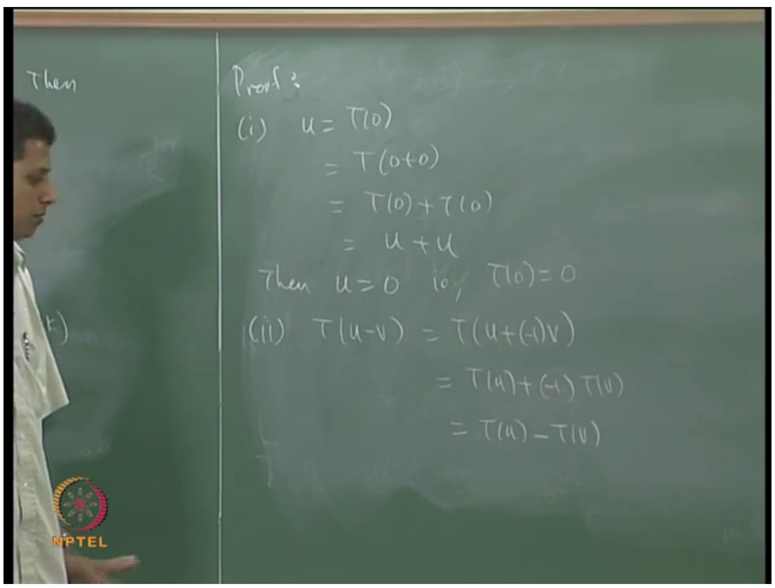

So our discuss the proof very quickly look at the first part I will u as T of 0 then u is T of 0 plus 0 T is linear so T of 0 plus 0 is T0 plus T0 I am calling this as u so I have u equals u plus u then from the first simple property of vector spaces it follows that u is 0 that is T of 0 is 0 that is the first property. Property 2, T of u minus V by definition this is T of u plus minus 1 times V minus V is minus 1 into V T is linear so T of x plus y so that is T of u plus the constant is outside T of V, minus 1 T of V that is happening in W, so this is Tu minus Tv that is property 2.

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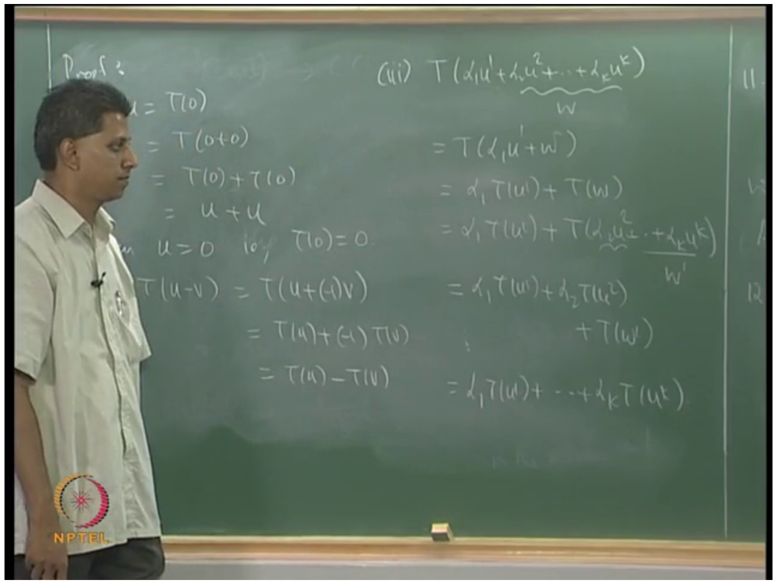

Property 3, T of alpha 1 u1 plus alpha 2 u2 etcetera plus alpha k uk, I will keep this as a vector maybe I will call it w then this is T of alpha 1 u1 plus w I know that this is alpha 1 T of u1 plus T of w then keep this as it is alpha 1 T of u1 plus T of w formula for w T of alpha 2 u2 plus etcetera plus alpha k uk I again have I will keep this as it is the rest of them I will call it w1 and proceed, alpha 1 T of u1 plus alpha 2 T of u2 plus T of w1 where w1 is alpha 3 u3 plus etcetera plus alpha k uk proceed by induction etcetera this is alpha 1 Tu 1 etcetera plus alpha k Tuk, okay so really simple property just making use of linearity definition of linearity, okay.

A little more non-trivial properties of a linear transformation we will discuss next, to motivate this property maybe I will give an example start with an example, let us look at the function sin x and cos x these are functions from R to R real valued functions of the real variable x these functions have the property that sin x equal to cos x at infinitely many points all those points starting from pi by 4 if you want and then you add 2 pi.

So there are many infinitely many points x for which sin x equal to cos x, okay for a linear transformation this kind of a think will not be true. For a linear transformation if you have two so I have two really two functions sin x and cos x which coincide at infinitely many points but if you have transformations T1 and T2 that coincide at all those basis elements then they must be the same linear transformation, okay this is one important property which separates a linear transformation from a general function let me make this clear.

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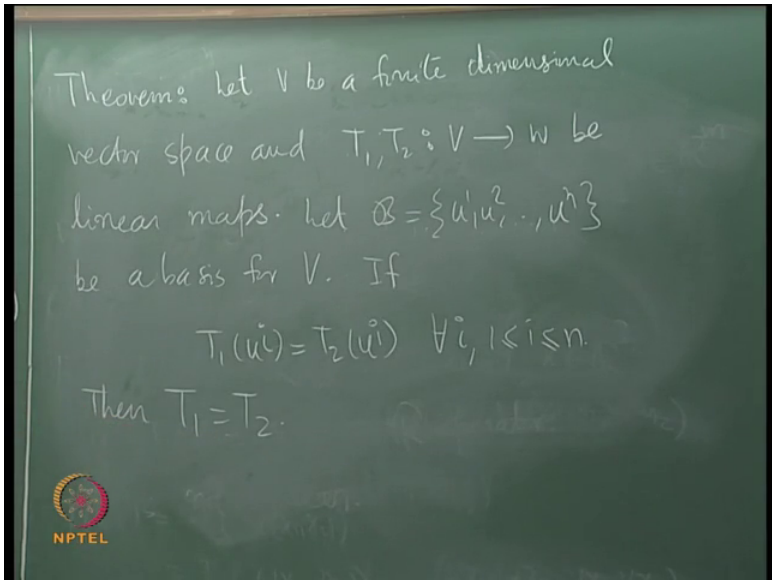

Let V be a finite dimensional vector space, and T1 comma T2 from V to W be linear transformations sometimes I will also call them maps, these are functions. So I have T1 and T2 linear maps from V into W no word about W V is finite dimensional, suppose that I have a basis b let us say u1, u2, etcetera, un let this be a basis for V. So V is finite dimensional there is a basis consisting of finite elements finitely many elements I am listing that basis, suppose T1 and T2 satisfy the following equation if T1 of ui equals T2 of ui for all i 1 less than or equal to i less than or equal to n that is the transformations T1 and T2 coincide for the basis vectors the transformation T1 and T2 coincide for the basis vectors then we can show that T1 is equal to T2 then T1 is equal to T2.

So now contrast this statement with the statement that I made to motivate this theorem sin x and cos x they are equal at infinitely many points but as functions they are not equal, okay remember that T1 is equal to T2 means as functions these two are equal that is T1 of x equals T2 of x for all x in V as functions these two are equal they are one and the same one can also make the following informal statement from this theorem, a linear transformation is completely determined by its action on any basis where I am assuming that the domain space is finite dimensional a linear transformation is completely determined by its action on any of its any of the basis of the domain space, okay let us prove this quickly I want to say that T1 is equal to T2 so I am prove that T1 of x equals T of x for all x.

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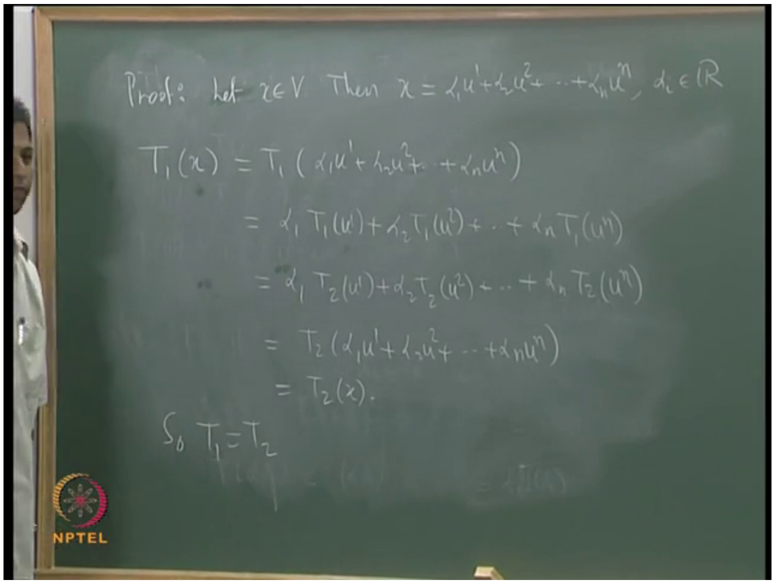

Let x belong to V, okay then I have a basis explicitly given script b so I can write x as alpha 1 u1 plus alpha 2 u2 plus etcetera alpha n un, okay let me know look at T1 of x, T1 of x is T1 of this representation alpha 1 u1 plus alpha 2 u2 etcetera plus alpha n un T1 is linear so this is alpha 1 T1 u1 plus alpha 2 T1 u2 plus etcetera alpha n T1 un. Now I will make use of the fact that T1 u1 is equal to T2 u1 T1 u2 is equal to T2 u2 etcetera that is what is given T1 and T2 coincide for the basis vectors, so this is alpha 1 T2 u1 plus alpha 2 T2 u2 plus etcetera plus alpha n T2 un again use the fact T2 is linear to rewrite this as T2 of alpha 1 u1 plus alpha 2 u2 plus etcetera plus alpha n un but this is the x that we started with, so this is T2 of x so what we have shown is that T1 of x is equal to T2 of x for all x in V and so T1 is equal to T2, okay.

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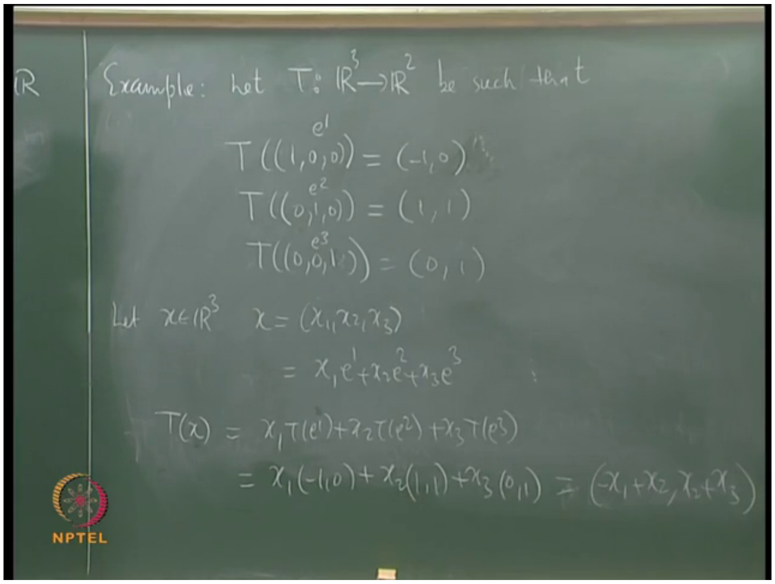

Let me look at a numerical example to illustrate this result I want to give an example let T from R3 to R2 be such that T of the first basis vector is this, T of the second basis vector I am in R3 to R2 let us say minus 1, 0, 0, 1, 0 let us say this is 1, 1 and T of 0, 0, 1 these three equation define T uniquely these three formulas define T uniquely. What is the general formula for T of x I can write down because any x can be written as a linear combination of these, okay so let us do that quickly let us take x in R3 then x is I am following this notation consistently x1, x2, x3 I can write this as x1 into, okay see in our notation this is e1, this is e2, this is e3 standard basis vector so this is x1 e1 plus x2 e2 plus x3 e3 any x is a linear combination of these the coefficient x1, x2, x3 are given by the components of x I want T of x that is the question what is the general formula for T of x given x.

So T of x by definition is x1 T of e1, x2 T of e2 plus x3 T of e3 just plugin these values you get the formula for T of x. So T of e1 is minus 1, 0 x1 into minus 1, 0 plus x2 into 1, 1 plus x3 into 0, 1 so you get a formula in terms of x this is minus x1 plus x2 second coordinate x2 plus x3 so this is T of x minus x1 plus x2, x2 plus x3, okay this is a general formula if you know x you just plugin here you get T of x, so the action of a linear transformation on a basis that is enough to determine the linear transformation completely, okay.

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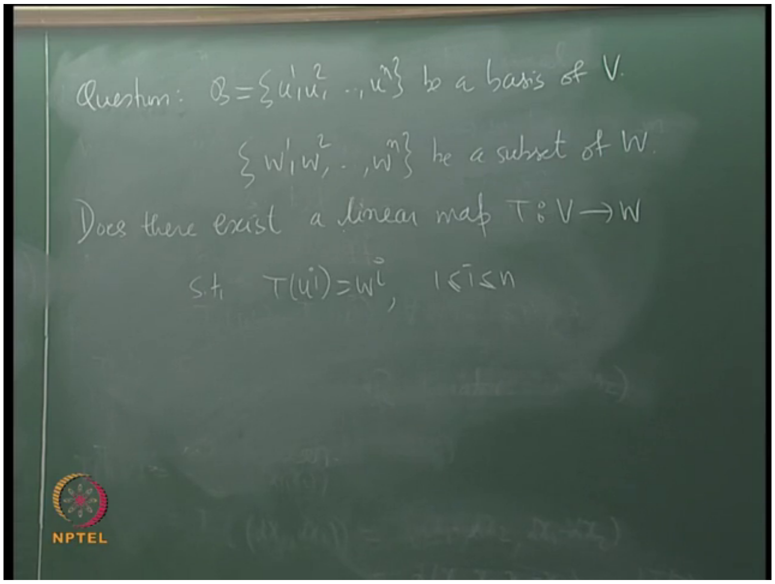

Let us also ask this question the answer will be given in the next, this is the question I have let us say a basis V u1, u2, etcetera, un as before this is a basis of V I am given a set of vectors not necessarily a basis of W let us call them w1, w2, etcetera, wn this is just a subset not necessarily a basis be a subset of w I have a basis for V and just a subset of W the question is does there exist a linear transformation a linear map T from V to W that takes the corresponding elements to the corresponding ui to the corresponding wi that is map such that T of ui is equals wi ui goes to wi, does there exist a linear map T from V to W such that this condition satisfied, okay.

For this to be satisfied do we need conditions on w1, etcetera, wn, okay if there exist a linear transformation is the transformation unique, we will answer these questions in the next lecture, I will stop.