Linear Algebra
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Lecture 13
Dimensions of Sums of Subspaces

Okay, so let us continue our discussion on dimensions dimensions of subspaces, I will like to prove a few results today on computing the dimension of certain subspaces but before that let me give you one or two general results which where hinted in couple of lecture ago when we discussed the notion of linear independence, okay.

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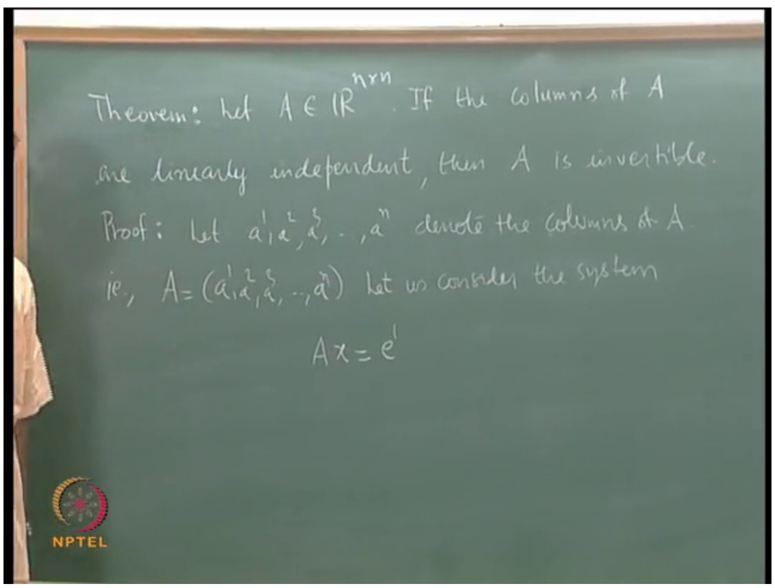

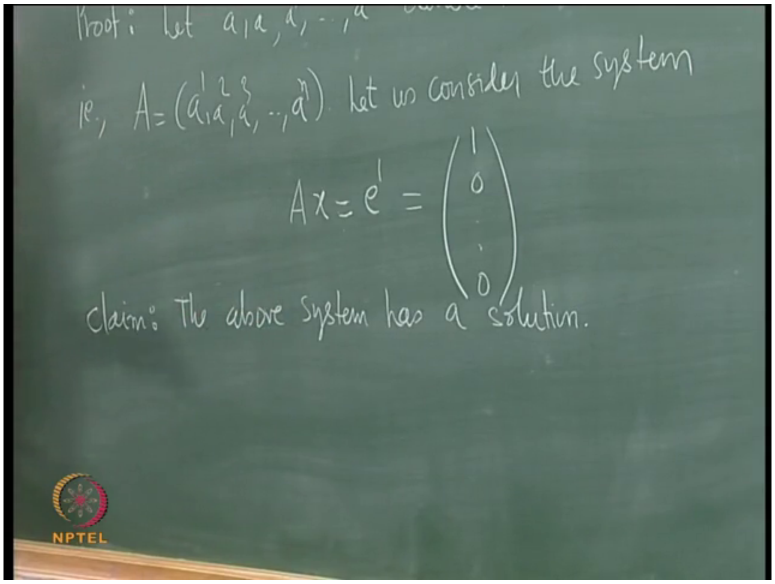

So let me start with this result I have a square matrix with real entries order n if the columns of A are linearly independent then the matrix A is invertible you might remember that we made use of this fact in a numerical example let us prove this general result, okay.

If the columns of A are linearly independent then A is invertible actually there is a corresponding results for the rows I have a square matrix the rows of A are linearly independent then it will be invertible, okay that is a result which you could prove using this result so I will not state that result I will only prove this result. Let me denote, okay super scripts 1, 2, 3 etcetera, n will denote the columns of A that is I can write A as a1, a2, a3, etcetera, an, okay what is given is that these columns are linearly independent you must show that A is invertible, okay we will make use of a result that we proved some time ago a matrix A is invertible if and only if the non-homogeneous equation Ax equal to b has a solution for all b, okay.

So let us take let us consider the system Ax equals e1, e1 is the standard first standard basis vector of Rn, okay so let me just write it is a column vector whose first coordinate is 1 all other entries is 0 the claim is the above system has a solution this is the claim we will prove this claim. Once we have this we can change the right hand side what we will show is that if the columns are linearly then this system will have a solution the next system Ax equal to e2 will have a solution, Ax equal to e3 etcetera Ax equal to en all these n systems will have a solution we will prove that, once we prove that these n systems have a solution it will follow that A is invertible, okay I will give the details but before that how do we show that this system has a solution.

I have this right hand side vector, okay this is a vector in Rn, okay let us now recall what we proved in the last lecture if I have a linearly independent subset of a vector space having the same number of elements as the dimension of the space then this linearly independent subset must be a basis.

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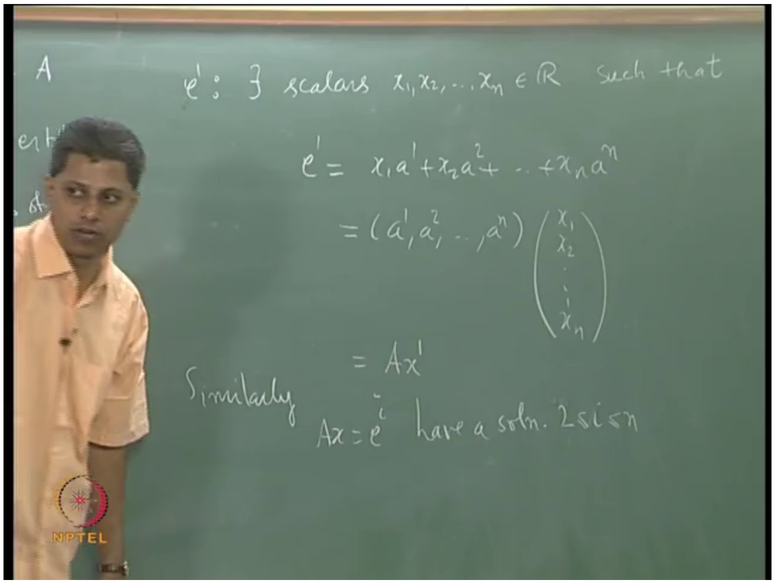

For e1 there exists scalars I will call it x1, x2, etcetera, xn such that e1 is in Rn the vectors a1, a2, etcetera, an the column vectors are linearly independent so e1 can be written as a linearly combination I will take the coefficient to be these, so e1 is x1 a1 plus x2 a2 plus etcetera xn an, okay this is because of the fact that any linearly independent subset of V having the same number of elements as the dimension of V must be a basis so it must be a spanning set in particular so any vector in Rn I must be able to write it as a linear combination of these vectors.

So I have these but let us rewrite this using the matrix A can you see that this is a1, a2, etcetera, an into the vector x1, x2, etcetera, xn now this is this can be easily verified it is x1 a1 plus x2 a2 etcetera plus xn an that is this expression but this is my a so this is equal to Ax what I have shown is that then e1 Ax equal to e1 that is this system Ax equal to e1 that has a solution that comes from the fact that e1 belongs to the span of the column vectors of A what I have done for e1 can be done for e2, e3, etcetera.

Let me simply say similarly Ax equals ei all these systems have a solution for 1 less than or equal to for i greater than or equal to 1 less than or equal to 2 i varying from 2 to n all these systems have a solution, okay what I will do is let me denote the solution for the first system by Ax 1 and denote the solution of the ith system by Axi I will collect the column vectors x1, etcetera, xi denote that by capital X.

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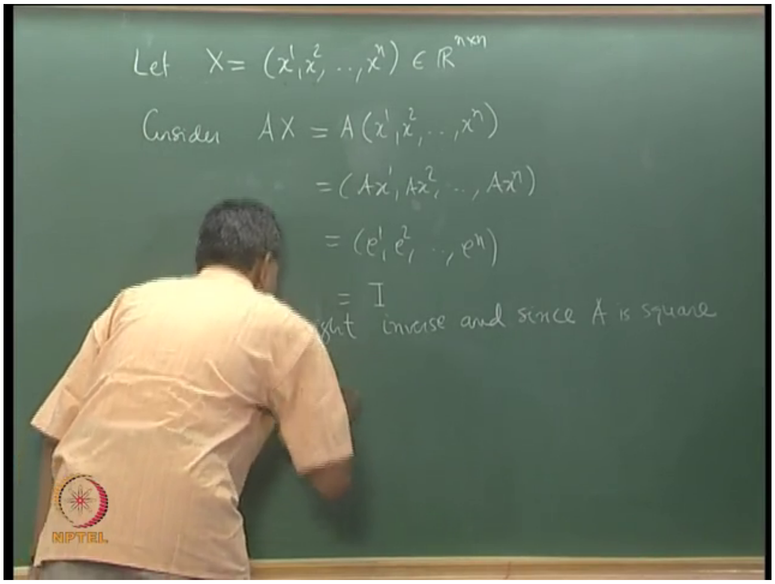

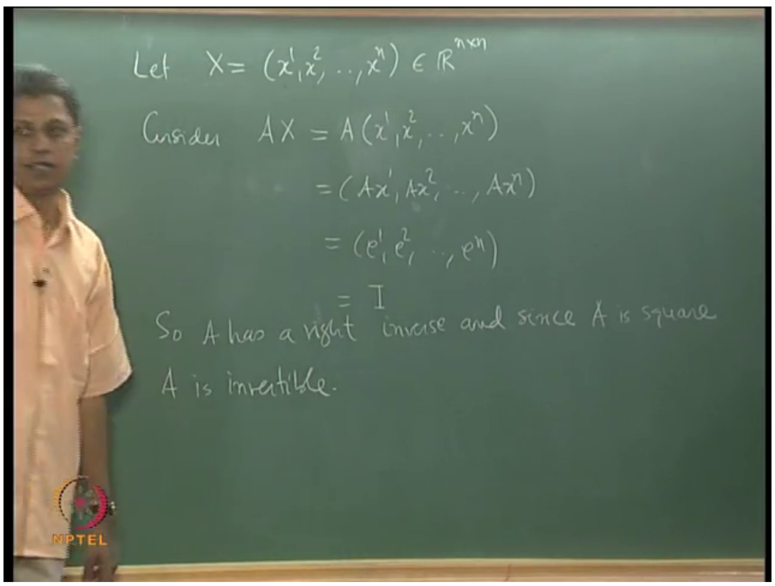

Let capital X equal x1, x2, etcetera, xn please observe that for a given vector it is components are denoted using subscripts and different vectors are denoted by using super scripts different vectors are denoted by using super scripts e1, e2, etcetera appearing here. Components will be x1, x2, etcetera, xn components of x1 are these, okay so I have a matrix now this belongs to Rn cross n remember that each of these is a column vector so this is an n cross n matrix.

Now consider AX A multiplied with the matrix X this is a1, a2, okay let me write this as x is given here it is Ax x1, x2, etcetera, xn using matrix multiplication you can verify that this is the same as Ax1, Ax2, etcetera, Axn using matrix multiplication this can be verified but this is e1, e2, etcetera, en e1 is 100, e2 is 010 etcetera so can you see that this is identity of order n, okay and so what we have shown is that this matrix X satisfies the equation Ax equal i again go back to a result that we proved if a matrix if I have a square matrix which has either a left inverse or a right inverse then it must be invertible so the conclusion follows.

So A has a right inverse and since A is square A is invertible, okay this is one of the results which I taught you must know, okay so let us move on I want to discuss the formula for the dimension of a sum of two subspaces, what is the formula for the dimension of the sum of two subspaces let me give a motivating example and then prove this general result.

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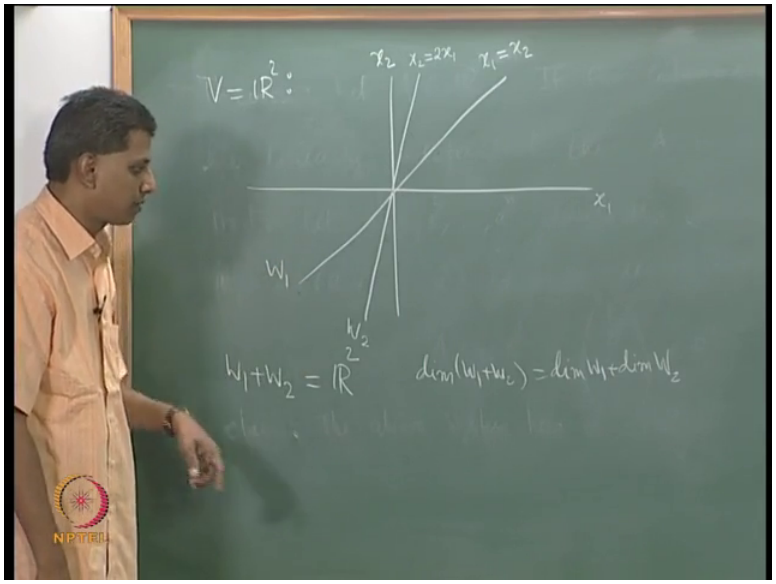

Let us look at the case of R2 and, okay R2 I have a horizontal axis I have a vertical axis let us look at this is a origin let us look at a line like this, okay this is so called y equal to x I will use x1 equals x2 let me use some other line x1 equals x2 equals 2x1 something like this this is x2 is the height x2 is 2x1, okay they are supposed to pass through the origin let us call this as a subspace what I know is that any line passing through the origin is a subspace so this w1 is a subspace this is another line w2 is a subspace, okay I want to look at w1 plus w2, okay the dimension of R2 is 2 dimension of w1 is 1 dimension of w2 is 1 what we observe here is that dimension of w1 plus w2 is equal to dimension w1 plus dimension w2, in this particular example w1 plus w2 I am saying has dimension 2 w1 plus w2 I am claiming is the whole of R2, what is the reason for that? The reason is as follows I must show that I must show that w1 plus w2 has a basis consisting of two elements then it will follow that w1 plus w2 is two dimensional that is equal to R2 but take any vector lying on w1 call that u take another vector on w2 call that V since they are in two different lines one is not a multiple of the other because if one where a multiple of the other they would lie on the same line.

So take any vector on w1 call that u on w2 u2 V u and V are linearly independent u and V span w1 plus w2, is that obvious? Anything in w1 can be written as a multiple of u anything in w2 can be written as a multiple of V. So anything in w1 plus w2 can be written as a linear combination of u and V and so w1 plus w2 is R2 in this example the sum of these two subspaces is equal to the vector space the original space that we started with and what we observed is that so dimension of w1 plus w2 it is equal to dimension w1 plus dimension w2, okay.

We look at a generalization of this particular example in a general vector space not necessarily R2 but this does not hold always in a general vector space what we have not observed what we have not made use of is the fact that the intersection is singleton 0, the intersection w1 intersection w2 in this example is a singleton 0. So we will show that this holds if w1 intersection w2 is singleton 0, okay.

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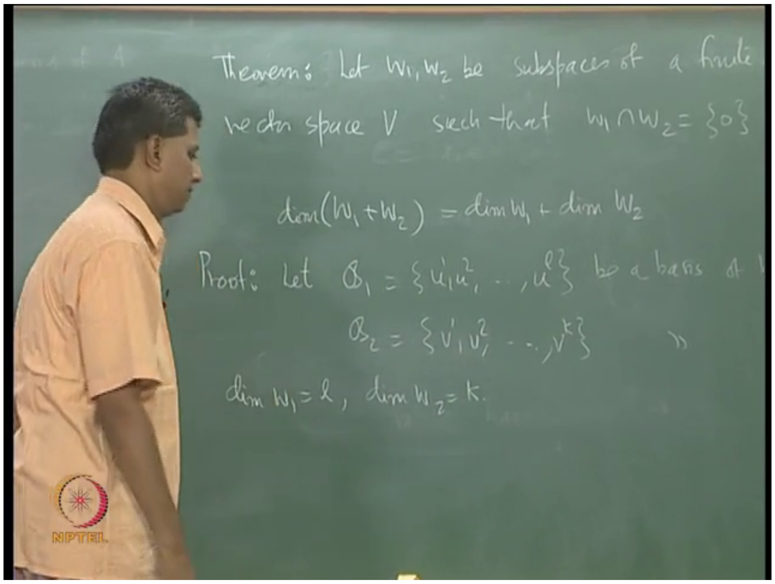

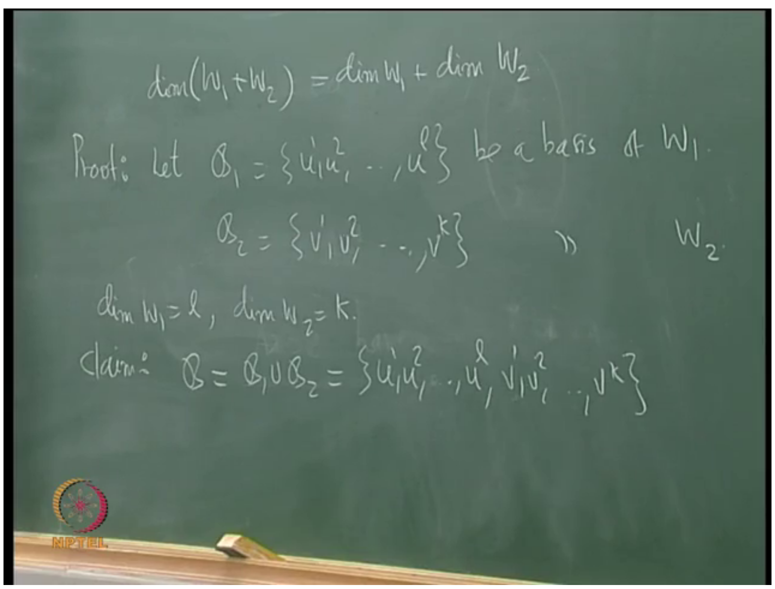

So we want to prove a result that generalizes this equation, first let me take the case when the intersection is singleton 0 so I would like to prove the following result, let w1 and w2 be subspaces of a finite dimensional vector space V such that w1 intersection w2 is singleton 0.

Remember that w1 intersection w2 must be a subspace intersection of two subspaces must be a subspace so it must have at least a 0 vector, here it has at most 0 it has precisely the 0 vector then we have the following formula dimension of w1 plus w2 this is dimension w1 plus dimension w2, okay we will next prove a result which is generalization of this but for that we need the notion of extending a basis for a subspace to a basis for the entire space, so let me first give this result give that a little later. So I want to prove this result let us observe that this is well defined w1 plus w2 was defined earlier and we know that it is a subspace, so one can talk about the dimension of a subspace what we also know is that dimension of the subspace cannot exceed the dimension of V, V is finite dimensional.

Similarly these two numbers these two are integers w1 and w2 are subspaces so these two numbers are well defined, okay how do we proof? The proof is probably along expected lines that is let us take the idea of the proof is as follows you take a basis for w1, take a basis for w2 simply join them that will turn out to be a basis for w1 plus w2 in this case because the intersection is singleton 0 otherwise it won't be. So let me take b1 to be u1, u2, etcetera, ul let b1 be a basis of w1 u1, u2, etcetera, ul that is a basis of w1, b2 will be v1, v2, etcetera, vk this will be a basis of w2, so I have taken a basis for w1 and a basis for w2 I know there dimensions I have written down explicit basis.

So dimension w1 is l, dimension w2 is k, what is a claim as I mentioned let me call it b, b is a union of these two basis b1 union b2 this is u1, u2, etcetera, ul v1, v2, etcetera, vk the claim is that this is a basis of w1 plus w2 this is the claim is that this is the basis of w1 plus w2, okay to prove that this is a basis of w1 plus w2 we need to verify the two conditions that this is linearly independent and a spanning subset.

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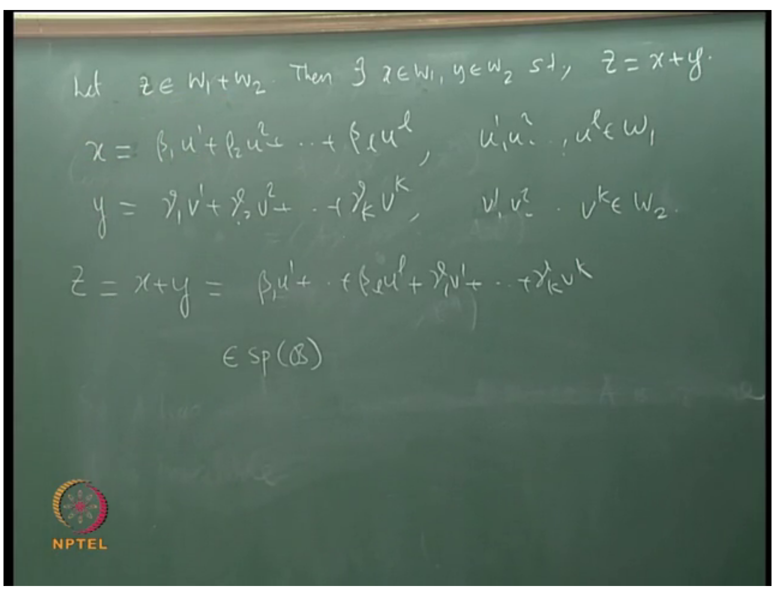

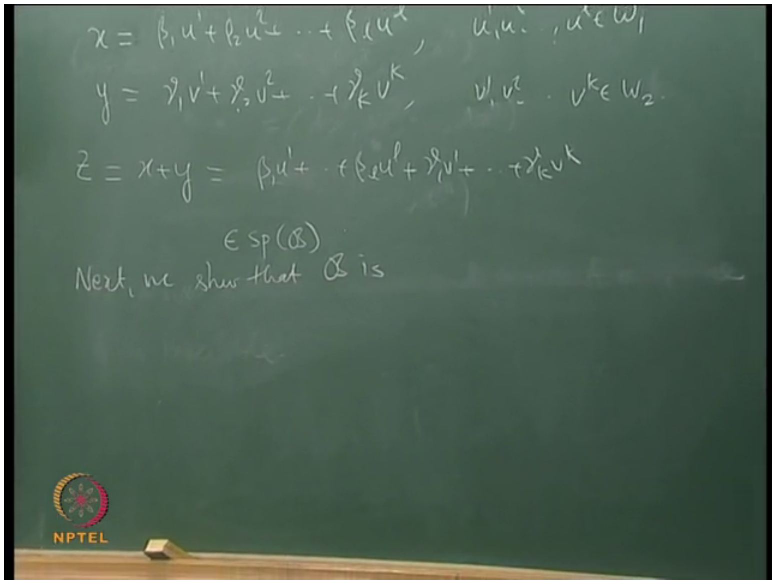

To verify that it is a spanning subset that is almost there let us take z to be in w1 plus w2, then there exist x in w1, y in w2 such that z can be written as x plus y that is a definition of w1 plus w2. Now x is in w1 and w1 has this as a basis so x is a linearly combination I will call it beta 1 u1, beta 2 u2 plus etcetera beta l ul where I observed that u1, u2, etcetera, ul of course belongs to w1 that is coming from the basis y is in w2 that is a linear combination of v1, v2, etcetera, vk gamma 1 v1 plus gamma 2 v2 gamma k vk of course v1, v2, etcetera, vk come from w2 just to emphasize.

Then look at z, z is x plus y that is beta 1 u1 etcetera beta l ul plus gamma 1 v1 plus etcetera plus gamma k vk no more interpretation of what the right hand side is I simply observe that this belongs to span of script b it is a linear combination of the vectors of b and so what we have shown is that this is a spanning set I started with an arbitrary z in w1 plus w2 I have shown that element is in span of b.

So w1 plus w2 is equal to span of b we need to now show that this is the basis the last step then is to show that b is linearly independent, okay next we show that this set b is linearly independent, that is u1, u2, etcetera, ul v1, v2, etcetera, vk are linearly independent vectors next to show that b is linearly independent, okay to show that they are linearly independent you must consider a linear combination equate that to 0 show that the scalars are 0 so let me do it on this side.

Remember till now we have not made use of the fact that w1 intersection w2 is singleton 0 we will use that now.

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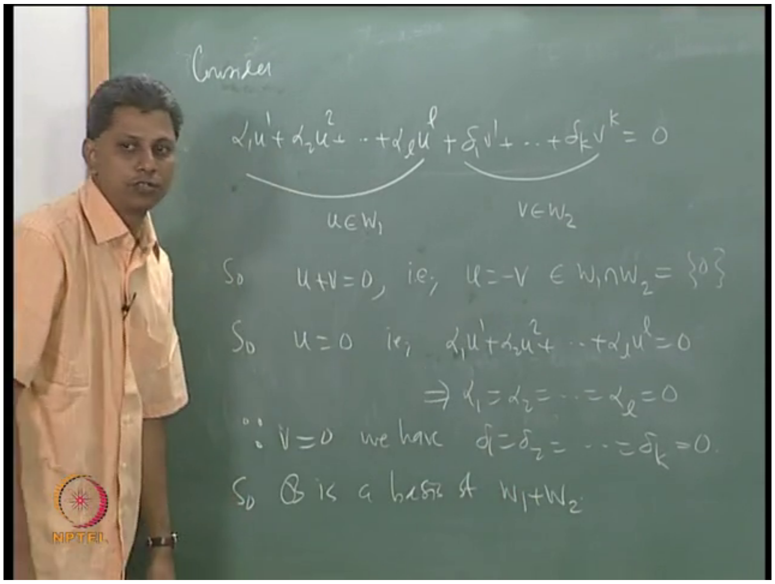

Consider a linear combination let us say alpha 1 u1, alpha 2 u2 etcetera alpha l ul plus some delta 1 v1 etcetera delta k vk equate that to 0. Now this is a vector in w1 I will call this u is a vector in w1 this is a linear combination of u1, u2, etcetera, ul which are in w1 so this is a vector in w1 this is a vector in w2 I will call this v, okay.

So what I have is u plus v equals 0 that is u equals minus v, okay but u is in w1 v is in w2 this means u for instance belongs to w1 intersection w2 which is a same as saying v belongs to w1 intersection w2 let me just say this belongs to w1 intersection w2, on the left hand side I have a vector in w1 on the right hand side I have a vector in w2 they are equal and so u or v belongs to w1 intersection w2 which I know is singleton 0 which means let us say u equals 0 u is a vector but go back and see what u is that is alpha 1 u1 plus alpha 2 u2 etcetera alpha l ul this is 0 but now I make use of the fact that u1, u2, etcetera, ul they form a basis so they are linearly independent so from this it follows that alpha 1, alpha 2, etcetera, alpha l they all must equal 0.

So I have taken care of the first part the coefficients corresponding to the part they are 0, similarly v is 0 because v is minus u since v is also 0 it follows that by a similar argument it follows that delta 1 equals delta 2, etcetera equals delta k equals 0, okay so what we have done is to start with a linear combination of the vectors un etcetera ul v1 etcetera vk equate that to 0 we have shown in the last step that this holds only if the scalars are 0 so the vectors are linearly independent so b is a basis of the sum w1 plus w2 now you see the formula holds, yes sir u belongs to w1 v belongs to w2 but u is equal to minus v which means that v is already in w2 but since v is minus u and u belongs to w1 v also belongs to w1 v is already in w2 by virtue of this equation v is minus u w1 is a subspace if u belongs to w1 minus u belongs to w1 and so v to w1 also.

So v belongs to w1 intersection w2 as well as u because again w1 intersection w2 is a subspace so if I have a vector which is in the subspace its negative will also be in that subspace, I hope it is clear, okay.

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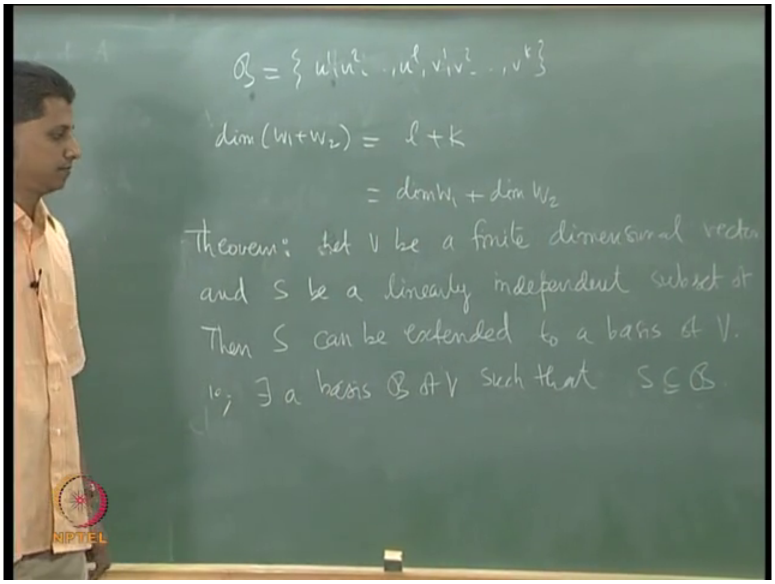

I have written down how does a formula hold I have written down the explicit basis for b in terms of basis of w1 and w2 basis of w1 plus w2 that is u1, u2, etcetera ul v1, v2, etcetera, vk we have shown that this is a basis of w1 plus w2 so look at dimension of w1 plus w2 l plus k but l is a dimension of w1 and k is a dimension of w2 u1 etcetera u2 etcetera ul is a basis for w1 v1 etcetera vk is a basis for w2 so dimension of w1 is l dimension of w2 is k so I have this equation, okay. There is a more general formula we will prove it a little later because we need a little result before that, okay.

Now what is a result that we will need to prove this general formula let me state this, let v be a finite dimensional vector space and S be a linearly independent subset S is a linearly independent subset of a finite dimensional vector space V then S is part of a basis of V that is S can be extended to a basis of V any linearly independent subset of a finite dimensional vector space can be extended to a basis of V, what is the meaning of this? The meaning of this is let S be a linearly independent subset of a finite dimensional vector space V let us script b be a basis of V, I am sorry there exists a basis script b of V such that S is contained in B that is there exists a basis script b of V such that this S is contained in script b, okay the proof will make use of a result that we proved earlier, okay.

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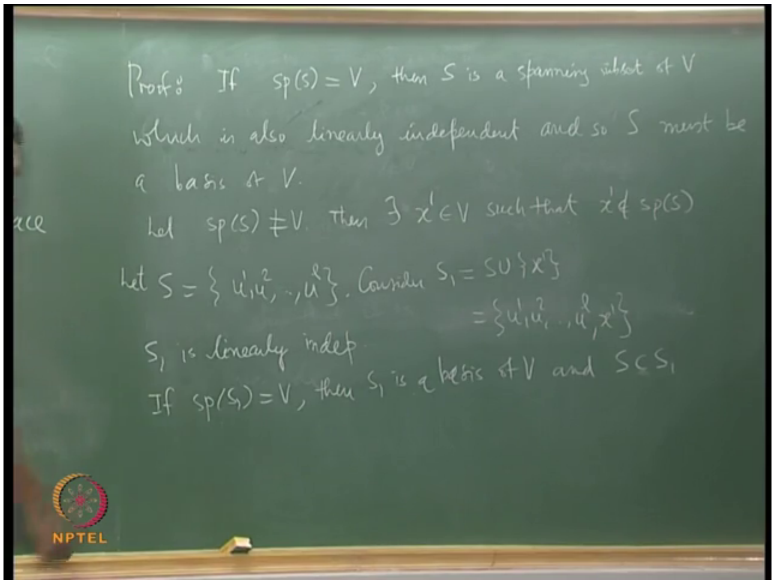

What is given is that S is linearly independent, okay if span of S equals V then there is nothing to prove, why? If span of S is equal to V then S is a spanning subset of V which is also linearly independent, sorry yeah which is also linearly independent and so S must be a basis, in this case that is a span of S is the whole of V then there is nothing to prove already S is linearly independent if span of S is equal to it follows that S is a basis.

So trivially S is part of a basis of V, if span of S is not equal to V so let us consider the case that span of S is not equal to V span of S is a subspace it is contained in V it is not equal to V means there is a vector in V which is not in span of S, then there exist a vector I will call it x1 in V such that this vector does not belong to span of S. Now let me write S explicitly I have S to be equal to let us say u1, u2, etcetera, ul I know that this is linearly independent I am now appending x1 to this set S consider S1 equals S union x1 this is u1, u2, etcetera ul x1 now we have encountered this situation before since x1 does not belong to span of S it follows that this set S1 since x1 does not belong to span of S it follows that set S1 is linearly independent.

If this set S1 where linearly dependent then there is a vector which is a linear combination of the preceding vectors that cannot happen for u1, u2, etcetera, ul because they are already linearly independent the only way that S1 is linearly dependent is that this x1 is a linear combination of u1, etcetera, ul but that does not happen because x1 does not belong to span of S so this S1 is linearly independent, okay.

S1 is linearly independent then it is like going back to this if then, if span of S1 equals V we are done it would then follow that S1 is a basis as before if span of S1 equals V then S1 is a basis of V, and what we want to show? We want to show that this x can be extended to a basis it is clear by the construction that S is contained in S1. So if S1 is basis then we know that S is contained in S1 and so the linearly independent subset S that we started with is a subset of a basis S1, if it is not equal to V we repeat the procedure, okay but remember what happens in this case, in this case in the first case let me go back and write down the inequalities for the dimension why should this process terminate, okay that is a question why should this process terminate.

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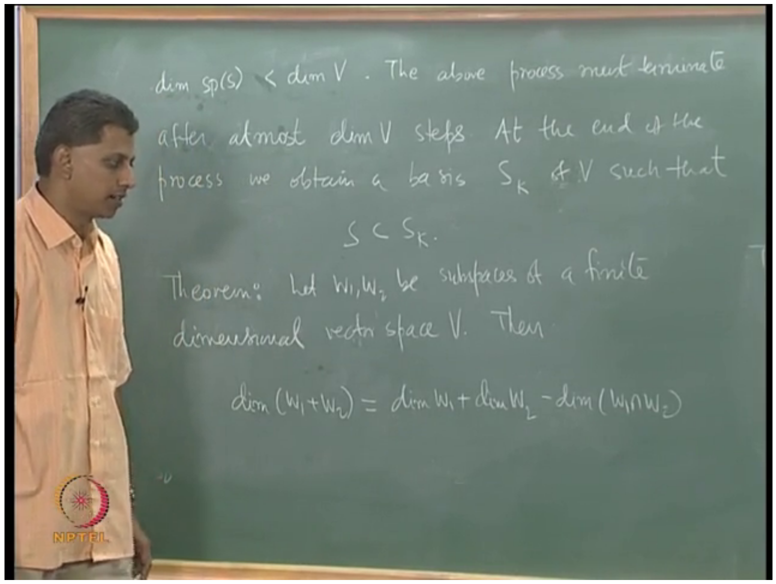

In the first case what is the dimension of the span of S if it is equal to the dimension of the original space then they are equal so in the second case this is less than or equal to dimension of V strictly less than dimension of V, that is this corresponds to dimension of span of S being equal to dimension of V so the subspace is equal to V so S is a basis in this case we know that there is nothing to prove we are looking at the case when the span of S is not equal to V this is a subset not the whole space, so the dimension of this must be strictly less than dimension of V.

In the next step the dimension of the dimension has been increased by 1, S1 has one element more than S so the dimension of this subspace has been increased so these are integers these are positive integers dimensions are positive integers, so from the previous step we have bridged the gap by one number one integer so span of S is now it is quite possible it is equal to V but if it is not equal to V we will definitely it is clear that the dimension of this subspace is one more than the dimension of span of S.

So this process has to terminate because V is finite dimension, since V is finite dimensional the above process that is argument the above process must terminate after at most dimension V steps in fact it is less than this it is at most after dimension of V minus dimension span S steps but in any case it does not exceed this number. Now since this is finite and every time you are increasing the dimension by 1 this procedure must stop, okay.

So at the end of the procedure you have a subset let us say Sk which is a basis, okay maybe I will just conclude at the end of the procedure at the end of the process we obtain a basis Sk such that a basis Sk of V such that S is contained in Sk which is what we wanted to prove that this S is a part of a basis of V, okay. So let us make use of this result there is a counterpart to this result probably I will leave that as an exercise the counterpart is as follows, remember this process in formally what it means is that you can start with a basis rather you can start with a linearly independent subset of a vector space if the vector space is finite dimensional you can go from this linearly independent subset to a basis.

A maximal linearly independent subset is a basis a minimal spanning set is also a basis, okay a maximal from a linear independent set you are adding one vector at a time. So a maximal linearly independent set as well as a minimal spanning set must be basis of a finite dimensional vector space, okay I will leave that problem as an exercise for you to solve. Now using this I would like to derive the particular identity for the sum of two subspaces in the general case when the intersection is not necessarily singleton 0, okay.

So what I want to prove is this theorem, let w1, w2 be subspaces of a finite dimensional vector space V then dimension of the sum w1 plus w2 is the sum of the dimensions remember this time I have removed the restriction that the intersection must be singleton 0 I need to bring in that here in the right hand side minus dimension w1 intersection w2, okay so this is the formula then for general subspaces w1, w2 formula for the sum of two general subspaces w1 and w2 you can now see that this is more general then the result that I proved today because of the fact that we also assume the dimension w1 intersection w2 is 0 in that case so we assume that w1 intersection w2 is singleton 0 so dimension is 0 so this number does not appear so I have this formula.

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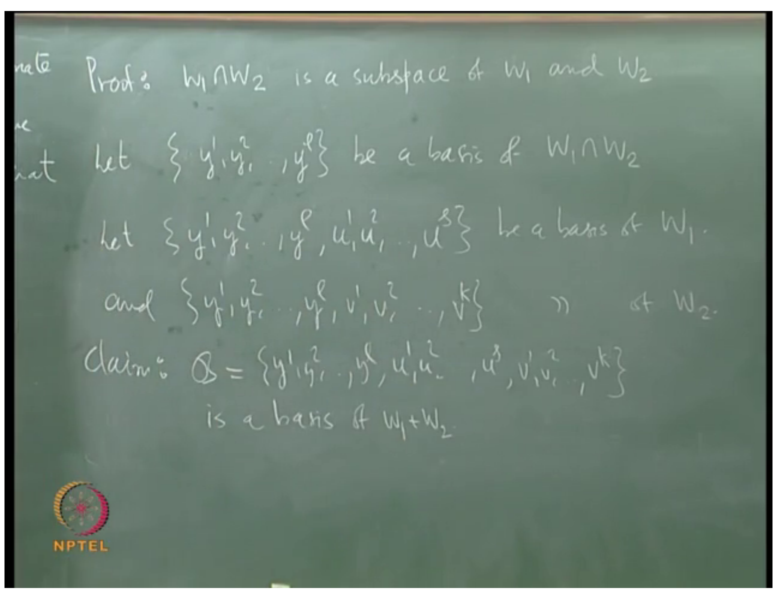

So this reduces to the earlier formula, okay the proof you will see is somewhat similar in the earlier idea of the earlier proof will be used, okay proof as before, okay this time I will do it little differently look at w1 intersection w2 this is not only a subspace of V it is a subspace of w1 as well as w2 is a subspace of w1 as well as w2, okay. So what I will do is start with a basis for w1 intersection w2, okay this time I will not use b1, b2, etcetera I will write down this explicitly let y1, y2, etcetera, yl be a basis of w1 intersection w2 this is a basis of w1 intersection w2 this is a subspace of w1, w1 itself is a subspace so w1 is a vector space in its own right I have a linearly independent subset of a vector space w1 this can be extended to a basis of w1 all the spaces are finite dimensional V is finite dimensional so w1, w2, wn intersection w2 all these are finite dimensional.

So I am now making use of the previous result that this being a linearly independent subset of w1 and w1 is a vector space this can be extended to be basis of w1, similarly w2. So let me write down a basis extending this explicitly let y1, y2, etcetera, yl comma I will u1, u2, etcetera, us let this be a basis of w1 this is possible by the previous result and y1, y2, etcetera, yl v1, v2, etcetera, vk I will use k be a basis of w2 I started with a basis of w1 intersection w2 I can extend these basis to basis of w1 as well as w2. So you observe the first part is coming from this basis of w1 intersection w2 for these two the first part comes from this, okay.

Now the claim is you see that this vector repeats so what I will do is I will collect y's, u's and v's I will call this now as script b collect y's, v's and u's y1, y2, etcetera, yl u1, u2, etcetera, us and v1, v2, etcetera, vk collect all these vectors the claim is that this is a basis of w1 plus w2. Let us assume for the moment that we have proved that this is a basis of w1 plus w2 let us quickly verify whether the consequence that this formula holds.

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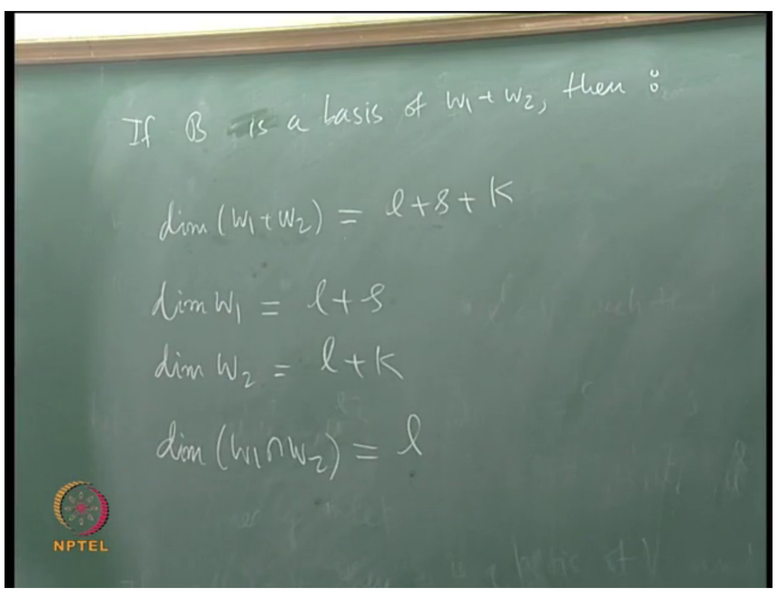

Suppose this is a basis of w1 plus w2 then if script b is a basis of w1 plus w2 then we have the following dimension of w1 plus w2 is equal to since this script b is a basis you observe that it has l plus s plus k vectors so this number will be l plus s plus k.

Now look at dimension w1 I want to write down dimension w2 also w1 has this as a basis, w2 has this as a basis so w1 dimension is l plus s w2 dimension is l plus k we stared with w1 intersection w2 dimension w1 intersection w2 please refer to the basis that I have there it is l, so you can see that this number l plus s plus k is this plus this minus this so if I prove that this b is a basis then I am through, okay this number is this number plus this number minus this that is the right hand side this is a left hand side, okay.

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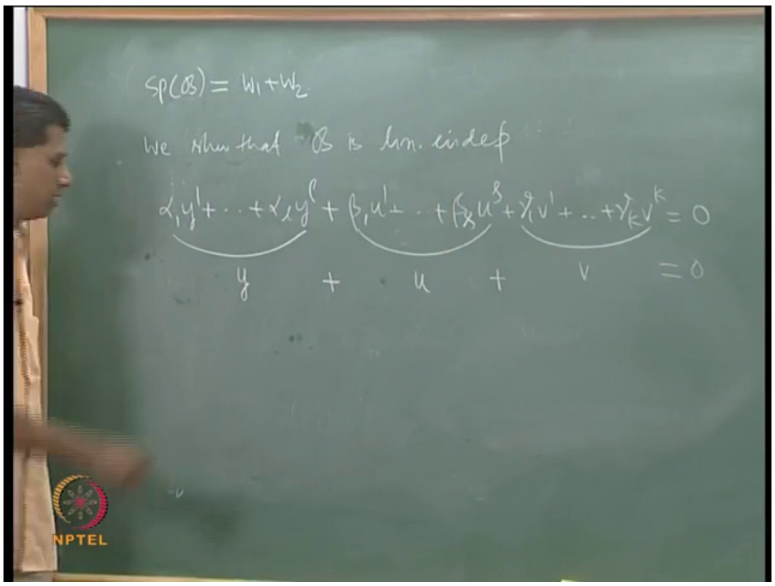

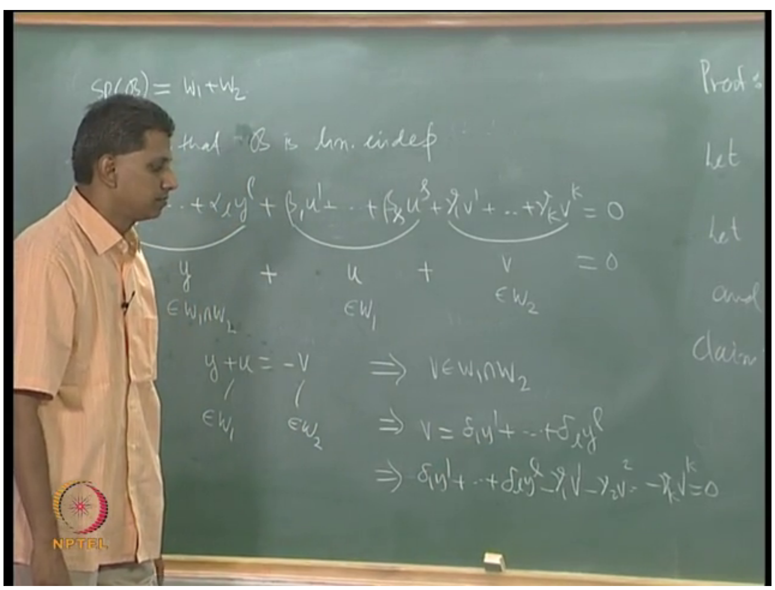

So we need to prove that b is a basis that it is a spanning set is easy to see that is as before, so I will not write the fact that I will not prove the fact that span of b equals w1 plus w2 that is easy as before we will only show linear independence, okay we show that b is linearly independent this spanning thing is as before there is no need to repeat the argument, okay we need to consider a linear combination then I will call it alpha 1 y1 etcetera plus alpha l yl plus beta u1 etcetera plus beta s us plus gamma 1 v1 etcetera plus gamma k vk I must equate this to 0 and then show that each of the scalar is 0, okay.

You must show that each of the scalars is 0 so this is what I have as before this is a vector in w1 intersection w2 let me call this y this vector I will call it u this is in w1 this is in w2 I will call this v, y plus u plus v equals 0 y is in w1 intersection w2 let me write this belongs to w1 intersection w2 this belongs to w1 u is a linear combination of u1 etcetera us this is in w1 similarly this is in w2, so let me rewrite this as follows y plus u equals minus v y plus u equals minus v y belongs to w1 u belongs to w1 so this belongs to w1 v belongs to w2 a similar argument as before this belongs to w2 so v belongs to w1 intersection w2, is that clear? v belongs to w1 intersection w2 v or minus v w1 intersection w2 is a subspace.

Now w1 intersection w2 has this as a basis, okay that is this means v can be written as delta 1 y1 etcetera plus delta l yl because y1, y2 etcetera, yl is a basis of w1 intersection w2 but look at what v is write down the expanded form of v this implies I can write this and then equate this or push everything to one side and write this as delta 1 y1 etcetera plus delta l yl minus v equals 0 that is minus gamma 1 v1 minus gamma 2 v2 etcetera minus gamma k vk equals 0, this minus v equals 0 is what I have written down here but look at this last equation this is a linear combination of y1 etcetera yl v1 etcetera vk look at what I have here y1 etcetera yl v1 etcetera vk that is a basis they are linearly independent.

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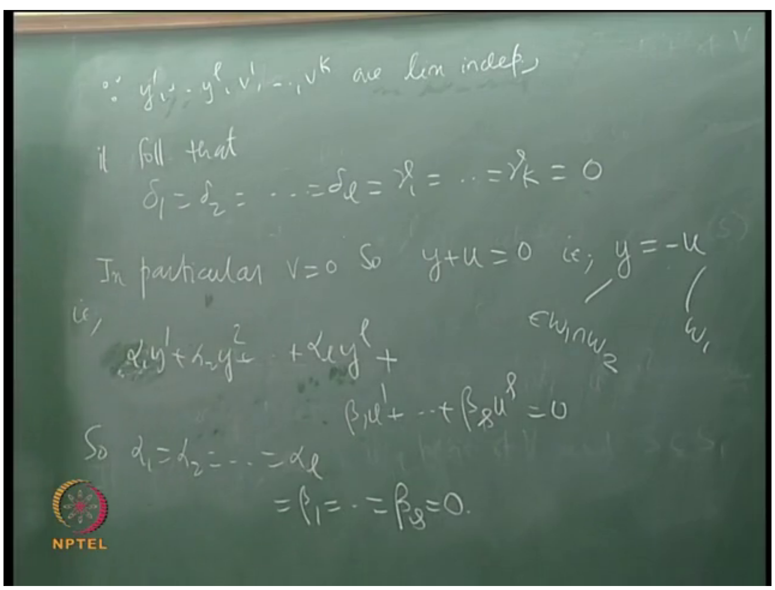

So this equation tells me each of the scalars here is 0, is that okay? Maybe I will use that part, since y1 etcetera yl v1 etcetera vk are linearly independent vectors it follows that these scalars delta 1 equals delta 2 etcetera equals delta l equals gamma 1 etcetera equals gamma k equals 0 in particular gamma 1 etcetera gamma k equals 0 what that means is that gamma 1 etcetera gamma k each of these scalars is 0 so I go back to this definition of v v is gamma 1 v1 etcetera gamma k vk so v is 0 from this v is 0 then I get the equation y plus u is 0, okay in particular v is 0 so y plus u equals 0 that is y equals minus u y belongs to w1 intersection w2 u belongs to w1, so both in particular belong to w1 write down the equations again you will be able to show that y as well as u are 0, okay that is, okay let us we already have the expressions for y and u and it follow immediate.

So let me write down this quickly that is I have y1, I am sorry what is this scalar alpha 1 y1 plus alpha 2 y2 etcetera plus alpha l yl this is y plus u there is no need to go to this plus u, u is what is the scalar for u, beta plus beta 1 u1 etcetera plus beta s us equals 0 this is what I have from the equation y plus u is 0 but y1, y2, etcetera, yl u1, u2, etcetera, us they form a basis of w1 so they are linearly independent so it follows that alpha 1 equals alpha 2 etcetera equals alpha l equals beta 1 etcetera equals beta s equals 0, I have simply used the definition of y and u and the fact that y1, etcetera, yl un etcetera us are linearly independent being part of being members of this basis, okay so let me stop with this, this concludes our discussion on vector spaces, subspaces, linear independence, basis, dimension.

From the next lecture onwards I will discuss the notion of linear transformations matrices of linear transformations, properties etcetera.