Linear Algebra
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Lecture 12
Dimension of a vector space

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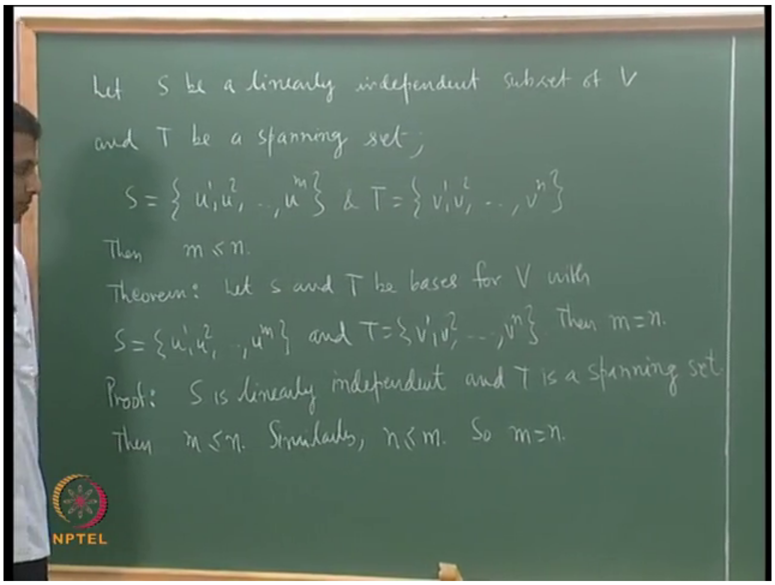

Okay, yesterday we had discussed this result let S be a linearly independent subset of a vector space V and t be a spanning set, also assume that S has finitely many elements as well as t S is u1, u2, etcetera, um and t is V1, V2, etcetera, Vn, okay. Then what we had shown yesterday was that m is less than or equal to n, okay. Let us use this result and prove that any two basis for a finite dimensional vector space will have the same number of elements.

So let us consider the following I want to prove this theorem let S and T be basis for a vector space V with S equal to u1, u2, etcetera, um and T equals V1, V2, etcetera, Vn that is I have two finites sets that are basis for a vector space V then they must have the same number of elements that is a conclusion then m is equal to n, okay two finite subsets of a vector space V if they satisfy the condition that they are basis for the vector space then they must have the same number of elements, okay the proof will use this result that I stated just now.

Now S is a basis so S is linearly independent so it must be a spanning set as well as a linearly independent set S is linearly independent similarly T I will exploit the fact S is linearly independent and the fact that T is a spanning set by the above result it follows that m is less than or equal to n, okay the next part is to reverse the roles of S and T, S is a spanning set T is a linearly independent set the number of elements in T must be less than or equal to number of elements of S, so let me just say similarly n is less than or equal to m and so m is equal to n, okay.

So the number of elements in any finite basis the number of elements in any two finite basis is the same, okay.

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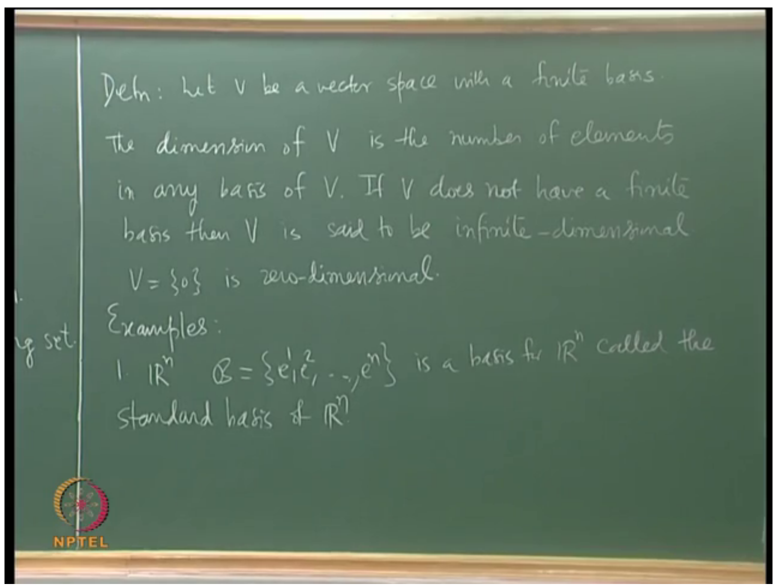

So this leads to the definition of the dimension let V be a vector space with a finite basis the dimension of V is the number of elements in any basis of V, so this is a definition of the dimension of a vector space. This number is unique by the previous theorem this number is unique and so this notion is well defined, okay if V does not have a finite basis then V is said to be infinite dimension.

Look at the trivial vector space V equal to 0 is 0 dimensional, okay the only vector space which is 0 dimensional is the trivial space single term 0. So remember it cannot be 1 dimensional because if it is 1 dimensional then the basis will have precisely one element and this one vector cannot be 0 because it must be linearly independent and so this cannot be 1 dimensional it is 0 dimensional, okay.

Let us then look at familiar examples of vector spaces that we have seen before and determine their dimensions, examples the first one is Rn for Rn we had written down the standard basis in the last lecture I will call this b, b is e1, e2, etcetera, en where e1, e2, etcetera, en are the vectors that where defined in the last lecture this is a basis this is the basis for Rn and it is called the standard basis of Rn, okay.

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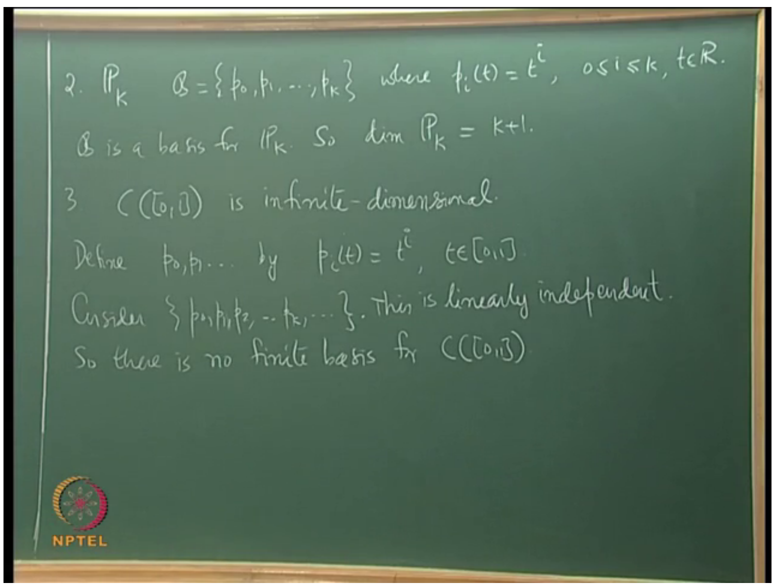

Example 2, let us look at space of polynomials of degree k for this look at the collection p not, p1, etcetera, pk where this was also defined before where pi of t is t to the i 0 less than or equal to i less than or equal to k t is a real variable, so this set of polynomials they are k in number this set is basis and so the dimension of Pk is k plus 1 there are k plus 1 vectors in that basis, okay. What is a dimension of C 0, 1? C 0, 1 is a space of all complex valued continuous functions on the interval 0, 1. What is a dimension of this space? This is infinite dimensional, what is argument for that? This is infinite dimensional, see the reason is as follows let us take the polynomials that we have defined earlier, defined p not, p 1, etcetera by pi of t equals t to the i this time t were is in 0, 1 define this polynomial that is 1, t t square etcetera.

Now this is an infinite subset consider p not, p1, p2, etcetera, pk etcetera this is an infinite subset I am claiming that this infinite subset is linearly independent. Now we had seen this definition yesterday that an infinite subset is linearly independent if and only if every finite subset of it is linearly independent, okay. So we need to show that every finite subset of this is linearly independent but is that something that we have proved before, take a finite subset, okay let us say I have the polynomials 1 t square t power 100 let us say t power 1000 then can we show that this is linearly independent you look at a linear combination equate that to 0 differentiate as many times as you want show that the coefficients are 0, okay same idea that we used earlier can be used here to show that this is an infinite subset, linearly independent subset this is linearly independent subset of C 0, 1 every polynomial function is continuous so these are first members in C 0, 1.

Now can you get a contradiction, if there is a finite basis if there is a finite basis then the previous theorem would be contradicted, so this cannot have a finite basis, okay. The number of if it is finite dimensional then the number of elements in any two basis will be the same but we have we need to use a lesser known fact which is if you have a linearly independent subset and the spanning set then the number of elements in the spanning set must be greater than or equal to number of elements in the spanning set must be greater than or equal to the number of elements in the linearly independent set that is clearly not possible here, if it has a finite basis then this is clearly violated, okay so there is no finite basis for this vector space, okay I have given the reasoning to fill up the details here.

If it has the finite basis then this would violate the inequality m less than or equal to n, okay. So this is an example of infinite dimensional vector space.

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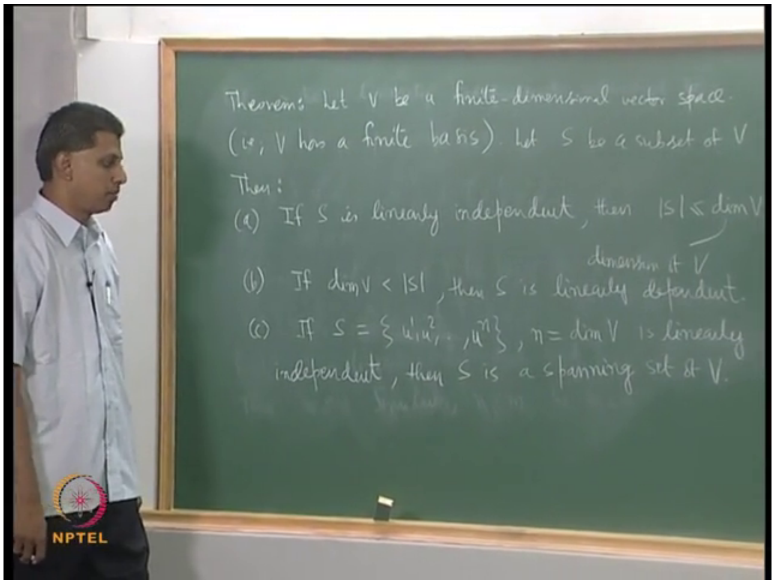

Let us also prove a few more consequences of the previous result, let V be a finite dimensional vector space that is V has a finite basis let S be a subset of V then we have the following if S is linearly independent then the number of elements of S cannot exceed the dimension of V if S is linearly independent then the number of elements in S this is the cardinality of S that cannot exceed the dimension of V this stands for the dimension this notation will be used for denoting the dimension that is one property, property 2 if dimension of V is less than the number of elements in any linearly independent subset in any subset S then S is linearly dependent, okay you will immediately notice that statement b is the contra positive equivalence of statement a so statement b does not really need to be proved.

Condition C, if S is property C rather if S is linearly, if S equals let us say u1, u2, etcetera, un with n denoting the dimension of V if this S is linearly independent that is I take a subset which is which has n elements where n is a dimension of the vector space then if it is linearly independent then it must be spanning set then S is a spanning set of the vector space V, okay a linearly independent subset having the same number of elements as any basis would have has to be a basis a linearly independent subset having the same number of elements as that of a basis of a finite dimension vector space must be a basis, okay that is property C, okay.

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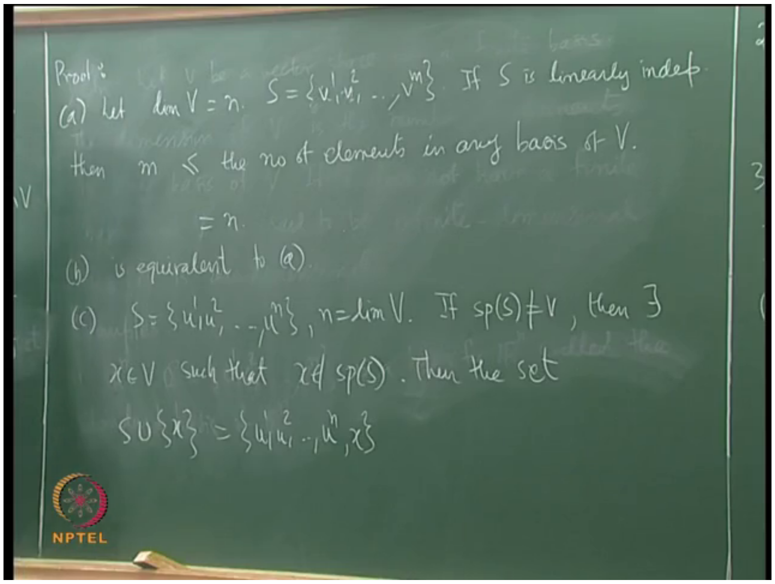

Proof the first one is something we have already proved, okay you have to just put the framework there, let us take dimension of V to be n this will be useful in part C also and write S as u1, u2, etcetera let us say V1, V2, etcetera, Vm then we have seen earlier that if S is linearly independent then the number of elements in S that is m that cannot exceed the number of elements in any basis of V, m is less than or equal to the number of elements in any basis of V this we have seen before, but what is a number of elements in any basis of V? That is precisely n because any basis will have n elements since the dimension of V is m, okay so this number is n so m is less than or equal to n that is condition a holds property a holds the number of elements in any linearly independent subset cannot exceed the dimension of the vector space.

b is the contra positive equivalence so I will simply say b is equivalent to a, is not prove for that. Property c, S is given to be u1, u2, etcetera, un where n is a dimension of V we must show that S is a spanning set suppose S is not a spanning set we will get a contradiction, okay. If span of S is not equal to V we want to show that S is a spanning set for the vector space V which means you must show that span of S is equal to V if span of S is not equal to V then there exist a vector let us call it x, x is in V such that x does not belong to span of S I have two sets which are not equal and one is contained in the other span of S is always contained in V that is a subspace in fact that we have seen.

So one is a subset of the other but these two sets are not equal which means the super set has an element which does not belong to that subset there exist an x in V such that x does not belong to span S I will look at this set S union x, okay include this x define a new set this has these elements u1, u2, etcetera, un and x. Now since x does not belong to span of S what it means is that x is not a linear combination of the preceding vector x is not a linear combination of u1, u2, etcetera, un so this is an linearly independent set this is linearly independent, okay let us recall the argument if this set S union x where linearly dependent then there is at least one vector which is a linear combination of the preceding vectors.

Now that cannot happen for u1, u2, etcetera, un because they are already linearly independent, so the only possibility if this set is linearly independent is that this x is a linear combination of u1, u2, etcetera, un we know that is not possible because x does not belong to span V so this set must be linearly independent. Now this is linearly independent and the number of elements here is n plus 1 so that cannot exceed n but this is a greater number and so you have a contradiction.

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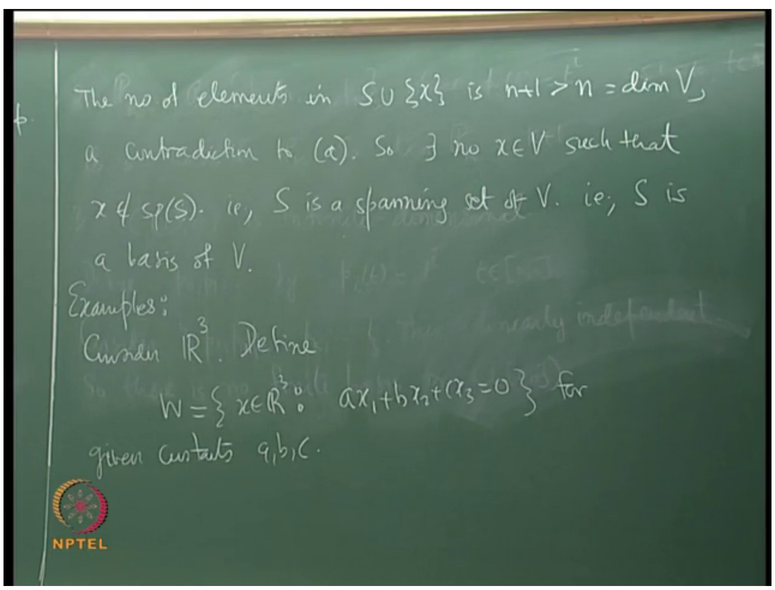

And so this cannot happen that is x there is not x in V which does not belong to span V the number of elements in the set S union x this is n plus 1 this is greater than n which is the dimension of V this contradicts there is a contradiction to property a that we proved just now the number of elements in any linearly independent subset cannot exceed the dimension of the vector space is a contradiction and so there exist no x in V such that x does not belong to span of V that is S is a spanning set of V that is S is a basis of the vector space V, okay so you can this is what we say the dimension is the number of elements in any maximal linearly independent subset or the number of elements in any minimal spanning set, okay.

Let us now look at some examples of subspaces and determine their dimensions, okay examples of subspaces there dimensions, let us first look at R2, okay let us say I start with R3 consider R3 look at the following subspace the set of all x in R3 such that ax 1 plus bx 2 plus cx 3 equals 0 for given constants a, b, c look at the collection of all x that satisfy this single condition what we know is that this is there is a certain plain passing through the origin the constant term is 0 there is a plain passing through the origin we have verified earlier that this is a subspace, okay let us determine its dimension let us determine the dimension of this subset that is a plain in R3 intuitively we know that the dimension must be 2 so let us prove that this has a basis consisting of precisely 2 vectors, okay.

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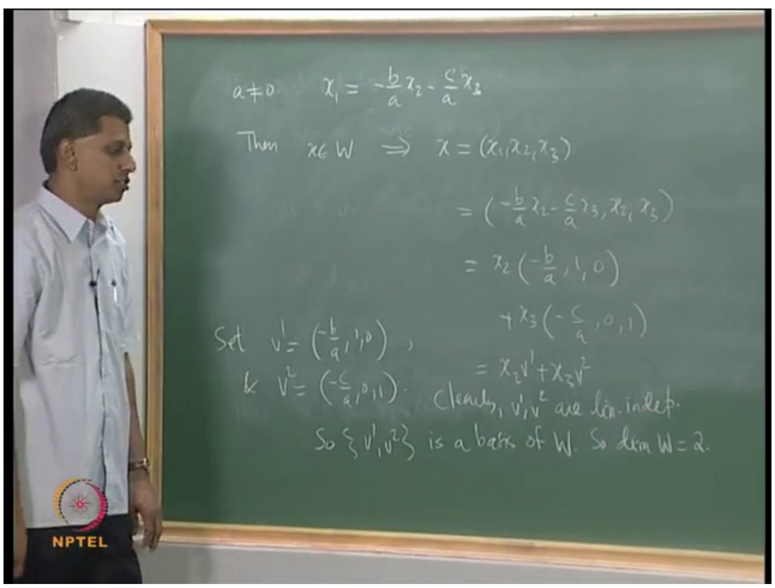

Without loss of generality let us assume a is not 0 at least one of these constant must be non-zero, okay this is a non b generate plain non-trivial plain. So take a to be non-zero then I can write x1 as minus b by ax 2 minus c by ax 3 remember I started with a single equation ax 1 plus bx 2 plus cx 3 is 0 single equation and three unknowns x1, x2, x3 a, b, c are given constants so I can fix one of them and determine I can fix two of them determine one in terms of the other two. So I have fixed x2 and x3 determining x1 in terms of x2 and x3 then any vector x is W can be written as x is x1, x2, x3 that is x1 is minus b by ax 2 minus c by ax 3 x2 x3 are arbitrary this can be rewritten as minus let us say x2 into minus b by a 1, 0 plus x3 into minus c by a 0, 1.

See all that I have done is to write x as a linear combination of these two vectors let us call this as V1, this as V2 set V1 to be minus b by a 1, 0 this time I am writing the vectors as row vectors set V1 to be this and V2 to be minus c by a 0, 1, okay then what is a claim? The claim is that these two vector form a basis for W so this must these two vectors must be linearly independent and they span W, okay.

The fact that these two vectors span W has been demonstrated here any x is W can be written as a linear combination of V1 and V2, is that okay? That is in this notation this is equal to x2 V1 plus x3 V2 I have written any x in W as a linear combination of V1 and V2 and so this is a spanning set these two span W is clear linear independence are they linearly independent, one is not a multiple of the other so these are obviously linearly independent clearly V1, V2 are linearly independent and so I have a basis, okay V1, V2 is a basis of the subspace W which is a plain passing through the origin.

So the dimension of any plain passing through the origin is 2, so dimension of W equals 2, okay we also prove the converse the converse is true if the dimension of a subspace is 2 then it must be a certain claim passing through the origin, okay. So let us prove that that is we are determining subspaces of dimension 2 precisely, okay we are saying that in R3 if a subspace has dimension 2 then it must be a plain passing through the origin and what we have just now shown is that if it is a plain passing through the origin then its dimension is 2, okay. So this is complete understanding of two dimensional subspaces of R3, okay.

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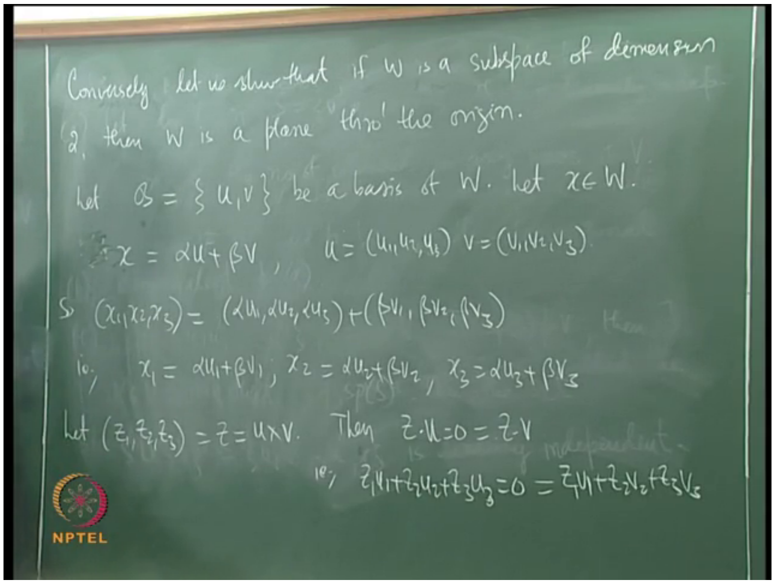

So the converse is what I want to show next, conversely let us show that if W is a subspace of dimension 2 then W is a plain passing through the origin, okay.

So let us look at W and call a basis a b of W consisting of elements u, v this be a basis of W I know that there are two elements because a dimension is 2 I must show that this W is a plain passing through the origin, okay. Let us take x in W I will take an arbitrary point in W I will show that this arbitrary point satisfies the equation of a plain, okay I will show that this point satisfies the equation of a plain so it follows that any point in W must lie on the plain and it will then follow that these two are the same, okay.

Let us look at, okay before that let me write down x as a linear combination x is let us say alpha times u plus beta times V let me use coordinates u equal to u1, u2, u3 V equals V1, V2, V3 and x is equal to x1, x2, x3, okay. So I have x1, x2, x3 being equal to alpha times u1, alpha u2, alpha u3 plus beta V1, beta V2, beta V3 this is my x that is x1 equals alpha u1 plus beta V1 x2 equals alpha u2 plus beta V2 x3 is alpha u3 plus beta V3 any x in V is a linear combination any x in W is a linear combination of the vectors u and V I have written down the expanded form of the coordinates of x of the components of x.

Let me also define a new vector I will call that as the vector z let me define a new vector z to be the cross product of u and V z is a cross product of u and V remember that u and V are basis vectors so none of them is 0 one is not a multiple of the other and so u cross V must be perpendicular to both u and V, okay vector calculus z is perpendicular the vector z that we have defined is perpendicular to u and V then perpendicularity is what a dot b equal to 0 and for 3 dimensional vectors a dot b is component wise multiplication and then addition a1 b1 plus a2 b2 plus a3 b3 so what I have is z dot u is 0 as well as z dot V that is z1 u1 plus z2 u2 plus z3 u3 this is 0 as well as the corresponding equation for z dot V equal to 0 z1 V1 plus z2 V2 plus z3 V3 this is 0, okay this z is perpendicular to u and V.

Now u and V are plain vectors z is perpendicular to u and V we are in 3 dimensions we will show that x belongs to the plain generated by u and V so we will show that x is perpendicular to z it would then follow that x must lie on the plain generated by u and V I will write down the equation of the plain explicit.

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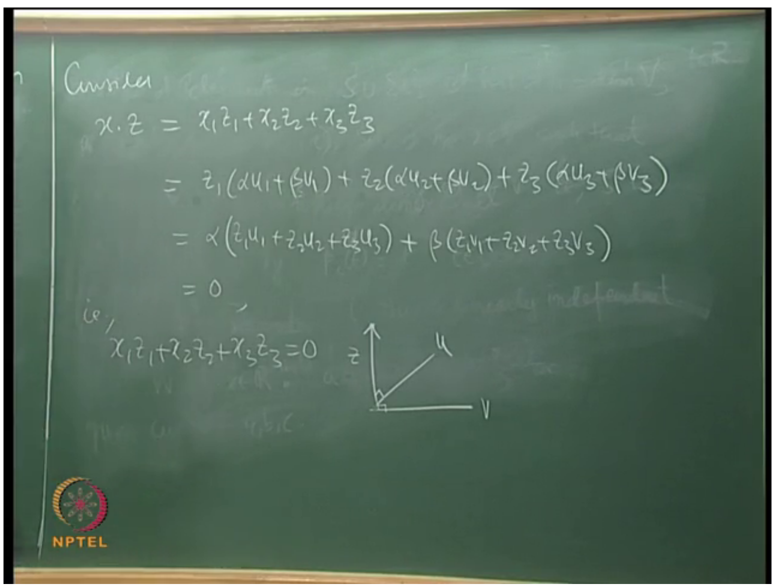

So the last step is to consider x dot z I will show that this is 0 x dot z is x1 z1 plus x2 z2 plus x3 z3 use the formulas for x1, x2, x3 so let me write z1 first z1 into alpha u1 plus beta V1 alpha u2 plus beta V2 plus z3 into alpha u3 plus beta V3, I have written down the formulas for x1, x2, x3 in terms of the coordinates of components of u and V.

So this is take alpha outside z1 u1 plus alpha outside z2 u2 the last term alpha has been taken outside z3 u3 plus the other set of terms beta into z1 V1 plus z2 V2 plus z3 V3 but I have just now observed that these two numbers are 0 so this is 0. So x dot z is 0 that is expand the expanded form gives me x1, z1 plus x2 z2 plus x3 z3 equal to 0 where z is a fixed vector z1, z2, z3 are fixed numbers they are fixed numbers because z is u cross V u and V are fixed the coordinates components of u and V are fixed so z is fixed that is the coordinates components z1, z2, z3 are fixed numbers so this is like ax 1 plus bx 2 plus cx 3 is 0 so x lies on a plain passing through the origin I have taken x as an arbitrary element here so any vector in W must lie on the plain passing through the origin, okay.

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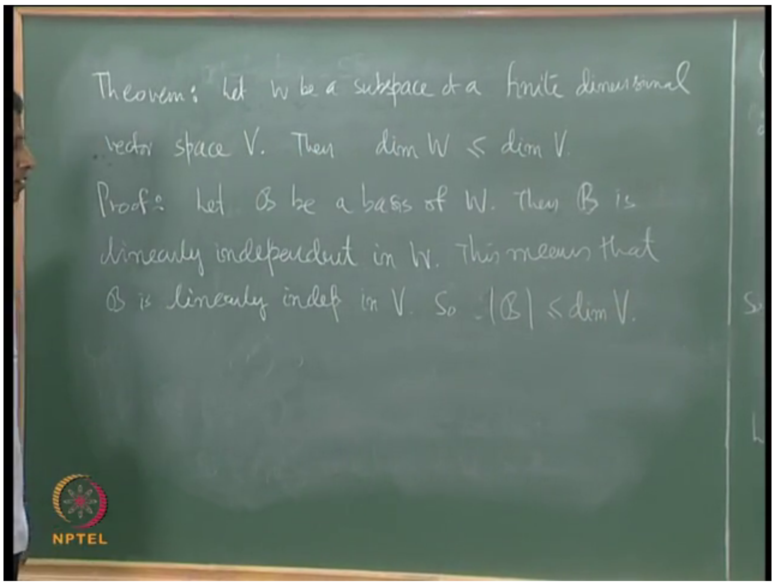

So we have determined all subspaces of dimension 2 in R3, okay so also look at the following result what are the following result is motivated by this question what are the possible dimensions of any subspace of a finite dimensional vector space all integers lying between 0, 0 excluded till the dimension of V, okay intuitively that is clear let us prove it quickly a quick proof of this fact let W be a subspace of a finite dimensional vector space V then dimension W cannot exceed dimension of V dimension W cannot exceed the dimension of V, okay intuitively this is clear a subset cannot have a greater dimension than the super set.

Proof is as follows let us take B to be basis of W then B is linearly independent in W any basis must be a linearly independent spanning subset. So this is linearly independent W but linear independence in W is the same as linear independence in V because it does not depend on anything that is got to do with W the only thing that I know is that the vectors are in W, so this means that B is script B is linearly independent in V but what we know is that the number of elements in any linearly independent subset cannot exceed the dimension of the space, okay. So the number of elements in B cannot exceed the number of cannot exceed the dimension of the space V but this number is the dimension of W that is dimension W is less than or equal to dimension V, okay so that is an easy consequence, okay.

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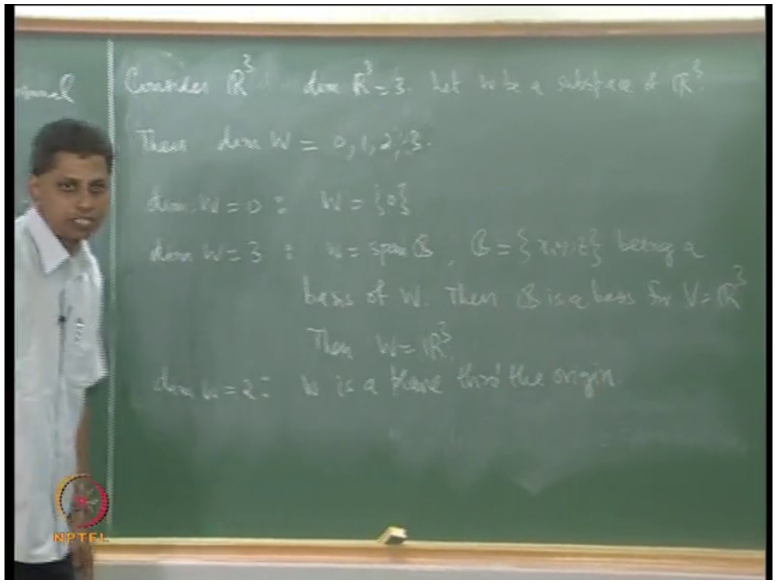

Let us now look at R3 and then list all the subspaces and there dimensions, we will use this result consider the vector space R3 the dimension of R3 is 3 let W be a subspace of R3 then by the previous theorem what we know is that dimension of W equal to 0, 1 or 2 it can be 3 also so let us say let us include that and then see what happens if it is equal to 3 dimension W 0, 1, 2 or 3 dimension W equal to 0 this means W is single term 0, 0 dimensional 0 is the only vector space.

Look at the other extreme dimension W is 3, okay in this is case W equal to span of B where B has three elements, okay let us say x, y, z if W has dimension 3 then it has three elements in any basis but now look at the result that we proved today, if I have a linearly independent subset that has a same number of elements as the dimension of the vector space then that must be a basis for that vector space if you have a linearly independent subset having the same number of elements as the dimension of the vector space then this must be a basis for that vector space, okay using that result it follows that B is a basis for R3, okay but this means W is equal to R3, is that okay?

In this case W must be the whole of R3, 0, 1, 2, 3 these are the four possibilities the dimension of the previous results says the dimension W does not exceed dimension V it can be equal, so these are the, yes yeah 0, 1, 2, 3 these are the four possibilities, V has dimension 3 dimension is a non-negative number dimension is 0, 1, 2 etcetera up to dimension V dimension of R3 is 3 so it is from 0 to 3 0 is really trivial because the only vector space which has 0 as its dimension is a 0 space so that is anyway trivial but we must include it here, it is a non-negative number it is a number of elements in a basis so it can be 0 so 0 should also be included even though it is trivial, okay.

So these are the two extremes if the dimension is 0 then the subspace is a trivial subspace the dimension is 3 the dimension of the original space then the subspace must be equal to R3 so we are left with these two possibilities dimension is 1, dimension is 2 dimension 2 we have disposed off dimension W equal to 2 we have disposed this means W is a plain a certain plain passing through the origin this we proved just now if dimension W is 2 then W is a certain plain passing through the origin.

So we are left with dimension W equal to 1 so what is your guess in that case? It must be a certain line passing through the origin, okay.

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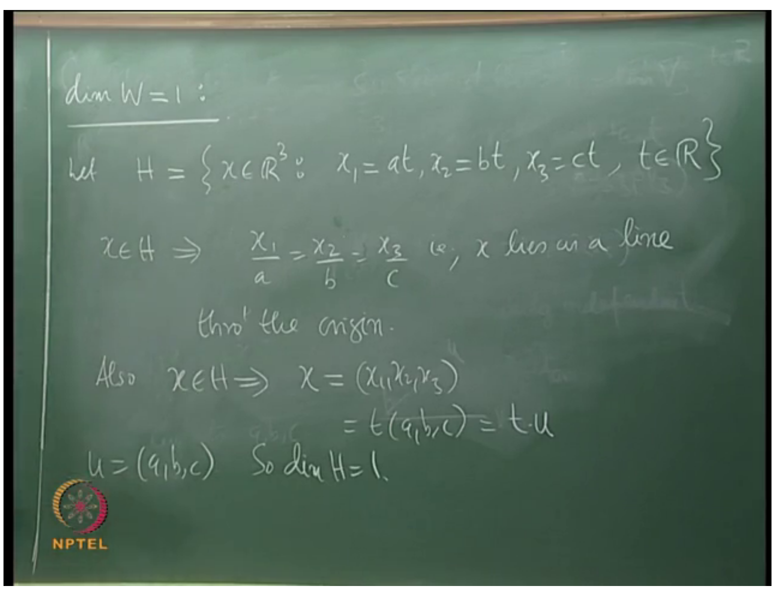

So let us dispose that off by giving a quick proof so we want to consider the case dimension W equals 1 let us first observe that if let us call something else let us call H let H be the set of all x in R3 such that let me first write down the set of all points lying on a certain line passing through the origin set of all x in R3 such that x1 is a times t, x2 is b times t, x3 is c times t where a, b, c are given constants t varies in R a, b, c are given constants t varies in R we have seen this before we have shown that there is a subspace, okay.

Now if x belongs to H then x1 by a equals x2 by b equals x3 by c that is this is a symmetric form of a line in R3 that is x lies on a certain line passing through the origin the direction rations are a, b, c, okay x lies on a line through the origin, what is also clear is that if x is in H then x can be written as x1, x2, x3 that is equal to use these equations I will take t outside t times a, b, c I will call it t times u where u is the vector whose components are a, b, c, okay.

Now this is a non-trivial line so there are three constants a, b, c at least one of them is not 0 so the vector u is not 0 a single vector non-zero linearly independent and what we have shown is that any x in H is the multiple of this any x is a linear combination of this and so this is a basis for H so dimension of H is 1 what we have shown is that if the collection of all points lying on any straight line passing through the origin that is a subspace of dimension 1, converse? Maybe I will leave it as an exercise.

If H is a subspace of R3 of dimension 1, show that H is precisely the set of all points lying on a certain line passing through the origin, okay that is going to be an exercise for you it the follows that we have understood all 1 dimension subspaces of R3 they are precisely lines passing through the origin, okay let me stop here.