Linear Algebra
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Lecture 10
Subspaces (continued), Spanning Sets, Linear Independence, Dependence

So in last lecture we were discussing the notion of subspaces, the topic for today is linear independence of vectors but before that there are certain aspects of subspaces that are important which I would like to cover, okay.

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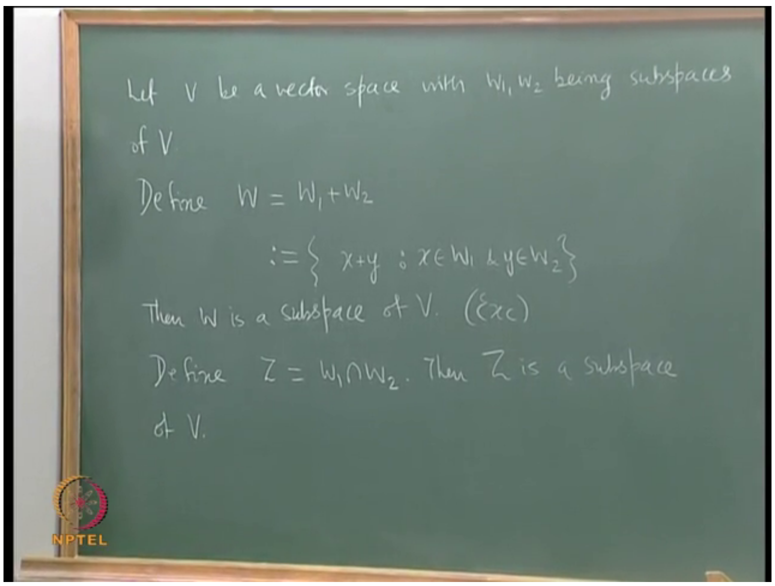

For instance how to get new subspaces from old? Let us first address this problem I have a vector space with to subspaces, okay then I get a new subspace by looking at the sum of these two subspaces I will define w as w1 plus w2 sum of sets sum of two sets what is the definition the definition is it is the set of all x plus y such that x belongs to w1, y belongs to w2, okay I am defining the sum of two subspaces w1 and w2 the notation is w1 plus w2 I am calling that as w this is the collection of all x plus y where x comes from w1 y comes from w2.

Now collection of all x plus y or y plus x does not matter because addition is commutative, okay but let us stick to this notation the first one will come from w1 second comes from w2 then this is the subspace of V, why it is a subspace? It is easy just appeal to the theorem that we proved last time we need to take two elements w show that there sum is in w take a vector in w take a scalar multiplication it slows with respect to that operation also, once you show that then it follows this is a subspace so I am going to leave as an exercise w is a subspace of V this is an exercise for you, okay.

Now this is in some sense larger than w1 and w2, let us look at something that is smaller than w1 and w2 that is another subspace let us call it z as w1 intersection w2 the set intersection, okay. Then I am going to leave this an exercise for you to show that this is also a subspace z is a subspace of V this has a property that this is contained in w1 as well as w2. So you can think of this as a subspace of w1 as well as w2, okay.

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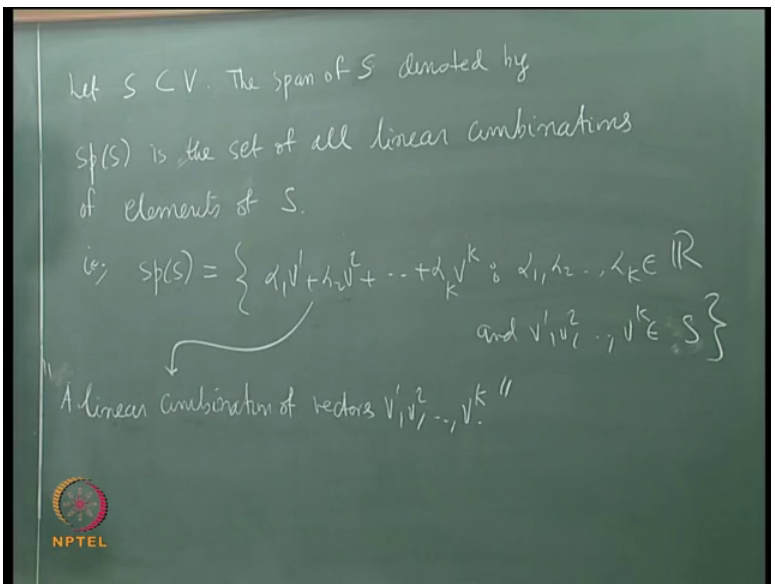

There is another method of getting subspaces let me describe that, let us take S as a subset of V I will define the span of V, I am sorry span of S are denoted by Sp of S I am going to define this is the set of all linear combinations of elements of S not defined what a linear combination is but I will do this now span of S the set of all linear combination span of S is the set of all linear combinations of elements of S. So mathematically what is span of S? Span of S is the set of all alpha 1 V1 plus alpha 2 V2 etcetera alpha k Vk where the alphas are scalars Vk's are vectors in S, okay.

So alpha 1, alpha 2, etcetera alpha k they come from the field and V1, V2, etcetera Vk they come from S, now this is what I am calling as a linear combination this will be specifically linear combination this is a linear combination of the vectors V1, V2, etcetera Vk this is my definition.

A linear combination of the vectors V1, V2, etcetera Vk will be called a linear is a typical element of spans and it is of this type a linear combinations of this form that is a typical element of span of S.

Now this span of S is a subspace, okay this span is a subspace of V to prove that this is a subspace of V again we will use that theorem close with respect to addition and scalar multiplication.

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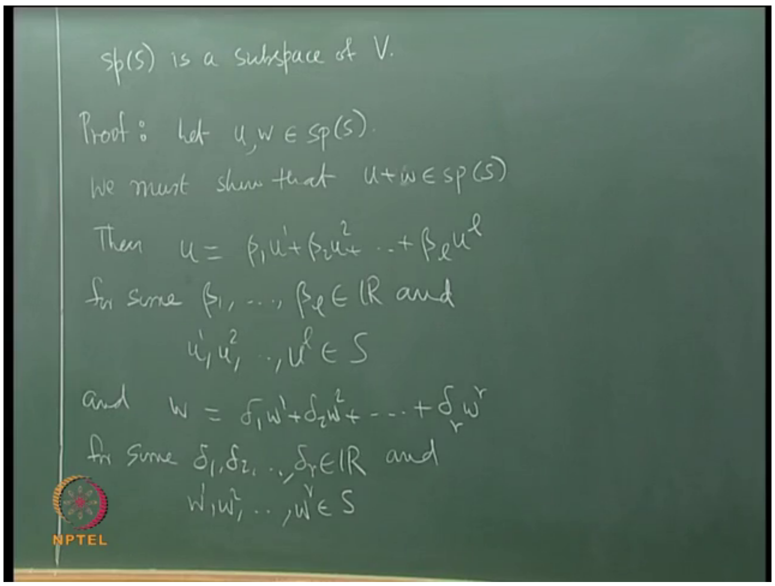

The proof is there once you write down the first step correctly, this is a subspace of V span of S is a subspace of V the proof you are going to tell me the first line and I will just leave it at that we want to show that this is close with respect to addition for instance scalar multiplication is similar close with respect to addition so I need to take two elements let you say u, comma w belong to span of S we must show that u plus w belongs to span of S, okay I want you to just tell me the first line after this, how do you prove this belongs to span of S, okay quite a few seem to know the proof let me just write down I am going to leave the other steps for you to fill up.

u is in span of S so it is a linear combinations so I have something like this u equals beta 1 u1 plus beta 2 u2 plus etcetera plus beta lul for some scalars beta 1 etcetera beta l, whereas the vectors u1, u2 etcetera ul they come from S, okay see it is just some linear combination please remember that it is not the same case that we have here finitely many terms V, sorry w I will write a similar expression for w delta 1 w1 plus delta 2 w2 etcetera plus delta R wR in general this R and l are different, okay.

Again for some delta 1, delta 2, etcetera delta R from the reals and w1, w2 etcetera wr from S, okay. This is the first step this is the important step the rest of the prove is obvious, u plus w now is a linear combination I can write it as beta 1 u1, beta 2 u2 etcetera beta l ul plus delta 1 w1 etcetera delta r wr where the scalars come from R the vectors comes from S so it is close with respect to addition scalar multiplication is simpler than this, okay so Span S is a subspace.

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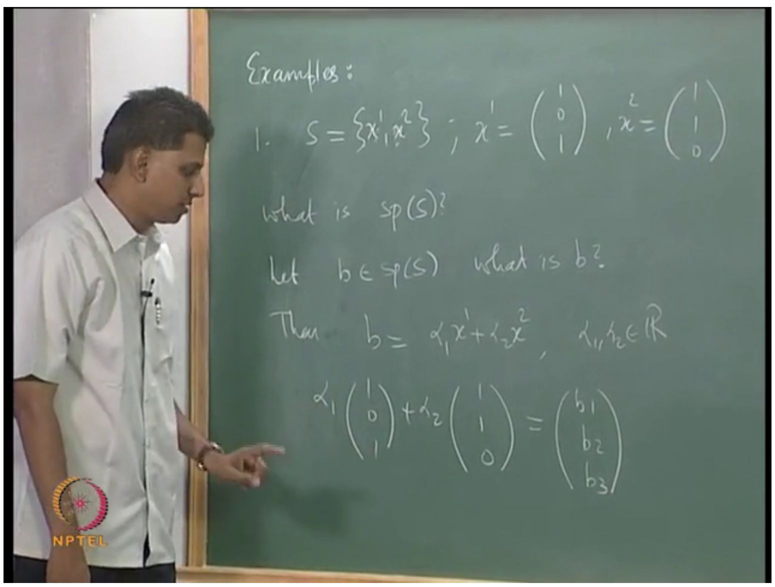

Let us look at one or two examples of span of a certain set, see remember this S could be an infinite subset, okay but for illustration let me take the following two examples, first one let us take S to be V1 x1, x2 where x1 is a vector 101, x2 is the vector 110 x1, x2 are vectors in R3 elements from R3, S is x1, x2 the question is what is span of S? Let us try to determine Span of S, now you will see that what we have learnt earlier with regard to linear equation that will be that will come handy, okay.

What is span of S is what we want to see, what we know is that span of S will contain linear combinations of x1 and x2 but what we want is a formula, a formula for elements in span of S, okay. Given a vector I should be in a position to determine whether this vector belongs span of S or not, okay given a vector is there a condition that I can impose on this vector in order for this vector to be in span of S if this condition is satisfied it is in span of S, if the condition is not satisfied it is not in span of S, okay.

Let us try to derive one condition you will see that this elementary row operations will be useful, okay so let we are trying to determine this question let b belong to span of S, okay then what is b? That is the question, okay what is b that is how does b look like that is the question. Now b is on span of S, okay so let us write down a definition b is something like alpha 1 x1 plus alpha 2 x2 for alpha 1, alpha 2 in R, okay if b is in span of S then by definition b is a linear combination of these two vectors, let us write it in full now this b is a general b in R3 we do not know how it looks like so let me rewrite it like this, I will write alpha 1 x1 plus alpha 2 x2 on the left and b on the right then familiar linear equations could be used alpha 1 x1 plus alpha 2 x2 alpha 1 into x1 is 101 plus alpha 2 into x2 110 on the right hand side I have the vector b so it has three components b1, b2, b3, okay alpha 1, alpha 2 are arbitrary what must be the conditions on these three numbers b1, b2, b3 in order for b to be in span of S in order for this equation to be satisfied but this equation you will see is precisely the following linear equation non-homogeneous linear equation.

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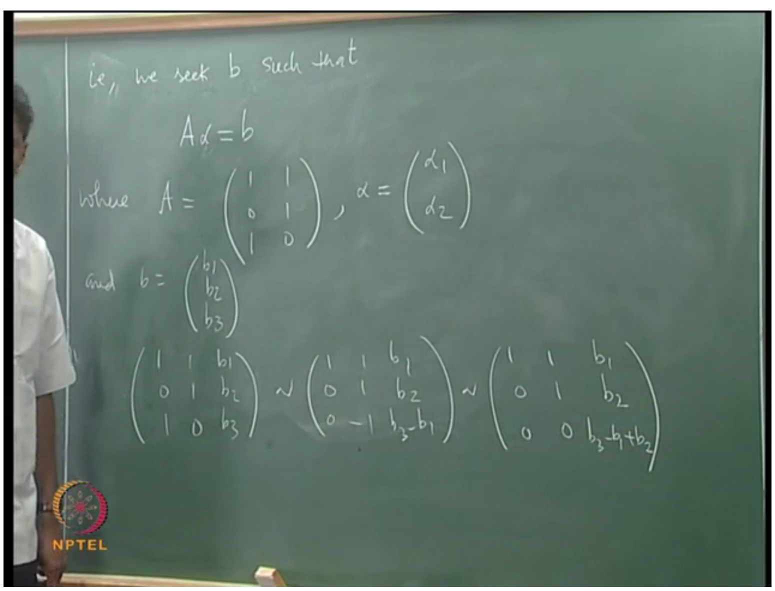

I can write it as follows, okay can you see it is precisely this, okay just tell me if this is okay that is we seek b such that A alpha equals b where A is a matrix whose columns are these two 101 110 alpha is the vector of unknowns alpha 1 alpha 2 and b is of course b1, b2, b3 we are seeking to solve A alpha equals b, do you agree with this? Just write down the three equations, first equation alpha 1 plus alpha 2 equals b1 alpha 2 equals b2 alpha 1 equals b1 those are the three equations in two unknowns alpha 1 alpha 2, okay.

Now let us do elementary row operation that is I want to now determine all b for which this system has a solution I know how to do it by using elementary row operations, so I need to look at A appended with b 101 110 b1, b2, b3, okay I do the elementary row operation reduces to this to row reduced echelon form the final step tells me what must be b, okay. So this is now row equivalent to I will keep the first one as it is 11b1 this will also be kept as it is minus this plus this 0 minus 1 b3 minus b1, okay next step this plus this one more operation I can actually stop here but I will reduce it to the row reduced echelon form.

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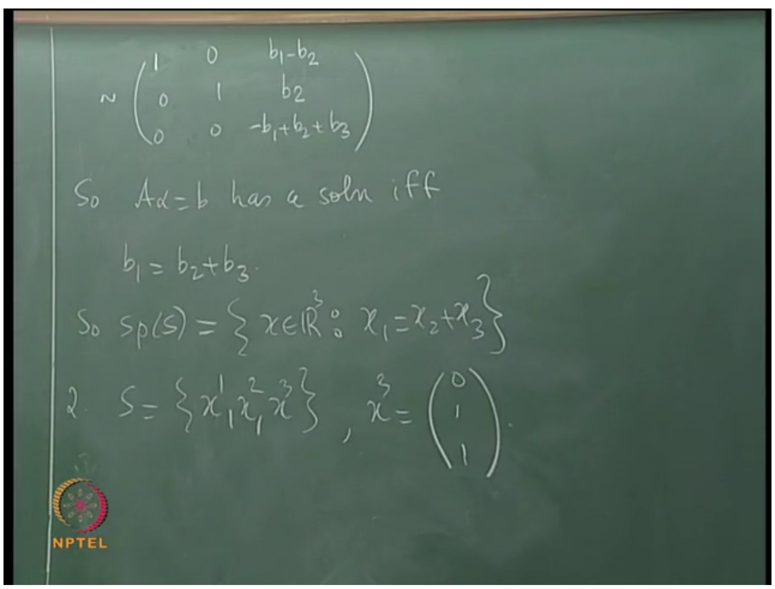

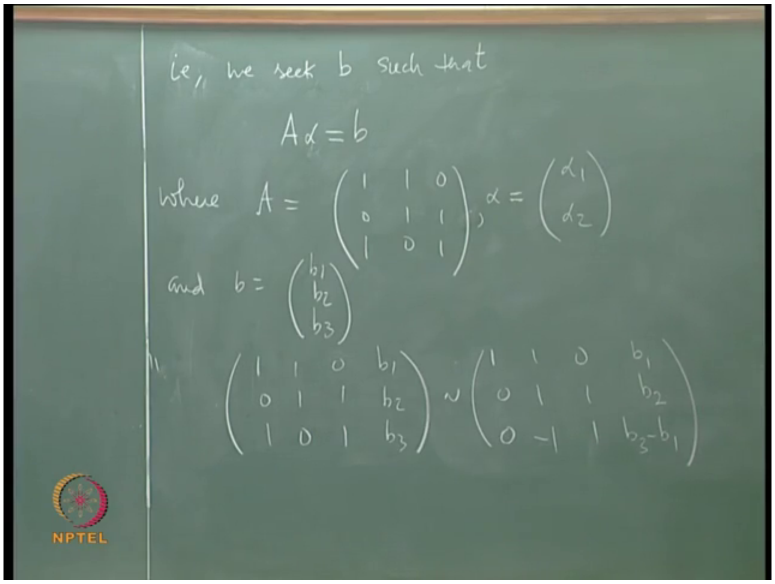

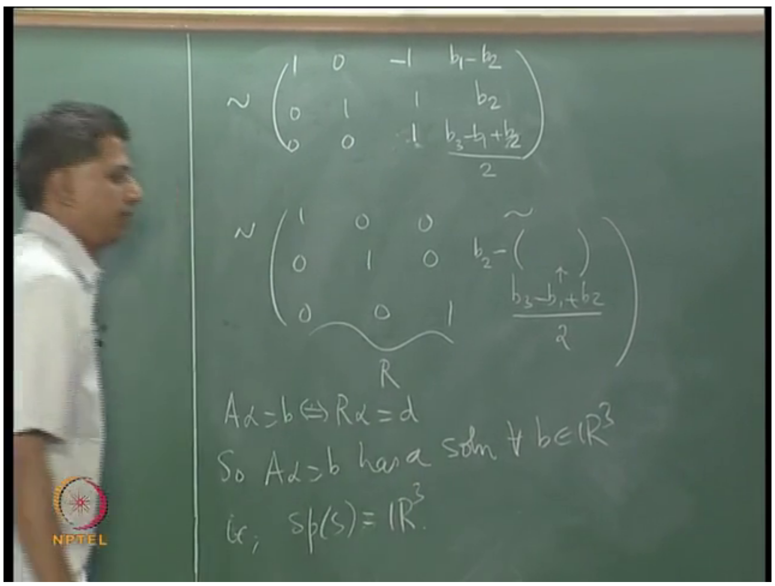

So let me just do one more operation for the sake of completeness, second row is 01 b2, okay now it is in the row reduced echelon form the number of non-zero rows of this is R for me this whole thing is R prime this is R for me this is d R I have R alpha equals d, right? This is R the number of non-zero rows is two that condition necessary sufficient condition for the original system to have a solutions are this must be 0.

So A alpha equals b has a solution if and only if b1 is b2 plus b3 that is the condition for any vector to be in span of S, so span of S is the set of all you tell me if this is clear now set of all x in R3 such that x1 is x2 plus x3, okay. So this determines the subspace completely, okay let us modify this example and just do one more you will see how the notion of homogeneous equations non-homogeneous equations invertibility etcetera that we had studied earlier will be useful once again.

So I want to look at another example modification of this this time S will be for me x1, x2, x3 x1, x2 as before, okay x1, x2 as before x3 is this vector 011 I would like to determine what is span S now, okay what is the span of this set. Now again we will have to look at something like this, okay we seek b such that A alpha equals b this time A is a 3 by 3 matrix 011 alpha is alpha 1, alpha 2, b is b1, b2, b3, okay so I will have to do the elementary row operations, let me remove this as before to determine the span S completely we need to solve the system A alpha equals b, okay this time A is the matrix whose columns are the three vectors that we started with, okay.

So we need to append this 011 and then b1, b2, b3, okay let us do this quickly this is row equivalent to first one I am going to keep it as it is write b1, okay next step this is row equivalent to, okay let us see what we could do, okay then I will divide this by 1 and then rewrite this like this and then keep this as the pivot row the operation performed with respect to this row this whole thing comes here, okay here I have to just add 100 I have another expression here please fill up this will be the sum of these two terms, okay but whatever it is now observe that this is R for me R alpha equals d this is R then I am reducing the system A alpha equals b to R alpha equals d the number of non-zero rows of R is 3 and so this is precisely the number of rows of R so there is no condition this di is equal to 0 for i greater than R that condition is vacuously satisfied.

So this system has a solution for all b, which means what? So A alpha equals b has a solution for all right hand side vectors b, so what is span of S? It is a whole R3, okay it is an improper subspace of R3 span of S is a whole of R3, okay if S is chosen as this set then span of S is R3, okay we will come back to this example this will be useful when we discuss the notion of linear independence, okay. So we are still discussing subspaces, in this example alpha is alpha 1, alpha 2, alpha 3 yes in this example it is alpha 1 here I have to make a correction, yes in this example there are, yeah that is correct the second example alpha has 3 coordinates, okay we are still discussing the notion of subspaces there is one small extension of this idea of span of a subset which is what the so called row space column space of a matrix, so let me describe this and then go to the next topic.

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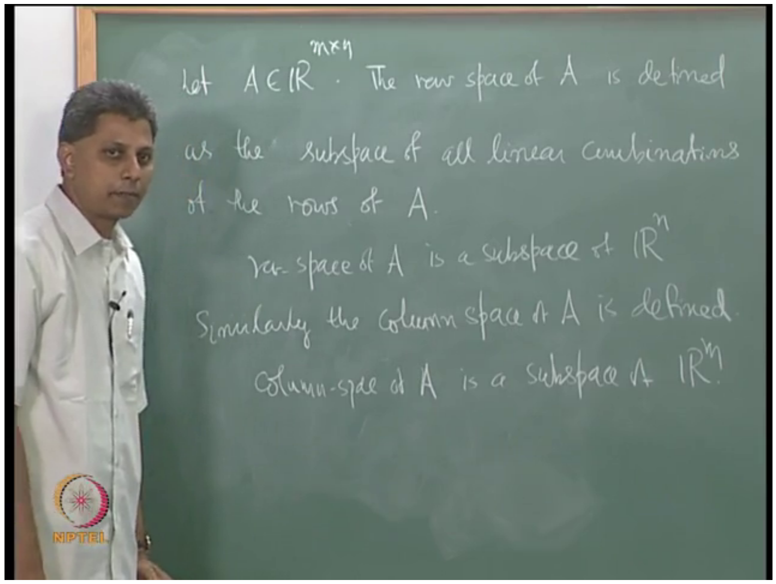

Let A be an m cross n matrix the row space of A is defined to be as the subspace of all linear combinations of the rows of A the row space of A is the subspace of a linear combinations of the rows of A. So what one does is to if A is an m cross n matrix there are m rows each row is in Rn each row has n coordinates so the row space so what one does is look at span of the rows, okay span of the rows that is set of all linear combinations of the rows of A that is the row space of A let us observe that the row space of A is a subspace because just now we have proved that the span is a subspace this is a subspace of Rn each row has n coordinates so this is a subspace of Rn one can define the columns space in the learning the set of all linear combinations of the columns of A the columns space is a subspace of Rm, so let me just say similarly column space of A is defined and what I want to say is that the column space of A is a subspace of Rm, okay.

So these two are subspaces of different vector spaces but they have a number that is the same there is a unique number associated with these two subspaces that number is a same for these two, okay we will see that a little while even though these are subspaces of different vector spaces we will see that there is an important number that one would like to associate with a vector space or vector subspace for that matter that number will be the same for these two, okay.

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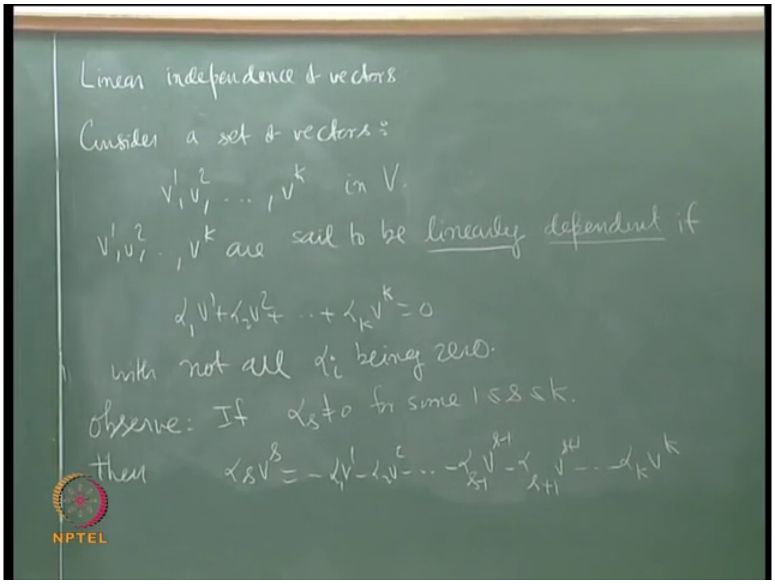

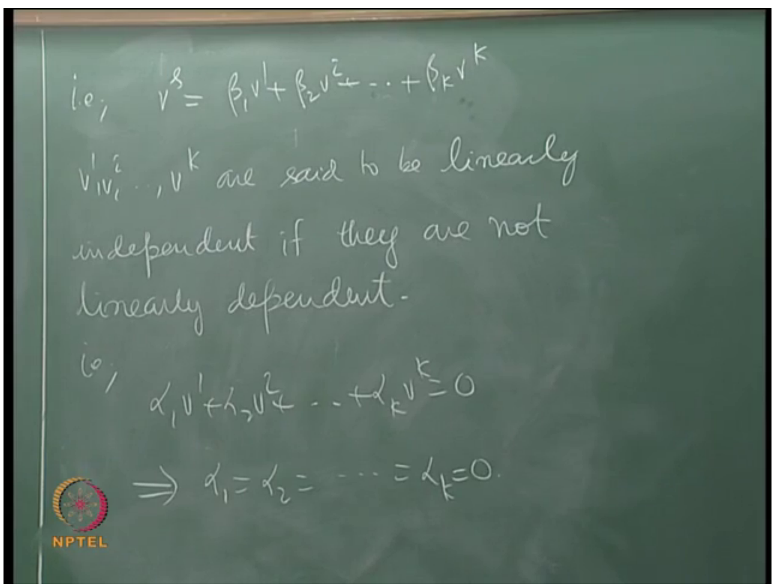

With this I would like to move to the notion of linear independence of vectors linear independence of vectors, okay once we have this notion we could define basis for a vector space and then the dimension of a vector space, okay. So what is linear independence? Let us first look at linear dependence, linear dependence among vectors let us take a collection consider a set of vectors let us V1, V2, etcetera Vk from a vector space V we would like to say that these vectors are linearly dependent if at least one of them depends on the others in formally at least one of them depends on the others one of them depending on the other at least one of them depending on the other means you can write one of them as a linear combination of the others, okay.

In other words, let us formulate this V1, V2, etcetera Vk are said to be linearly dependent if I have this equation to be satisfied alpha 1 V1 plus alpha 2 V2 etcetera plus alpha k Vk if this is the 0 vector so I am defining linear dependence, okay linearly dependent if this holds without all of them being 0, yes and write it like this this holds with not all alpha i's being 0 that is there is at least one alpha i such that this equation holds, okay.

Now you will see that if there is at least one alpha i which is not 0 let us say alpha S is not equal to 0 for some S at least one scalar is non-zero then one could write down see all that I want to demonstrate next quickly is that there is some dependence that is at least one of the vectors here can be written as a linear combinations of others that is why it is called dependence if this holds then I have linear dependence, okay.

Then observe that alpha S is not 0 I push all the other vectors to the right hand side and just have alpha S Vs on the left minus alpha 1 V1 minus alpha 2 V2 etcetera minus alpha S minus 1 Vs minus 1 minus alpha S plus 1 Vs plus 1 etcetera minus alpha k Vk I simply push the other vectors to the right hand side, since alpha S is not 0 I will divide this by alpha S that is Vs is I will use beta 1 V1 plus beta 2 V2 plus etcetera plus beta k Vk you fill up the details I have written Vs after dividing by alpha S I have written Vs as a linear combination of V1, V2, etcetera Vk on the right side I will not have the vector Vs this is linear dependence, okay.

So this is a definition that confirms to what we feel intuitively about linear dependence, okay. What is linear independence? Linear independence is a negative of linear dependence V1,V2, etcetera Vk are said to be linearly independent if they are not linearly dependent, okay of course what does it mean in terms of that equation alpha 1 V1 plus etcetera alpha k Vk equal to 0 we have the following alpha 1 V1 plus alpha 2 V2 plus etcetera plus alpha k Vk equal to 0 this holds where I do not have linear dependence so each scalar must be 0, okay.

So this implies this notation this implies that alpha 1 equals alpha 2 etcetera equals alpha k equals 0 that is linear independence because even if one of them were non-zero one could push the other vectors to the right divide by that scalar to get this non-zero to get the vector on the left as a linear combination of the others, so none of them should be 0 that is linear independence, one also says that the only way to get the 0 vector by means of these vectors V1, etcetera, Vk is by the trivial linear combination.

Then, we say that the vectors V1, V2, etcetera, Vk are linearly independent, okay.

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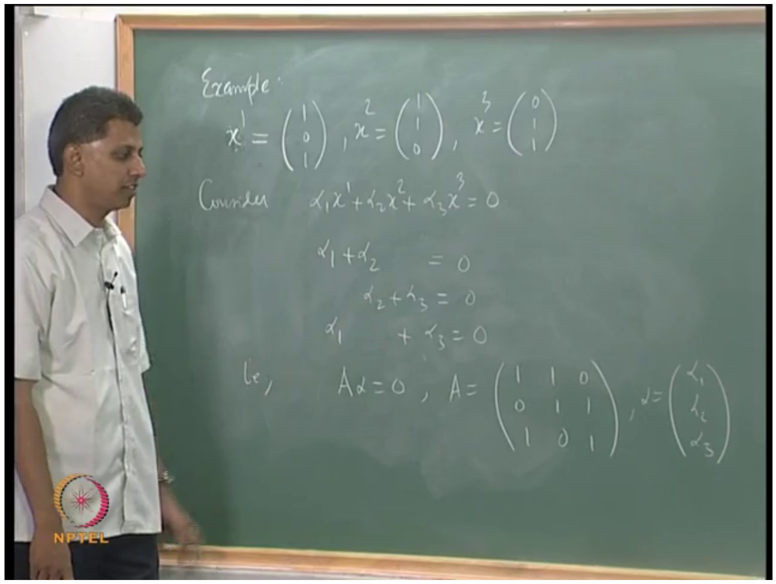

Let us look at some examples, let us look at some examples to consolidate take a example of the previous one the span thing let me stick to the same notation x1 is 101, x2 is 110, x3 is 011, okay I would like to verify if these vectors are linearly independent. So I must start with a linear combination, okay like that so consider a linear combination alpha 1 x1 plus alpha 2 x2 plus alpha 3 x3 equal to 0 the 3 dimensional 0 vector the vector with 3 components each being 0, then again, okay.

Let me write in this example to make it transparent, so I have alpha 1 plus alpha 2 equals 0 alpha 2 plus alpha 3 equals 0 alpha 1 plus alpha 3 equals 0 these are the 3 equations, of course Gaussian elimination can be applied and then you can get the solution immediately but I will pretend as though I cannot do that I want to use elementary row operations once again, okay but let us recall what happened for this particular example this is the same as I will again use the same A that I used in that previous example I am looking at the system A alpha equal to 0 I want to know whether this system has a non-zero solution I want to know whether this system has a non-trivial non-zero solution, what is A? A is the coefficient matrix here just write down these 3 vectors as columns you get A, what is alpha this time? There are three unknowns, okay I want to know if this system has a non-trivial solution but look at what we did earlier for this matrix A we had shown that the right hand side look at the non-homogeneous system A alpha equal to B we had shown that the right hand side needs no condition, whatever be b whatever be the right hand side this system A alpha equal to b has a solution. Remember the result that we proved with regard to linear equations A is invertible if and only if the homogeneous system Ax equal to 0 has 0 as the only solution if and only if Ax equal to b has a solution for all right hand side vectors b, okay.

So from what we have done earlier Ax equal to b has a solution for all b means that Ax A alpha equal to 0 as 0 as the only solution which means if this system has a solution then 0 is the only solution which means alpha 1 equals alpha 2 equals alpha 3 equals 0 that is linear independence.

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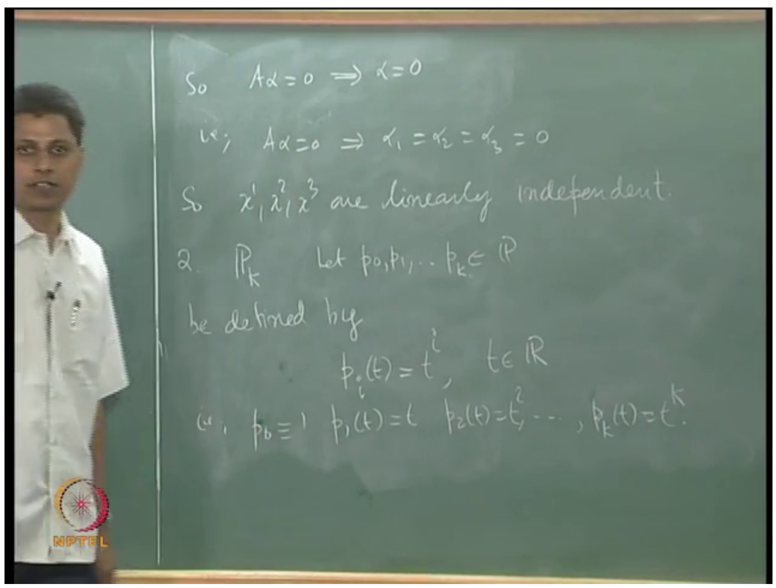

So A alpha equals 0 implies alpha equals 0 the computations have been performed earlier I am making use of that, that is A alpha equals 0 implies alpha 1 equals alpha 2 equals alpha 3 equal to 0. So the vectors x1, x2, x3 that we started with they are linearly independent, okay.

Now what is a guess about the other example the example where we had only the first two vectors x1, x2, just guess do not look at the numbers, do not look at the coordinates that is what is given is that you have three vectors x1, x2, x3 you have shown these three are linearly independent the previous example is x1, x2 my question is they are linearly independent, okay the reason is something that we can prove in general any subset of a linearly independent set is linearly independent, okay so like a precursor you can show that the vectors x1, x2 from just if you take in fact you take any of the vectors from this set they will be linearly independent that follows from a more general principle, okay.

What about different vector space instead of R3, so I am going to look at example 2 really let us look at Pk, Pk is a vector space of all polynomials of degree less than or equal to k with real coefficients, okay let me pick these vectors so I pick k plus 1 polynomials pi of t equals t to the i that is p not is a constant polynomial 1 p1 is the polynomial which satisfies p1 p equals t for all t p 2 t is t square etcetera pk of t is t power k. Let me conclude by showing that these vectors that is these polynomials are linearly independent.

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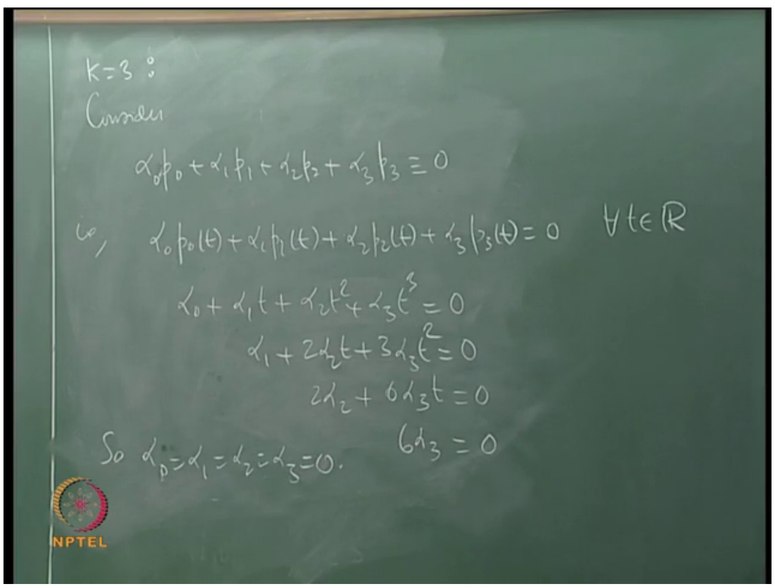

Let me take the case k case equals 3 just for simplicity and consider a linear combination alpha I will start with alpha not alpha not p not plus alpha 1 p1 alpha 2 p2 plus alpha 3 p3, I start with this linear combination I must see if I can show that all the coefficients are 0, then I could conclude that these vectors that is these polynomials are linearly independent, okay. Now this means what this means see the set of polynomials is a vector space so this is also a polynomial this polynomial takes the value 0 this is identically the 0 polynomial means at every t this is 0 that is alpha not p not of t plus alpha 1 p1 of t plus alpha 2 p2 of t alpha 3 p3 of p this is a 0 this number is 0 for all t in R that is a meaning that is a previous one is identically 0 I should write which means you evaluate this at any t then the value is 0.

Now you write it in full and see what you get, you get alpha not plus alpha 1 t alpha 2 t square plus alpha 3 t cube equal to 0 this is true for all t, how to conclude from this that each of the scalars is 0 that is the claim I am making, yes one could use heavy machinery fundamental theorem of algebra a polynomial of degree k has precisely k roots, okay but we can do without the fundamental theorem of algebra that is what I wanted to illustrate in this example. See each is a polynomial polynomial we know are differentiable functions differentiate this equation as many times as you want and get the desired conclusion differentiate once you get alpha 1 plus 2 alpha 2 alpha 2 t plus 3 alpha 3 t square once more, 2 alpha 2 plus 6 alpha 3 t one last time 6 alpha 3, okay.

So look at the last equation alpha 3 is 0 the (())(42:32) equation alpha 2 is 0 etcetera backward substitution gives you alpha not alpha 1, alpha 2, alpha 3 each of them is 0, okay so this is the proof which does not use the fundamental theorem of algebra just use differentiability of this polynomials, okay let me stop here.