Linear Algebra
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Lecture 9
Elementary Properties in Vector Spaces, Subspaces

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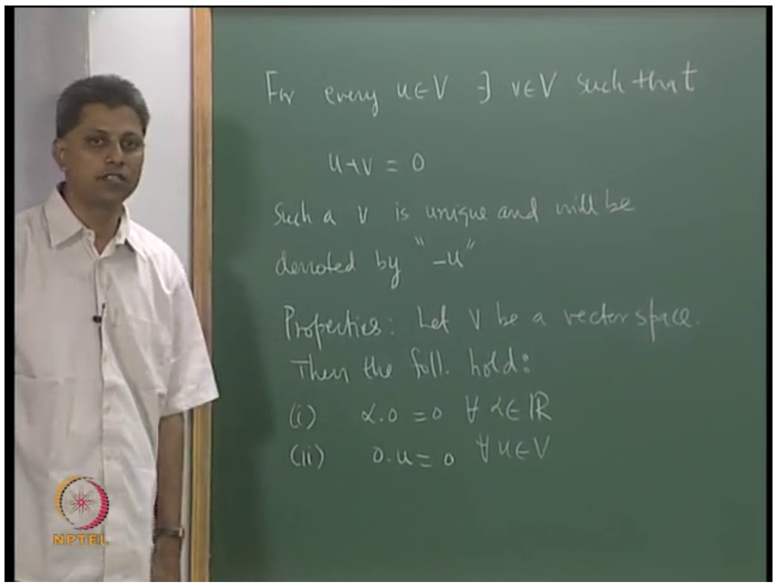

Okay, we are discussing the notion of vector spaces we had seen several examples, let us now look at a few properties of that hold in a vector space before that let us go back to the schemes the existence of negative element, okay that is schemes corresponding to the existence of negative element for every u element of v I mentioned that there exist V element of V such that u plus v equals 0, okay.

Now this V such a V is unique that can be proved easily I want to introduce a notation (())(1:19) this notation is just to appeal to our intuition its additive inverse so we will call it the negative element we will denote it by minus u, see this is just a notation presently we do know what this means, okay this what we know is that this is a vector in V which satisfies this equation, so minus u is that unique vector v that satisfies this equation, okay. I want to discuss 4 properties that hold in a vector space four elementary properties which we will be using on many occasions let V be a vector space in the following hold in the following properties hold first property alpha into 0 equals 0 this is true for all alpha element of F hereafter I will write R for F these properties hold for C also, okay.

So unless I specify the underlying field will be the field of real numbers for all alpha element of R let us observe that this is scalar multiplication, okay scalar vector so this 0 is a 0 vector the right hand side vector is also the 0 vector, okay. Property 2, 0 times u equals 0 for all u element of V this time you need to differentiate the 0 on the left is a scalar 0 that comes from the field the additive identity the 0 on the right is a vector 0 this holds for all u.

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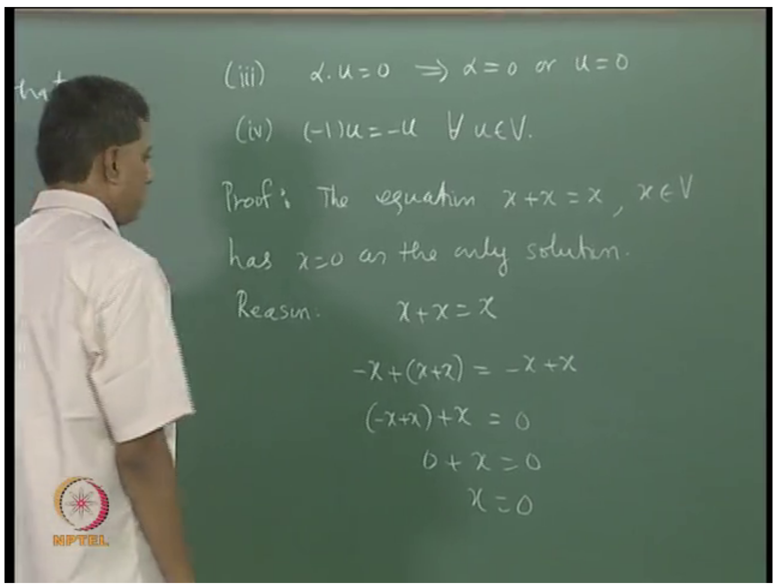

Property 3, if alpha into u equal to 0, then either alpha equal to 0 or u equal to 0, okay. Again we are using the same 0 for scalar as well as vector so let us understand that this 0 is the vector 0 this is scalar 0 this is again vector 0 that is property 3. Property 4, it is a only with regard to property 4 that I wanted to use this notation, so what property 4 says is minus 1 times u is minus u for all u element of V.

So minus u is precisely minus 1 times u minus 1 into u, okay these are elementary properties quick proof for proving 1 and 2 I will use the following result the equation x plus x equals x for x in V has, so can you say something about x? It must be 0 the equation x plus x equal x for x in V has x equal to 0 as the only solution as x equal to 0 as the only solution, okay let us prove this once we have this properties 1 and 2 will follow immediately, okay let us prove this quickly what is the reason? So I have x plus x equals x and I will add minus x to it I know that minus x in is in the vector space that is minus x is the negative of x just now I introduce that notation, so I add minus x plus x plus x that is minus x plus x, minus x plus x I know this is the negative element so this is 0 I know that addition is associative so minus x plus x plus x again use the fact that this is 0 0 plus x equals 0 but I know that 0 plus x is x so x is equal to 0.

So if this holds for x then x must be 0 and clearly 0 satisfies this equation, okay so 0 is only solution of this equation.

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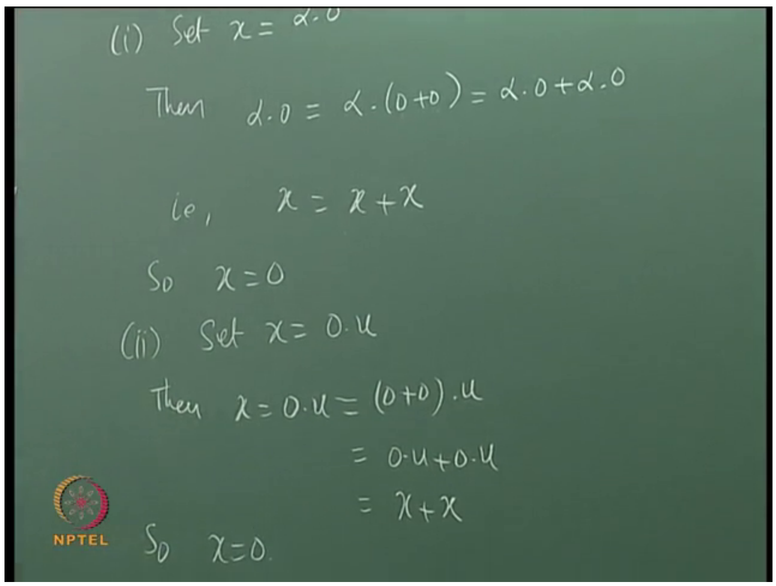

Let us now consider the proof of 1 and set x to be alpha into 0, okay then alpha into 0 is alpha into 0 vector which can be written as 0 plus 0 that is distributivity alpha into 0 plus alpha into 0 that is x is equals x plus x just from what we have seen now follows that x is 0, okay that is a first part I hope this is clear it follows essentially from the fact that the equation x plus x equals x has x equal to 0 as the only solution that is first one, proof of the second I will again set x as 0 into u then previously we used the vector 0 now we will use the scalar 0 and use the property that alpha plus beta times u is alpha u plus beta u. So look at x that is 0 into u that is 0 plus 0 this is scalar 0 into u that is 0 into u plus 0 into u which is x plus x and so x is equal to 0, okay so that is property 2 that (())(8:07) elementary, okay.

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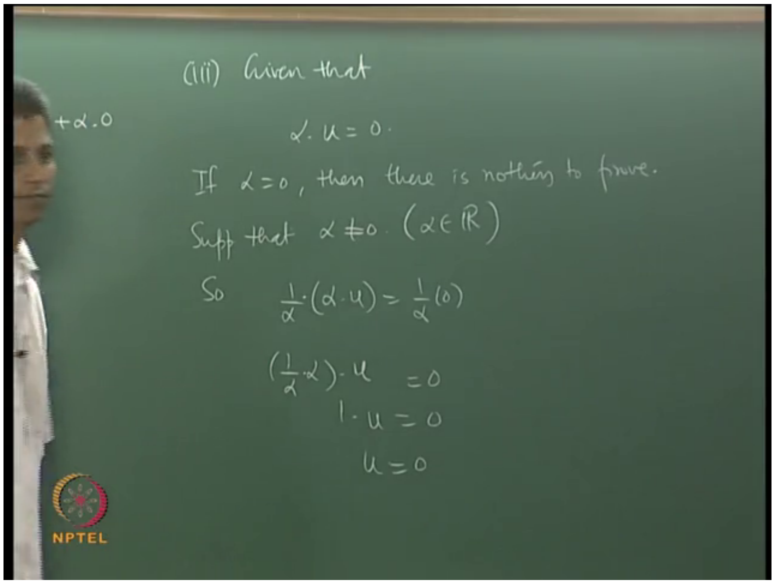

Property 3, we need to show that I am given that alpha into u is 0 we must show that either alpha is 0 or u is 0 or both, okay. If alpha is 0 then there is nothing to prove let us take the case that alpha is not 0, remember that alpha comes from R alpha is a real number non-zero every non-zero real number it is a field, so as a multiplicative inverse that is 1 by alpha. So let us look at this equation I pre-multiply this equation by 1 by alpha 1 by alpha into alpha into u is 1 by alpha into 0 1 by alpha into 0 is 0, 1 by alpha into alpha I am using schemes 7 alpha into beta u is alpha beta into u 1 by alpha into alpha into u 1 by alpha is the inverse of alpha, so this is 1 1 times u condition scheme 8 1 into u is u u is 0, okay so that is proving 3.

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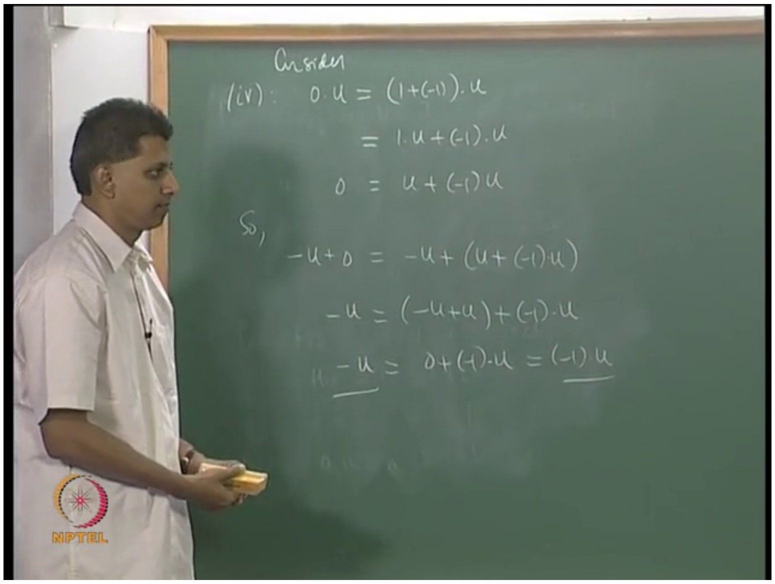

Proof of 4 (())(9:51) 0 times u I start with this 0 into u that is I can write this as 1 plus minus 1 into u this happens in the field 0 is a scalar here this happens in the field 1 plus minus 1 is 0 in the field this is 1 into u plus minus 1 into u that is u plus minus 1 into u, I will add minus u on both sides 0 into u I know is 0, okay so minus u plus 0 on the left minus u plus u plus minus 1 into u minus u on the left minus addition is associative this is 0 that is minus 1 into u that is just minus 1 into u.

So we have shown that minus 1 into u is minus u, so the negative element that we denoted by minus u is minus 1 into u, okay.

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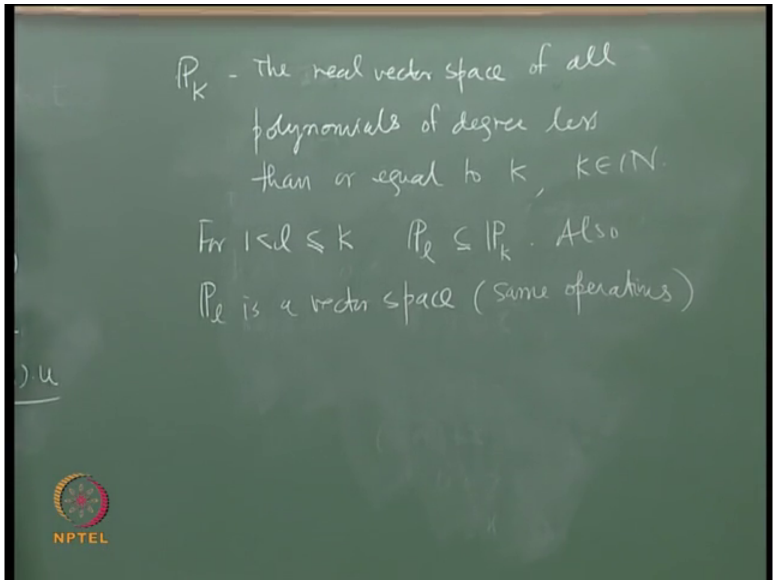

So this is quite elementary, okay let us proceed we had looked at several examples yesterday several examples of vector spaces I will pick up one of them, we used this notation Pk I think I used Pn yesterday but I want to look at Pk Pk is the vector space the real vector space of all polynomials of degree less than or equal to K for a fixed positive integer n or a fixed positive integer K this is the real vector space by which I mean the coefficients coming from the field of real numbers.

It is easy to see that for l less than or equal to K take the non-trivial case it is easy to see that Pl is contained in Pk strictly and that this holds also Pl is a vector space in its own right Pl is a vector space in its own right, okay since I have include equal to I will include equal to here also it is a vector space in its own right with respect to the same operations let us observe this with respect to the same operations as you have in Pk same addition and scalar multiplication, okay.

So this motivates a definition of a subspace, okay which is what I want to discuss next, from this we will look the notion of a subspace it makes sense to say that Pl is a subspace of Pk whenever l is less than or equal to K Pl is a subspace of Pk whenever l is less than or equal to K, what is a subspace?

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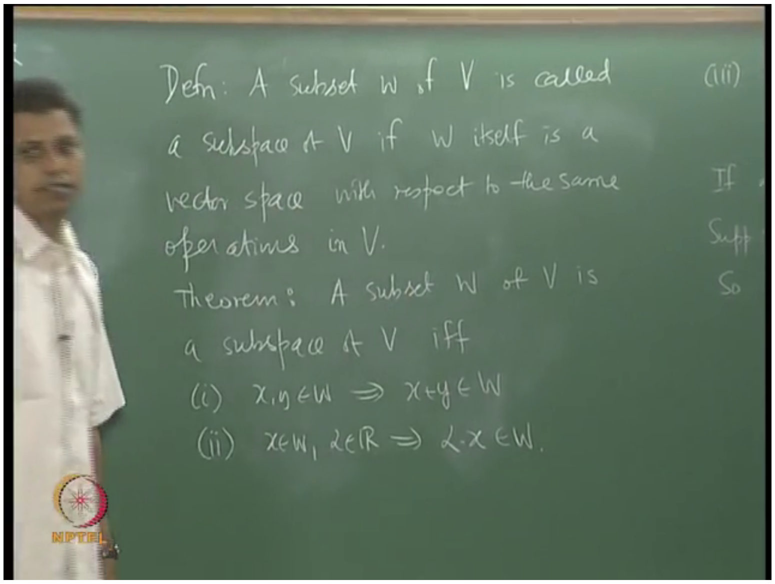

A subset w of a vector space V is called a subspace of V if w itself is a vector space with respect to the same operations by which I mean vector addition and scalar multiplication from V with respect to same operations in V such a subset this is a special subset which is also a vector space such a subset is called a subspace of V, okay we have already seen an example Pl is contained in Pk Pl is a subspace of Pk whenever l is less than or equal to K.

Let us look at other examples but before that to verify that these examples are indeed examples of subspaces we need a device so let us prove this result which will be useful in the examples. So we have the following theorem for a subspace for those of you who have studied group theory this will not come as a surprise, a subset w of a vector space V is a subspace of V if and only if the following two conditions hold.

First condition, x comma y in w implies x plus y is in w it must be closed with respect to the vector addition where this plus is the same as the addition in the vector space V only to emphasize x element of w alpha element of R implies alpha into x belongs to w this is closure with respect to scalar multiplication. So one needs to verify just these two conditions to show that certain subset is a subspace, okay just these two closure schemes really, okay.

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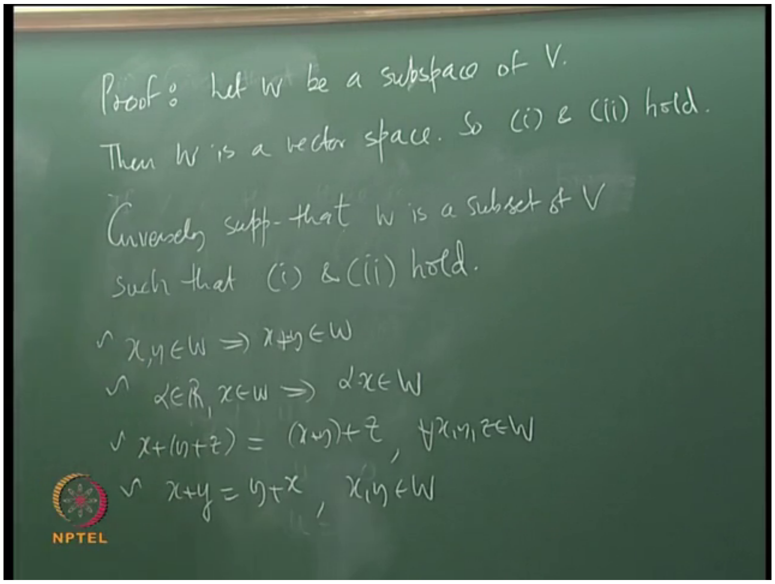

Let us prove this first and then look at examples of sort of prove this theorem, there are two parts its if and only if there is a necessity part there is a sufficiency part if w is a subspace that is a easy part if w is a subspace then this condition holds that is the easy part let w be a subspace of V then obviously w is a vector space so 1 and 2 hold trivially because it is a vector space it is a vector space in its own right, so one part is easy if w is a subspace then these two condition hold, okay it is a converse that is interesting conversely suppose that w is a subset of V such that conditions 1 and 2 hold we must show that w is a subspace, okay.

Let us quickly write down the schemes of a vector space, okay this is also useful in recalling the definition of a vector space, what are the schemes? First we must verify that it is closed, okay so x, y in w implies x plus y is in w then second condition alpha in R x in w implies alpha x is in w these two are the closure schemes these two schemes hold because conditions 1 and 2 have been assumed. Then, associativity x plus y plus z equals x plus y plus z for all x, y, z this holds because these are elements of V and in V associative holds so this equation holds in w commutativity x plus y equals y plus x that is I am picking elements x and x, y and z see to emphasize this is true for all x, y, z in w I am writing down the schemes for w being the vector space, x and y are from w but it does not matter I am looking at x plus y in V so these two are the same.

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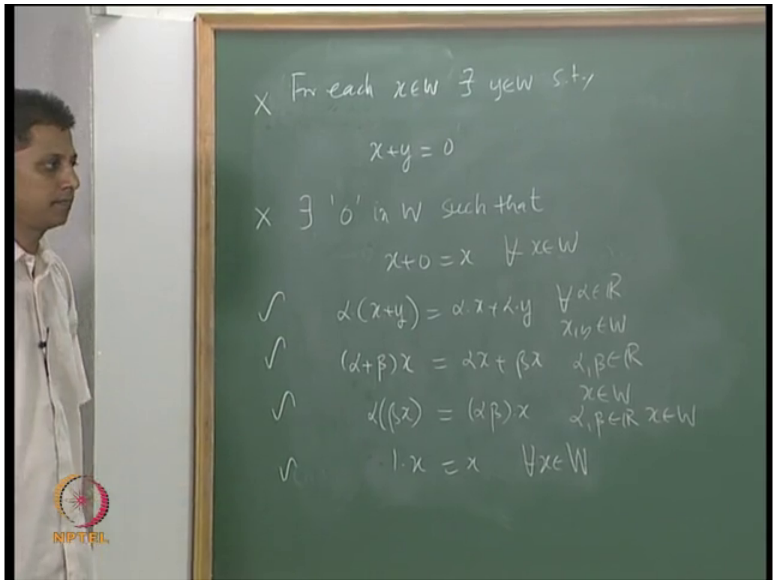

So I have associativity I have commutativity, so let me say this is there, this is there, this is there, this is there I have all these four let me write down the other conditions the other for each x element of w there exist y element of w such that x plus y is 0 this we do not have presently we will prove this, okay presently we do not have this we will prove this. What we have is x is in w so x is in V so I know there exist y in V such that x plus y is 0, why should this y belong to w we will show that this y will indeed be in w if those two conditions hold, okay but this right now we do not have the next one is also not well even before this look at this schemes there exists 0 in w such that x plus 0 equals x for all x in w we need to very this even before verifying this, what I know is that there exist 0 in V why should that 0 we will actually prove that 0 is in w also, okay.

So this we do not have presently so we will prove that this the 0 indeed belongs to w and then prove this existence of additive inverse, what are the other schemes then? With respect to scalar multiplication I have written down schemes 1 scheme 2 is what distributivity alpha times x plus y this is alpha x plus alpha y for all alpha in R x, y in w again this is true because this holds in V and so this is through, condition with regard to distributive over the fields alpha plus beta into x that is alpha x plus beta x alpha beta and R x in w this again is through, what else do we have? Alpha into beta x this must be alpha beta into x alpha beta in R, x and w this again is true because this happens in the vector space V, finally 1 into x equals x for all x element of V for all x element of w we want this to hold in w this is also true because if it is in w it is V and for elements in V this holds so this is also through we need to only verify these two conditions, first 0 belongs to w and that for every x in w there is a negative element, okay.

Yes? w is a subset in which the conditions 1 and 2 hold we must show that w is a subspace by definition we must show that w is a vector space I have written down all the schemes of a vector space and the ticks correspond to those which we do not have to prove the first two ticks we already have as part of the conditions of the theorem the others hold for any subset not just a subspace the other schemes hold for any subset so we do not have to prove the others they are already there for you we need to prove only these two I will prove this first and this.

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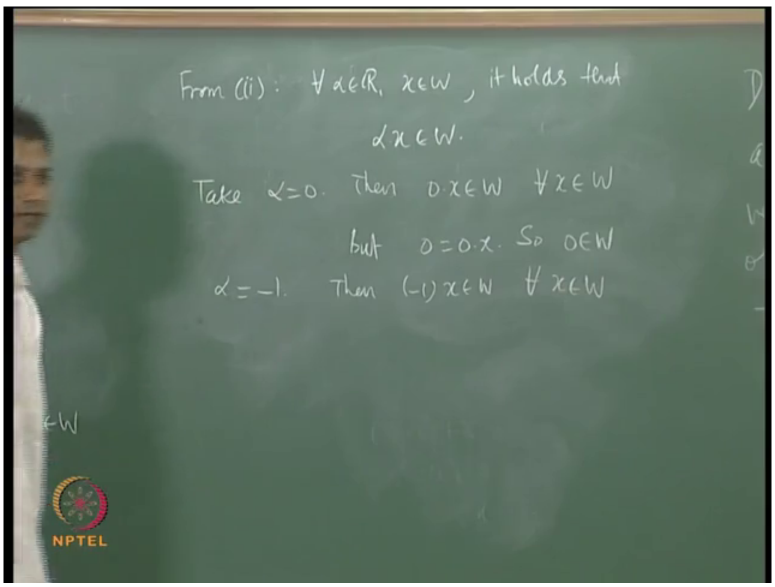

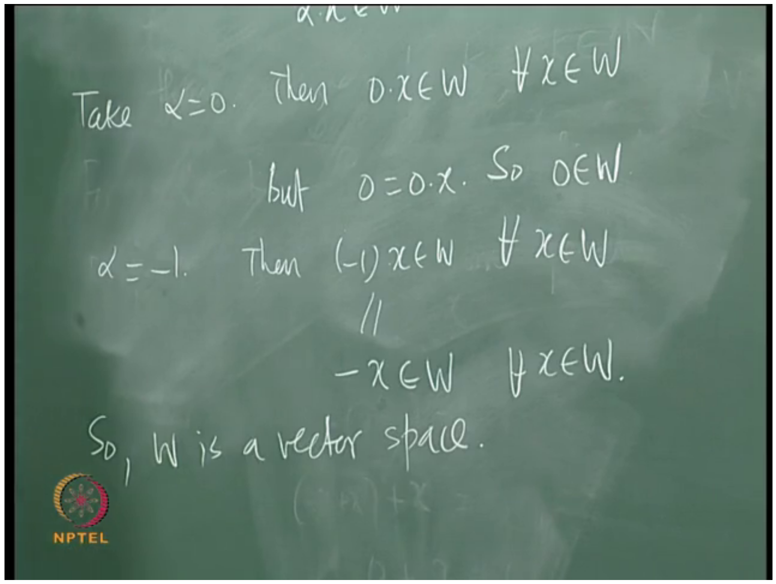

I will prove that 0 belongs to w where the 0 comes from V, okay so they share the same additive identity that must happen but that is easy, look at condition 2 from condition 2 which takes that for all alpha in R and x in w I have, okay just choose alpha to be 0, then 0 into x belongs to w for all x in w but the property first property that we proved 0 into x is 0, so 0 belongs to w, okay that is easy take the scalar to be 0.

So this also holds so let me remove, I am sorry this so let me remove this we have proved that 0 belongs to w and this is easy it is enough just to prove that 0 is in w this holds in the entire V we need to show existence of additive inverse, okay how do you do that take alpha equal to minus 1 consider the alpha equal to minus 1 I am again looking at condition 2 that is it is close with respect to scalar multiplication take alpha equal to minus 1 then minus 1 into x belongs to w whenever x belongs to w property 4 that we proved last time minus 1 into x is minus x minus x belongs to w for all x in w so that was easy, okay.

So existence of additive identity that is existence of 0 and existence of negative element and so w is a vector space in its own right with respect to the same operations of V and so w is a subspace by definition, okay. So you need to only verify that to verify that a certain subset is a subspace you need to only verify that the closure schemes are satisfied, okay.

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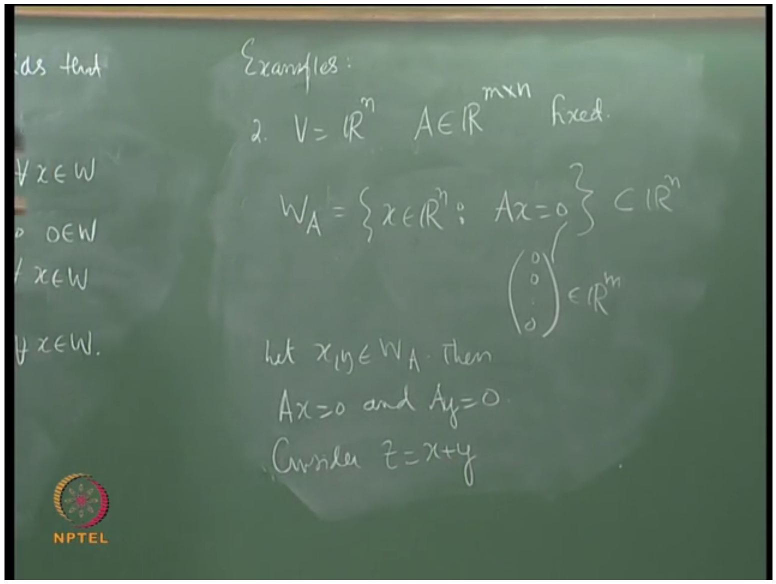

So lets us now look at examples quite a few of them examples of subspaces I have already given 1 let us look at 1 coming from linear equations homogeneous linear equations so I want to list a few subspaces I will take this as a example 2 look at V as Rn by which I mean a real vector space the underlying field is R I am given a fixed matrix of order m by n m rows n columns I will define the subspace WA I will define the subset W which depends on A as the set of all x in Rn such that Ax is equal to 0 this example was given as an example of a vector space yesterday's class I hope you remember this is a subspace of Rn remember this is contained in Rn, okay A is m cross n x is in Rn so the product is m cross 1.

So this belongs to Rm this is a 0 vector of Rm, okay to write this in detail this is 0, 0 etcetera 0 that has m coordinates then this WA is a subspace let me prove this very quickly by using the previous theorem. Let us take we only need to prove that it is close with respect to addition and scalar multiplication so let us take two vectors x, y and WA then we have Ax is equal to 0 and Ay equal to 0 solutions of the homogeneous equation I must show that x plus y belongs to WA as well as alpha x. Consider z as x plus y I will call x plus y as z I must show that Az is 0.

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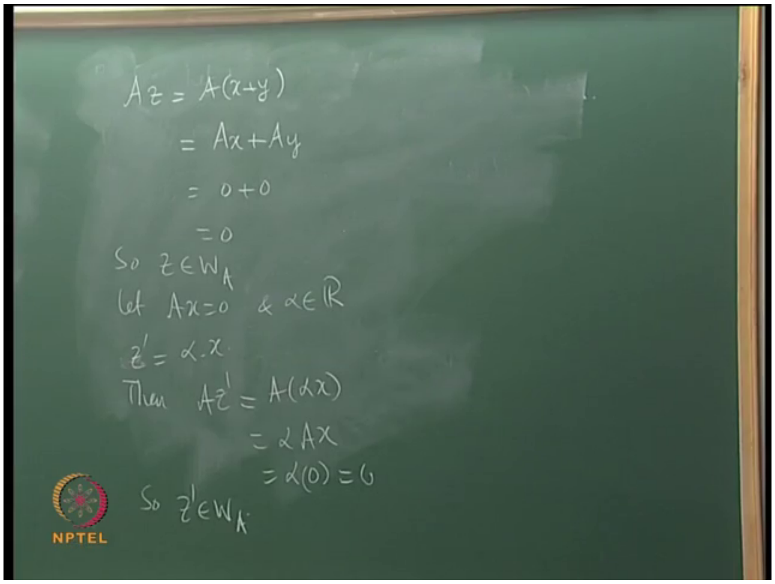

So let us look at Az Az is Ax plus y but I know that matrix multiplication is distributive this kind of a things can be split Ax plus Ay that is 0 plus 0 that is 0.

So Az is 0 that is we have shown that z belongs to WA and so it is closed with respect to addition scalar multiplication still easy, suppose Ax equal to 0 and alpha is a real number let us call w, w is not a good idea z prime as alpha into x look at Az prime that is A of alpha x but I know from matrix multiplication alpha is a scalar this is alpha into Ax alpha into 0 that is 0 so I have shown that z prime belongs to w, okay I am treating this as the first example of a subspace and proving it in detail that it is indeed a subspace.

From the next example onwards I will just wave and it means it is an exercise for you, okay. So this is a subspace, example 3 you have any questions?

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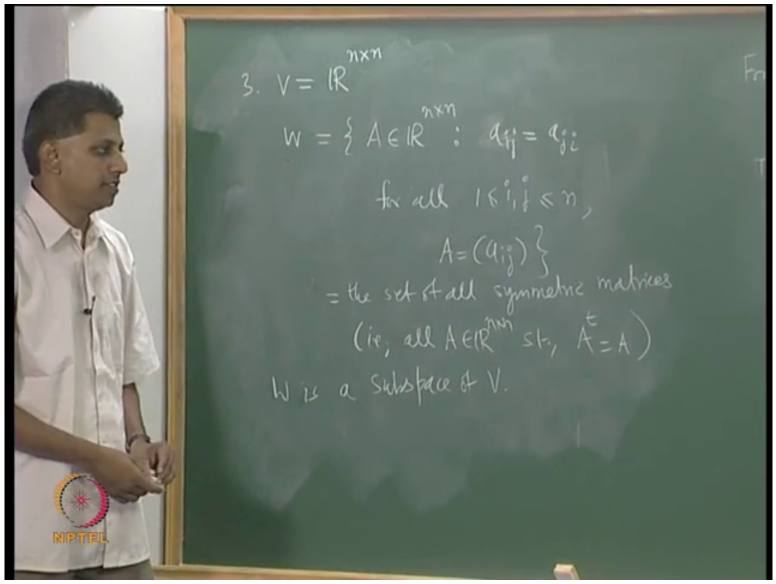

Example 3, V is R n cross n this time the space of square matrices with real entries let me define w as the set of all matrices A that satisfy this condition, okay let me write like this the notation probably I could include that also here, what is aij? For me aij is the ith row jth entry of A just to confirm this notation, okay ith row jth element is a jth row ith element these matrices are called symmetric matrices A equal a transpose, okay set of all matrices that satisfies the condition set of all matrices A that satisfies the condition A equal A transpose set of all symmetric matrices all A such that A transpose equals A if you are familiar with this operation of taking transpose the rows becomes columns as a result the columns becomes rows.

Look at this subset it is obviously a subset this is a subspace w is a subspace that is to verify that w is a subspace you must ask whether this question has an affirmative answer, if A and B are symmetric matrices then A is A plus B symmetric A plus B is symmetric obviously you can verify that even if you do not know this equation, if A is symmetric alpha is scalar alpha times A is that symmetric, yes, okay. So by the previous theorem these two conditions are enough to verify so this is a subspace.

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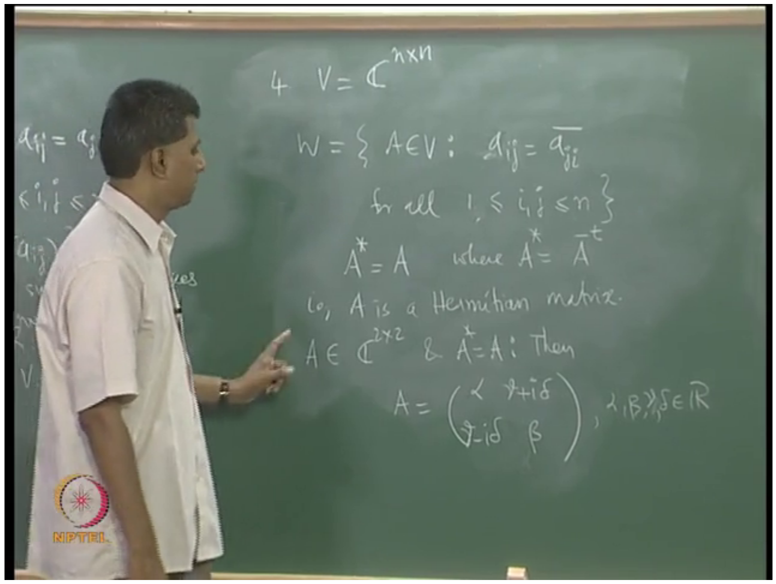

Let us try and do a similar thing for complex matrices, okay this time the underlying field is the field of complex numbers, so V is C m cross n set of all n cross n whose entries are complex numbers the underlying field is understood to be C, in this let me define w as the set of all A such that l give a similar definition, Aij is aji bar this is called the conjugate transpose, okay for all 1 less than or equal to i comma j less than or equal to n, that is A star equals A where A star is the conjugate transpose A bar transpose to write down an equation similar to A transpose equal to A I have A star equals A that is A star is you take the complex conjugates of the matrix A first and then take the transpose that is A star, such matrices are called Hermitian matrices, okay.

A is a Hermitian matrix sometimes we also use the word self-adjoin A is a Hermitian matrix, is this a subspace of V? Intuitively we would say yes, okay the answer is no, this is not a subspace of V the reason is the following I will not give the proof but you try to fill up the gaps you take a Hermitian matrix the diagonal entries of a Hermitian matrix must be real numbers, okay. Let me just write down a 2 by 2 Hermitian matrix this is a way to look like where this is easy to see a 2 by 2 Hermitian matrix just to give you a feel of the fact that this is not a subspace a 2 by 2 Hermitian matrix has this form alpha, beta, gamma, delta are real numbers this will be the complex conjugate of this the diagonal entries if you go back and check this Aii equals Aii bar so diagonal entries will be real numbers.

So if A is Hermitian then the diagonal entries are real numbers I this is a complex vector space so the scalar comes from C look at i times A i into A the diagonal entries are not real, so it is not a subspace this is not a subspace of C but this is a subspace of R this time the underlying field is R then this is a subspace, okay. So what we have done is to write down a formula similar to the real case but we observe that this is not a subspace but it is a real subspace, okay that is example 4.

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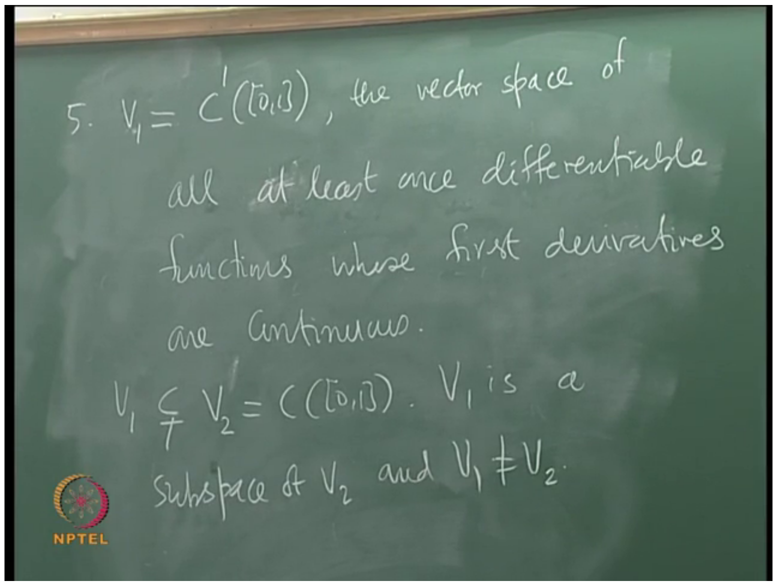

The look at I introduced this notation yesterday V is, okay let me introduce w first and then V let me say V is C1 0, 1 C1 0, 1 this is the vector space I have not included this yesterday but you can easily verify that this is a vector space and what are the elements, the vector space of all differentiable of all once at least once differentiable functions whose first derivatives are continuous similar to C 0, 1 C 0, 1 is vector space of all continuous function complex valued continuous functions this is a complex valued functions which satisfies the property they are at least once differentiable and that the first derivatives are continuous you can verify that this a vector space, okay basically two schemes to be verified if f and g are functions that belong to V then f plus g belongs to V and alpha times f belongs to V essentially that is what you need to verify.

So this is a vector space we also know from calculus that if a function is differentiable then it must be continuous and so this is a subset V prime V is C prime 01, it is not a very good notation let us say this is V1 for me and this is V2, V2 I will use C 0, 1 which I introduced yesterday, this is obviously true because every differentiable function is continuous this is a subspace V1 is a subspace V2 and V1 is not equal to V2 because there are continuous functions it is not differentiable, okay that is example 5.

I want to go back to example 2 and then specialize it for R2 and R3 the reason why I do this is these examples will be useful a little later when we discuss the notion of linear independence and dimension, okay.

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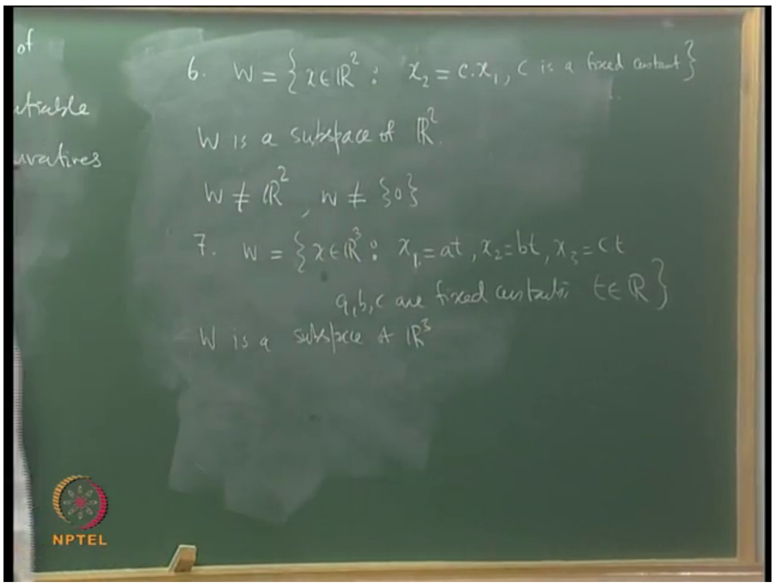

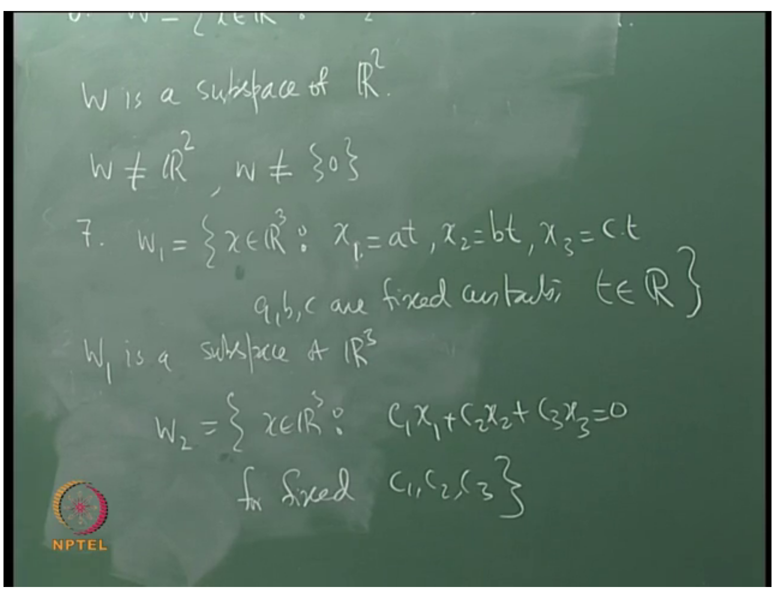

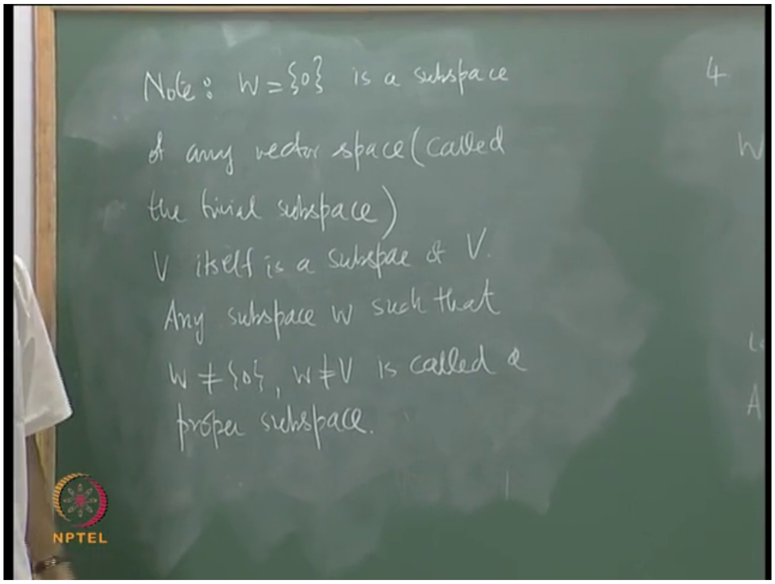

So let us go back to example 2, example 2 is set of all solutions of the homogeneous system, okay I am looking at a particular case I want look at this example 6 which is w is the set of all x in R2 the plane such that X2 equals a fixed constant C x1 C is a fixed constant real constant, okay this from (())(41:06) geometry of two dimensions we know that this is like y equal to nx plus C the constant term is 0 so this is the set of all points lying on a certain line whose slope is C lying on a certain line passing through the origin passing through the origin is important, okay w is a subspace of R2, can you see that w is not the entire R2 and also w is not singleton 0, okay.

Example 7, w is a set of all x in R3 such that so here I am following the notation that if x is in R2 then x is written as x1 comma x2, okay that is a notation that I am using always, x in R3 now x3 equals A times t x2 equals, okay let us say x1, x2 equals b times t x3 equals C times t A, B, C are fixed constants t is in R look at the set of all vectors that satisfy this condition, so can you tell me what this subset of R3 looks like? Is it similar is it not similar to the previous example this is also the set of all points lying on a straight line, this is a set of all points lying on a straight line passing through the origin if you want you can write down this symmetric form of the line x1 by A equals x2 by B equals x3 by C, okay please verify that this is a subspace set of all points lying on a certain straight line passing through the origin w is a subspace of R3 I will call this w1 I will define w2 next w2 is a set of all x in R3 such that let me use different constants now C1 x1 plus C2 x2 plus C3 x3 equals 0 for fixed constants C1,C2, C3 I collect all x that satisfy this condition.

C1, C2, C3 are fixed numbers collect all x that satisfy this condition, what is geometric interpretation this is a plane passing through the origin this is the set of all points lying on a certain plain passing through the origin, please verify that this is also a subspace, okay w2 is a subspace w1 or w2 they are not the entire R3 they are not 0, okay there are non-zero vectors in each of these let me include this and conclude today's lecture w equal to 0 is a subspace of any vector space and the rest will be called the trivial subspace just 0 that is the minimum you need to have for a subspace, okay.

For instance if you want to show that a subset is not a subspace you show that 0 does not belong to that this may work out in certain examples. So this is a subspace because 0 plus 0 is in w then whatever be alpha alpha into 0 is in w, so these are subspace the trivial subspace V itself is a subspace V will be interested in subspaces which are neither V nor singleton 0 those are called proper subspaces any subspace w such that w is not singleton 0 n not V will be called a proper subspace is called a proper subspace if you go back to the previous examples you will you have examples of proper subspaces, example 4 this is a proper subspace not every matrix is a Hermitian matrix there are continuous functions whose first derivative there are continuous functions which are not once differentiable there are points in R2 which do not lie on a certain line passing through the origin, similar here this is a plain passing through the origin there are points in R3 which do not lie on this particular plain passing through the origin, okay these are all proper subspace, okay.

I will stop here, tomorrow's lecture we will look at the notion of dimension that is the notion of linear independence dimension and then the notion of basis.