An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimentional Tori and Elliptic Curves

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Lecture – 08
Homotopy and the First Fundamental Group

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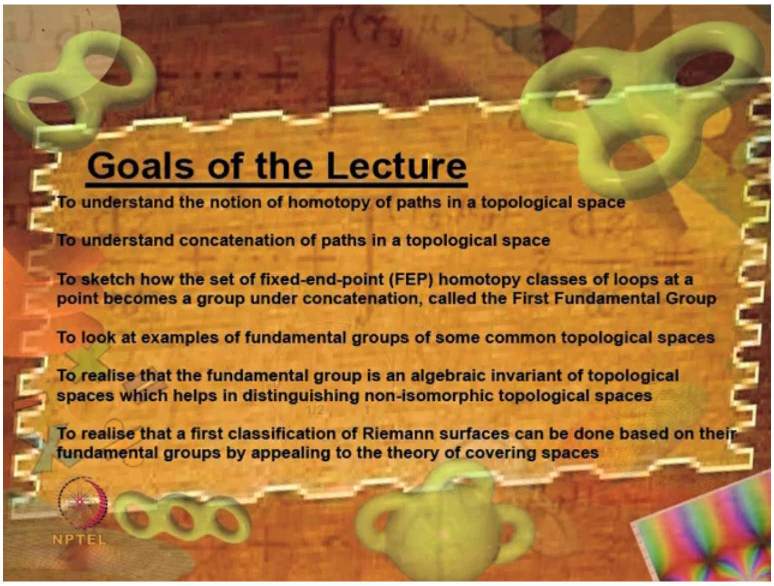

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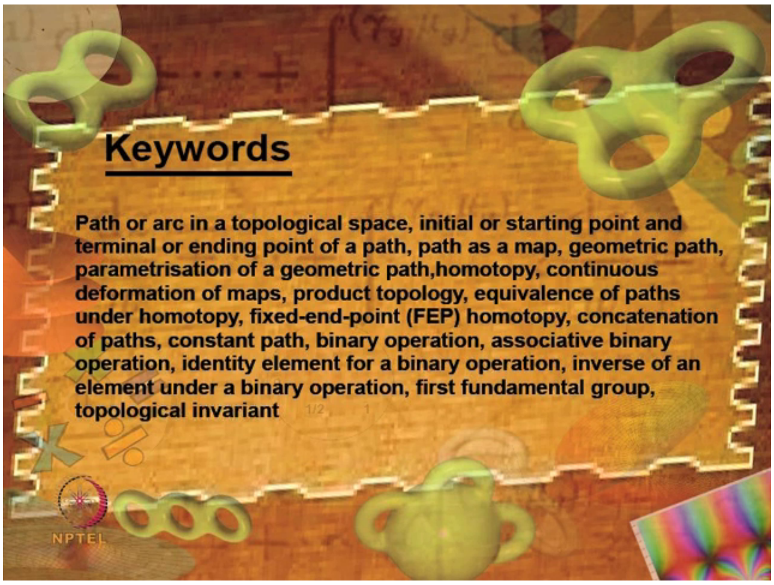

Welcome to this 8th lecture. So, in continuation with what I was saying in the last lecture the whole study of Riemann surfaces reduces to studying Mobius transformations and the device we use is or the concepts that we use are basically concepts of the covering space, covering space theory which is intimately connected with notion of a fundamental group.

So, I will first begin by quickly trying to recall what the fundamental group is give you an idea of what the fundamental group is and then I will tell you the basic classification of surfaces that you can more or less easily arrive at. So, let me first give you a brief idea of the fundamental group.

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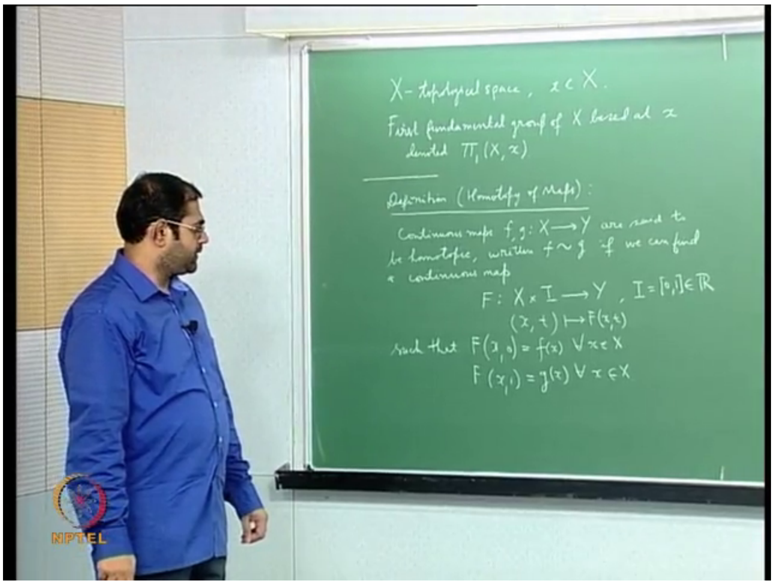

So, I start with X a topological space start with X topological space of course, you know to define the fundamental group X could have been any topological space, but please remember that at least for all practical purposes in this course the topological spaces that we are going to consider are going to be certainly Hausdorff.

So, I take a topological space and I fix a point X and I am going to define the first fundamental group of X based at the point X and it is denoted pi 1 capital X comma small x and to define this basically 1 needs to go back and understand the notion of homotopy. So, let me define what homotopy is. So, it is it is a concept in topology which tells you when 2 maps between a pair of spaces can be continuously deformed from 1 to the other.

So, I will. So, let me define Homotopy of Maps. So, continuous maps f and g from a topological space X into a topological space Y as it to be homotopic as it be homotopic as it to be said to be homotopic written f homotopic to g. If we can find a map we can find a continuous map capital f from X cross I to Y where I is the unit interval on the real line. So, it is a continuous map.

So, we write a point here as x comma t and of course, it is mapped to the point F of x comma t of Y such that F of x comma 0 gives you the first map f for all X and F of x comma 1 gives you the second map for all x in X. And so this definition actually tells you that f is homotopic to g and you can guess that this is going to be an equivalence relation eventually. So, this will also tell you or you can reduce very easily that this will also give you the g is homotopic to f, but the point is that what we have is a continuous map which is to be thought of as follows you think of t as time and you think of t as time varying from 0 to let us say 1 unit of time if you want to say 1 second. So, as time varies from 0 to 1 for each fixed time f of x comma t will give you a map from X to Y.

So, what you are actually getting is a family of maps from X to Y if you freeze if you if you think of t as a as a time parameter varying on the interval. So, so let me write that down.

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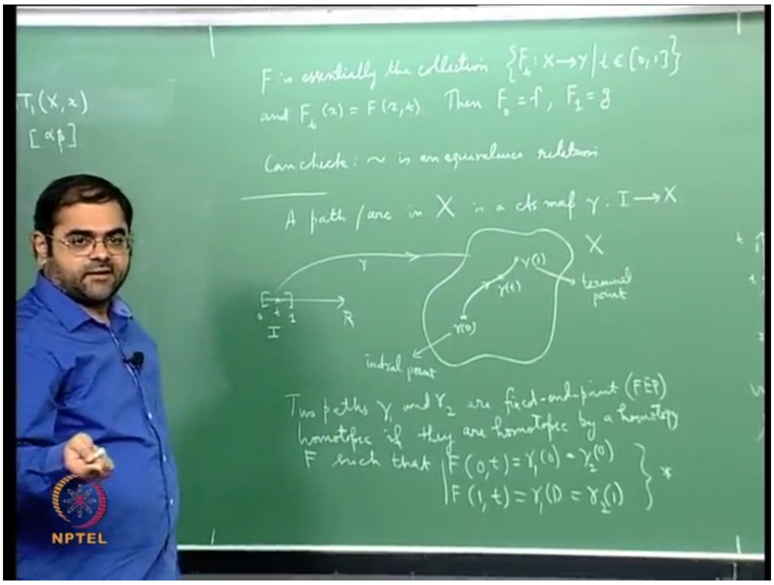

So, you know F is essentially the collection let me write as F sub t from X to Y where t is 0 is the time parameter and Ft and F sub t of x is just f of x comma t and what the definition says is that F sub 0 is small f and it says F sub 1 is small g.

So, F then capital f sub 0 is small f capital F sub 1 is a small g. So, what is happening is that you have family of maps and the family of maps starts with at time t equal to 0 it is equal to the map small f at time t equal to 1 is a map it becomes a map g and then in between every other map Ft is also a continuous map and the way the Ft varies with respect to t is also continuous that is a whole point of saying that capital F is continuous.

So, in some sense what it means is that the map f in time is being deformed into the map t into the map g as time varies and actually it is a little bit of an exercise that 1 has to do and then 1 can check that this is an equivalence relation you can check that this is an equivalence relation and can check. So, can check an equivalence relation namely you can check that it is a reflexive symmetric and it is transitive.

So, now we adapt this to the situation when the maps are actually maps from I it itself and maps from I are going to be continuous images of the close interval 0 1 and they will be of course, paths. So, from as a special case of homotopy of maps you have going to you are going to get homotopy of paths in in a topological space. So, that is what I am going to go to next. So, let me do that A path or an arc in X.

So, X is the topological space a path or an arc in X is a continuous map gamma from I to X. So, basically you picture it like this. So, here is a real line that is this closed interval 0 1 which is I and you have this map gamma and it is image is going to for you. So, this is my topological space X and this image is going to trace out a continuous curve if you want a continuous path or continuous arc on the topological space.

And you know this is this is going to be gamma of 0 which is called the initial point is called the initial point and of course, this point is going to be gamma of 1 is going to be called the terminal point and at any instant of time let us say t the path is that the point that you get here is gamma of t. So, as t varies from 0 to 1 the map gamma traces the curve the image gives you the curve.

So, we often also talk of geometrically this curve as the path where a where this is actually just a directed you know think of it is a directed or directed curve on the topological space, but of course, you know topological space by itself is very abstract. So, you do not know whether you know it really looks like a some subset of r and so that you can really think of something like this, but nevertheless 1 imagines it like that.

So, we often call the we often you know abuse language by saying that this is a path and sometimes we also say the path is also a map this is just abuse of language. Now of course, if you have 2 paths which are you know if you have 2 paths then you can of course, define homotopy between these 2 paths. So, what one can do is you can define when 2 paths are homotopic.

So, let me do that 2 paths gamma let me call them as gamma 1 and gamma 2 are fixed end point for short FEP homotopic, if they are homotopic by a homotopy F. So, I must say here that this continuous map capital F is called the homotopy is called a homotopy is the homotopy that gives you this this relation of f being homotopic to g.

So, let me add here called a homotopy. So, if they are homotopic by homotopy F such that you know F. So, you know the endpoints have to be fixed. So, it means that always F of 0 comma t is always the same point is the same point which is the initial point of both the paths gamma 1 of 0 and should also be gamma 2 0 and F of 1 comma t. So, I should be careful about my they are write it.

So, this is going to be my space here and I want I want the time variable going on the second I wanted to be going on the second coordinate. So, which means no matter what the second coordinate is my intermediate path should always have initial point and final point the same. So, this is right. So, let me let me rewrite it properly gamma 1 of 1 is equal to gamma 2 of 1.

So, if I, what I mean here is capital F is map from I cross I into X.

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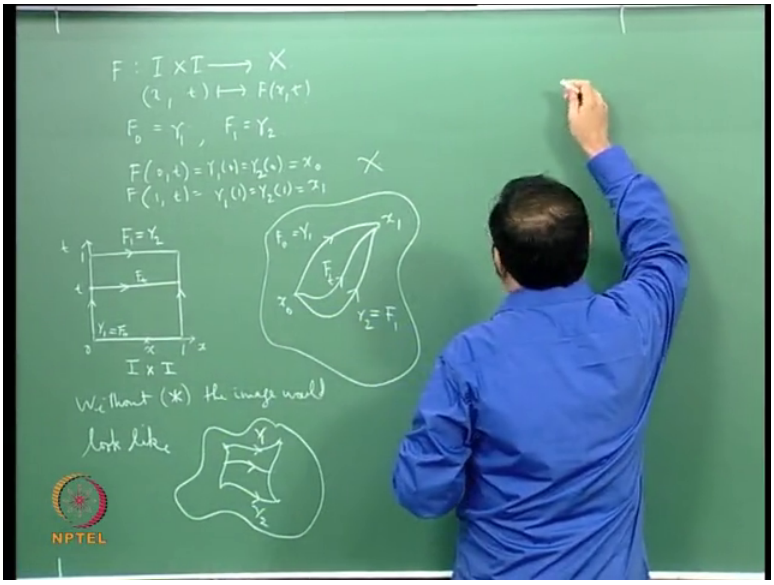

So, you know this X has been replaced with Y and this Y has been replaced with X I am sorry for the sorry if it causes any confusion, but let me again write it here. So, you know this is let me write this is again as x comma t going to f of x comma t and of course, you know F F sub 0 has to be gamma all right gamma 1 and F sub 1 has to be gamma 2 that is a saying that the 2 paths are homotopic, but of course, I want also that the end path the endpoints of each intermediate path also to be the same as any of the any of gamma 1 or gamma 2 and that is given by the condition by these 2 conditions F of 0 comma t is always gamma is a point gamma 1 of 0 is equal to gamma 2 of 0 and F of 1 comma t is gamma 1 of 1 to gamma 2 of 1.

So, in effect what is happening is the following diagram here that helps a lot to visualize. So, this is I this is just I cross I. So, here I have you know here I have x along this axis and I have t along this axis all right and this is just in the r 2 plane this is just the unit square which gives you I cross I and you know and then you have the topological space here X and what happens is that if you freeze a certain time t then you get so the, this is the second coordinate this is the first coordinate. So, if I freeze the time t then if you vary X you get this you get this line and on this. So, at time t you get the path F sub t.

So, at time t equal to 0, so this this edge will correspond to the path F 0 which is which is gamma 1 and this edge that will correspond to t equal to 1 will correspond to F 1, which will be gamma 2 and along this along this segment what is happening is x is 0, but t is varying. So, what you will get is F of 0 comma t and; that means, that you know F of 0 comma t should be the initial it should be the initial point for both of the paths and so this edge is mapped to 1 point and then this edge which corresponds to x equal to 1.

So, when x is 1 you get F of 1 comma t as t varies and that is mapped to the terminal point. So, finally, this whole diagram will look like this. So, I will have you know gamma 1 I will have gamma 2 here gamma 1 is the map at F 0 time 0 gamma 2 is a map at time 1 and at any at any intermediate point of time the map is going to be F sub t and this point is going to be well if I call this as X 0 this is going to be the point X 0 and I call this point as X 1 this is going to be the point X 1. So, this whole line segment is going to be mapped to this point this whole line segment is going to be mapped to that point and every inter this line segment is going to be essentially gamma 1, this line segment is gamma 2 and then if you take anything in between that is going to give you the path F sub t.

So, this is again a homotopy of gamma 1 with gamma 2 by a continuous family of maps parameter is by the time by the time parameter and of course, the endpoints have been fixed and in fact, the way I got this diagram you can you can think about it that if I did not fix the endpoints what I will get is something that looks like a twisted type of image of the square.

So, you would have got you would have got if I had not put this condition if I had not put this condition. So, let me let me call this condition as say star if I had not put without star without the condition star the image would look like you know something like the following you would have got. So, this would be gamma 1 you know and this would be the initial point this would be the final point and then gamma 0 gamma 2 would be something else and the intermediate paths would have gone on to be like this.

So, it would have look like this, but then the fixed endpoint condition is that all of this collapses to a point all of that collapses to a point and you get a diagram like this. So, these are all diagrams which help you to you know fix ideas individualize. So, now, you know what homotopy of 2 path is then we what we do next is we try to we what we want to do is somehow, we want to capture the idea that you know suppose you are for example, on a surface and suppose as a whole for example, something like an annulus or punctuate plane or punctuate disc. If I have a loop surrounding that I cannot continuously collapse it to a point because at some point there is a hole and I cannot move the loop across the hole. Therefore, I will have to study you know homotopies of loops and how do you interpret a loop you interpret a loop as a path starting from a given point and coming back to the same point.

So, and then you one would like to make a group out of these things. So, you will have to combine loops to form newer loops and in general that would mean that you should be able to combine paths and form new paths and this thing is called concatenation of paths and this is the way it is done it is done in a very obvious way. So, let me do the let me write that down here concatenation of paths.

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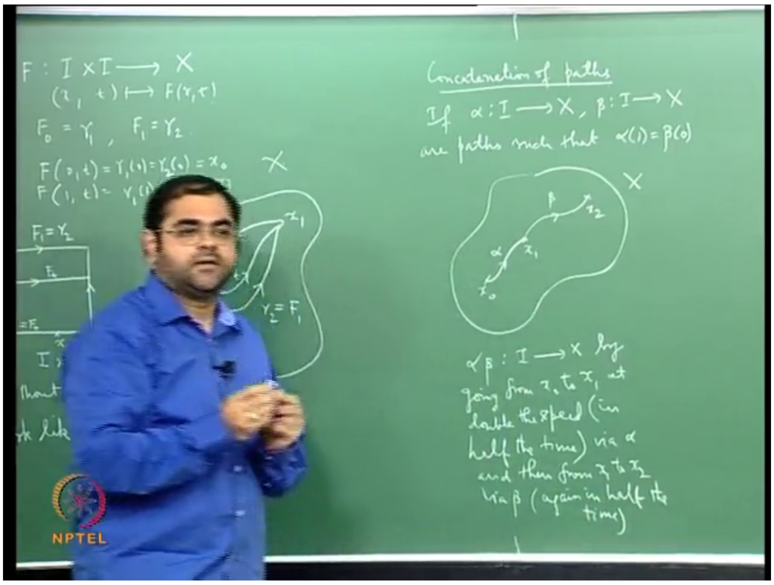

So, suppose you know alpha is a path from is a path in X and beta is another path in X are paths such that the end point of alpha is the initial point of beta. So, the situation is as follows you have you have X here you have a path alpha which starts let us say at x 0 and ends at x 1 and then you have part b you have a path beta let us say which starts at x 1 and ends at x 2 . So, the end point of alpha which is alpha 1 is the same as the initial point of beta which is beta of 0 and that is that is x 1 in this diagram. There is an obvious way of combining these 2 paths namely take the path gotten by first going along alpha and then going along beta that is obvious.

So, if you want to really write it down as a path the only thing is that you have to scale the time parameter because the way I have written it alpha is supposed to go from here to here in unit time and beta is supposed to go from here to there in unit time if I compose it and I want a path again to go in unit time I should do it faster. So, what I do is I define the path alpha beta the path from I to X by going from x 0 to x 1 at double this double the speed if you want at double the speed or that is the same as in half the time.

And then go from and of course, go from x 0, x 1 via alpha via alpha a double speed and then from x 1 to x 2 via beta again in half the time half the time or if you want double the speed. So, how does 1 can write down equations for this it is very very simple?

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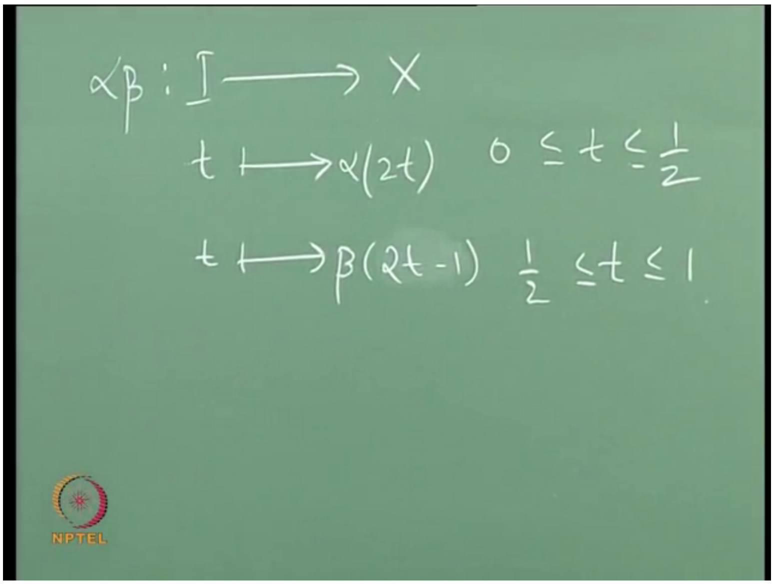

So, alpha beta is the map from I to X it is defined like this for time varying from 0 to half I want the whole of alpha to be covered.

So, what I do is I will send it to alpha of 2 t if I do this at t equal to 0 I will get alpha of 0 and at t equal to half I will get alpha of 1. So, in half the in half the time which you can think of as trace the speed you have you have gone half way by alpha and then you send t to beta of 2 t minus 1 for t varying from half to 1, probably I should write 1 minus 2 t I should write 1 minus 2 t 1 minus you can see that it is; obviously, scaled. So, that when t is half I get beta of 0 and when t is when t is 1 no that was right was it was not it 2 t minus 1 was right yeah 2 t minus 1 when t is half it is 0 when t is 1 it is beta of 1.

So, and that the point is that you can check that this is continuous function because the only problem is you have to check continuity at half, but at half the values match that is because alpha of 1 is beta of 0. So, this is the way you multiply 2 paths, you compose 2 paths and then you know you can check that this concatenation of paths is behaves well under homotopy.

So, can check the following that if alpha is homotopic to alpha 1 and beta is homotopic beta 1 then alpha beta is homotopic to alpha 1 beta 1.

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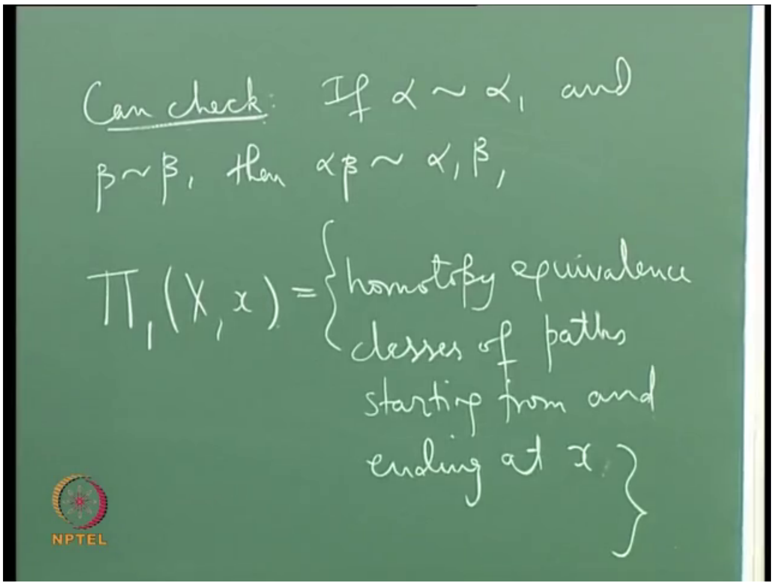

So, the moral of the story is that the concatenation of paths respects homotopy classes and further ah. So, as I told you we finally, we want to look at loops. So, we put the condition that you have paths starting from a fixed point and ending at the same fixed point.

So, what you do is that you define pi 1 of X comma x. So, to be the homotopy equivalence classes of paths starting from and ending at x. So, it is a set of homotopy equivalence classes of paths starting from and ending at the point x.

So, we essentially these are loops starting at X and ending at x all right. So, and then this fact that homotopy respects I mean the concatenation the concatenation of paths the multiplication of paths respects homotopy tells you that this gives you a binary you know operation on the set. So, the point is that that makes setting to a group and that is called the fundamental group of X base at the point small x.

So, let me write that down.

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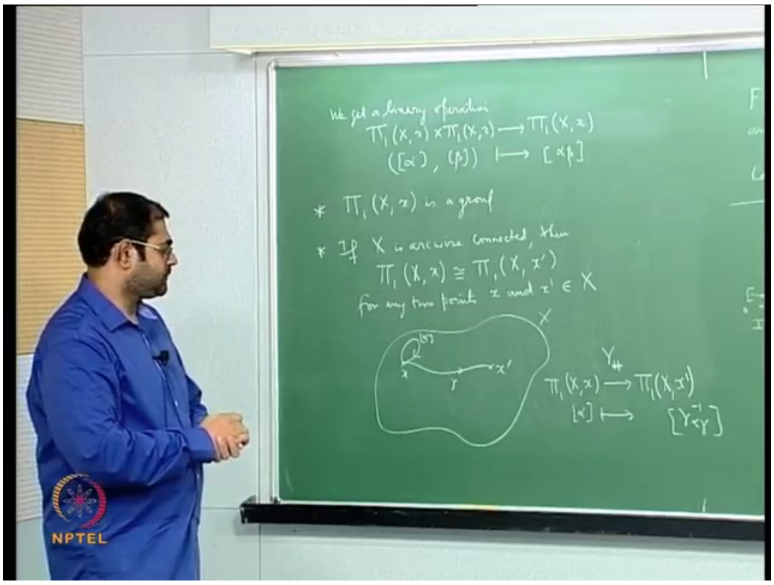

So, we get a binary operation pi 1 X comma x cross pi 1 of X comma x to pi 1 of X comma x, which is well alpha equivalence class comma beta equivalence class going to the equivalence class of alpha beta that this is so, here when I put this square bracket I mean the equivalence homotopy of equivalence class of the path alpha and this is the homotopy equivalence class of the path beta and this is the concatenation that this that you get a well-defined map like this itself depends on this fact that you know your concatenation respects homotopy.

So, the point is that for under this operation pi 1 X comma capital X comma small x becomes a group. So, what are the other things that you will have to verify you will have to verify that this this operation is associative, then you will have to verify that there is an identity element for this operation and that is very easy to see the identity element is just the constant path see because I am looking at path starting at x and ending at X I can simply look at a path which is always at X that is a deep degenerate path you map the whole interval to the point X that is a path by definition because it is a constant map is always continuous.

So, that is by definition of path it is called the constant path at X and that path simply does not do anything it just it just stands there it stays put and; obviously, if you concatenate with any path you are going to get back the path you concatenate it with. So, it is going to act as an identity element and therefore, the constant path will act as an identity element and then associativity is something of an exercise that you can try it is not. So, difficult, but the point is why do you get inverses it is very very simply the if you give me a path starting at x and ending at X the inverse path is just traversing the same path, but in the reverse direction.

So, essentially in the equation of the path you replace t by 1 minus t. So, that you would have see if I replace if in a map like this for example, if in a map like this a gamma of t if I take gamma of 1 minus t then you can see that I am going to get I am going to get the path in the reverse direction. So, the moral of the story is that that provides replacing t by 1 minus t automatically provides you with a path which is inverse to the given path because if you concatenate it you will get you can see that it is going to be the trivial path and it is going to be homotopic it is going to be homotopic to the constant path.

So, basically the argument is that I am saying take a path or in this case I can even say it in terms of loops. So, take a loop and then it is inverse will be going the other way and then I am saying this followed by this can be deformed to the constant path in the way I do it is just is just by you know by just pulling this loop out at time t. So, that I just come back to 0.

So, it is very easy to see that you know you can you can you can easily see that the there is an inverse element and maybe some maybe it requires some writing down all right. So, you get an inverse element for each path and then this becomes a group. So, so let me right that here pi 1 of X capital X comma small x is a group this becomes a group.

And, so this becomes a group and this is called the fundamental group of the topological space capital X based at the point small x all right. Now why is it helpful it is helpful because it helps you for example, distinguish between topological spaces for example, first of all you should notice that this this group for it to be meaningful 1 has to be kind of I am not. So, dependent on the point x and that can be easily achieved by putting an extra condition on the topological space that is making X arc wise connected.

So, the point is that if capital X is arc wise connected then pi 1 of capital X comma small x is isomorphic to pi 1 of capital X comma small let me say x prime for any 2 points x and x prime in X. So, by putting this arc wise connectedness you can show that the fundamental group is the same of the isomorphism and this is achieved in a very very simple way, if you take an element of pictorially you know. So, this is my this is my topological space and you know this is the point small x this is a small this is a point x prime.

And if I take an element here this is a loop based at x. So, I am going to get something like this and it is going to be represented by alpha the equivalence class will say let me write it as square bracket alpha and how do I get a get an isomorphism from here to there what I do is that since X is arc wise connected any 2 of it is points can be joined by a arc by an arc. So, I join this by an arc and let me call that arc is gamma that is a definition of arc wise connectedness.

So, I can get a path from x to x prime choose 1 such path and then what you do is that you first go by gamma inverse, which is just the reverse of the bath gamma and then you go by x and then you follow it by gamma and this in principle gives you a loop base at x prime. So, mind you when I say a loop you should also understand that if I go from the point to a certain by a path and come back by the same path, that is also called that also qualifies to be called a loop, because my only condition is that my starting point should be the same as the ending point and that is the reason why because that is the reason why if I have a path like this a loop like this it is inverse is this way and because the combination, I can pull out this end and shorten it to paths smaller paths going back and forth and then finally, I can shorten it to get the constant part at the at the point.

So, what happens is that the isomorphism in this case is given by pi 1 X comma x to well pi 1 X comma x prime is given by. So, this is let us call it gamma hash this just takes any alpha to well first like gamma inverse, then apply gamma inverse followed by alpha followed by gamma and the equivalence class of at .

So, when I write something like this mind you I am not thinking of gamma as a map. So, this should not be this should not be compose confuse with composition of maps. So, when I write something like this I mean first go by gamma inverse then go by alpha and then go by gamma. So, do not confuse it with composition of maps right and you can check that this is a these an isomorphism of groups the constant path will go to the constant path. In fact, if I take the constant path here it will go to gamma inverse gamma and gamma inverse gamma is again homotopic to the constant path that x prime.

So, you can check this a group a homomorphism you can check it is an isomorphism because there is an obvious way to define an inverse by using the path gamma inverse I can define I can send any loop here to a loop there in the same way. So, and that will be an inverse to this map and it is a group isomorphism and the moral of the story now is that if x is arc wise connected I do not care which point I will fix the fundamental group up to isomorphism is the same.

So, we normally work with arc wise connected spaces and do not care about the point. So, well now why is the fundamental group? So, important as I told you it is important because it helps you to distinguish between topological spaces and that is because it is what is called a topological invariant.

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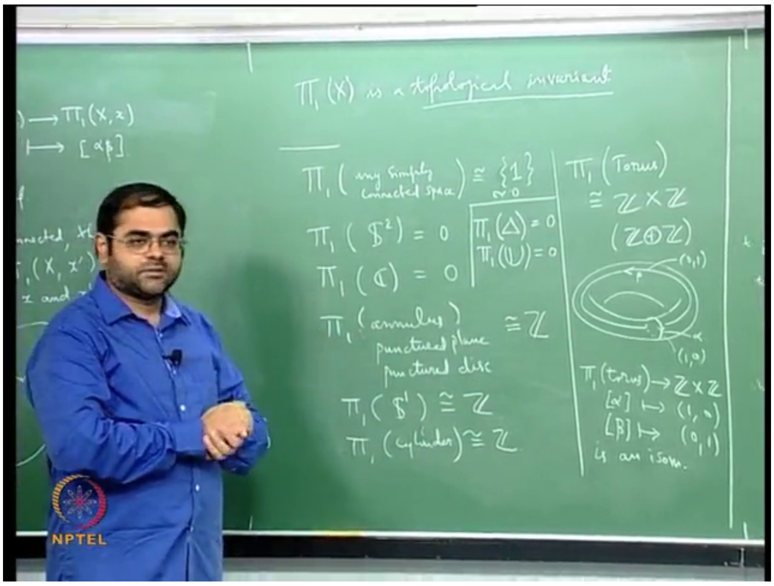

So, pi 1 of X comma x so, let me simply write pi 1 of X.

So, pi 1 of X is a topological invariant, what does this mean; it means that you know if you change the topological space by a homeomorphism, then the fundamental group first fundamental group will change only up to isomorphism. So, in particular if 2 topological spaces are homeomorphic the fundamental groups have to be isomorphic of course, I am assuming they are all arc wise connected. So, I am not worried about the base point.

So, if 2 2 topological spaces are homeomorphic the fundamental groups are isomorphic. So, if I want to decide the 2 topological spaces are not homeomorphic all I have to do is to show that they have different fundamental groups. So, you know for example, if you want to show that let us say the torus is not homeomorphic to the sphere the easiest way is to say that the fundamental group of the sphere is trivial is a trivial group whereas, the fundamental group of the torus is non-trivial and saying the fundamental group is non-trivial is the same as trying to find loops that cannot be continuously shrunk to a point and on the torus of course, you know I can I can I have 2 types of loops which cannot be shrunk to a point.

So, that argument will tell you immediately that the torus and the sphere the real 2 sphere and torus they are not homeomorphic. So, that is the advantage of having the fundamental group and this is a key ingredient in the theory of covering spaces. So, let me write down some simple cases. So, pi 1 of any simply connected space is a 0.

So, when I write this of course, I am assuming that is a topological space which is you know arc wise connected and by simply connected I mean that every loop can be continuously shrunk to a point and if I can shrink a loop continuously to a point it means that I can deform that loop to the constant map at that point which is the identity element. So, you are saying that every element is the equivalence class of every element is the same as the equivalence class of the identity element that was if you take equivalence classes there is only 1 equivalence classes.

So, the group reduces to 0 maybe the only they of course, I write 0 are maybe I want to be it is written 0, but maybe if you want I should write one. So, usually the you use 0 if you know the group is additive, but and then you use additive notation only if the group is Abelian, but fundamental groups can be highly non abelian and well, but in many books you will see pi 1 0.

So, in particular for example, the first fundamental group of say the sphere is going to be 0 first fundamental group of the complex plane is going to be 0, the same thing will happen to any space that is that can be contracted to a point any contractible space is a space that can be continuously contracted to a point. So, it will happen to this it will happen to this it will happen to the say unit disc a half plane, but of course, it would not happen to an annulus or a cylinder.

So, what happens is that if you take pi 1 of annulus or if you take you know punctured plane or if you take punctured disc all of these will have pi 1 isomorphic to Z and of course, in a sense the 1 set plane is also an annulus with inner radius Z tending to 0 and the inner radius 0 and outer radius infinity in a certain sense and if you keep the inner radius 0 and fix the outer radius you are going to get a puncture disc.

So, all of these and this is also the same as this also the same as pi 1 of S 1 which is the unit circle the set of uni modular complex numbers that is also going to be Z and you can see that pi 1 of of course, this is also the same as pi 1 of the cylinder that is also going to be Z then you can see that if you take a torus pi 1 will be Z cross Z or Z direct some Z.

So, let me write that down pi 1. So, so here let me also write pi 1 of delta is 0. So, here r 0 on pi 1 where delta is of course, in a disc pi 1 of the upper half plane is upper half plane which is by holomorphic to the unit disc by any Mobius suitable Mobius transformation that is also 0 and pi 1 of the torus is going to be isomorphic to Z cross Z which is written as Z direction Z.

So, essentially what is happening is that if you take the torus you can the fundamental group is Z cross, Z which means that Z cross Z as a group is generate has 2 generators namely 1 comma 0 and 0 comma 1. So, 1 comma 0 should correspond to a particular kind of loop and 0 comma 1 should correspond to a particular kind of loop and these are the obvious ones namely 1 comma 0 is a loop like this a loop like this it could be you can label it 1 comma 0, corresponding to 1 comma 0 and you can take loop like this and call this loop as 0 comma 1 and you can see that or rather what I mean is that you take the equivalence suppose I call this group as suppose I call this loop as alpha and suppose I call this loop as beta this 1, what I mean is that from the fundamental group of the torus to Z cross Z if you define the map by sending alpha to 1 comma 0 and beta to 0 comma 1 of course, I should be careful I should say homotopical class and if you want I fix a base point I that that can be done.

So, I can take a base point which is common to the which is the intersection of these circle and that circle. So, that I have loops based at 1 point and then the point is this map is an isomorphism is an isomorphism. So, these are all computations that you can look up in any basic book on algebraic topology and you are invited to justify at least these groups are as they are as I have written it these are very basic exercises you can you can do that.

So, having done this this is a very brief introduction to the idea of a fundamental group and then let me see let me tell you how it continuous how the; however, I mean discussion continues it continues in the following way you take topological space X and what you do is you are trying to look at a space which sits on top of this, which is such that it is like a bundle of fundamental groups of this space based on this space itself and that is what the that is what gives rise to a covering space.

So, basically you have the topological space X capital X and for every point small x you put a copy of it is fundamental group above and then you take this union disjoint union of all these copies of the fundamental group together all right along with the natural projection, the fact is that this fits into a very nice picture namely the picture of a universal covering space for X and of course, for this to happen you have to put some conditions on X and the conditions on X are that X has to be you know connected, arc wise connected, locally arc wise connected and also locally simply connected.

So, if this is true then this this procedure really fits into the picture of a universal covering space for X and as I told you studying these universal covering spaces for Riemann surfaces is what helps you to classify Riemann surfaces.

So, I will stop here.

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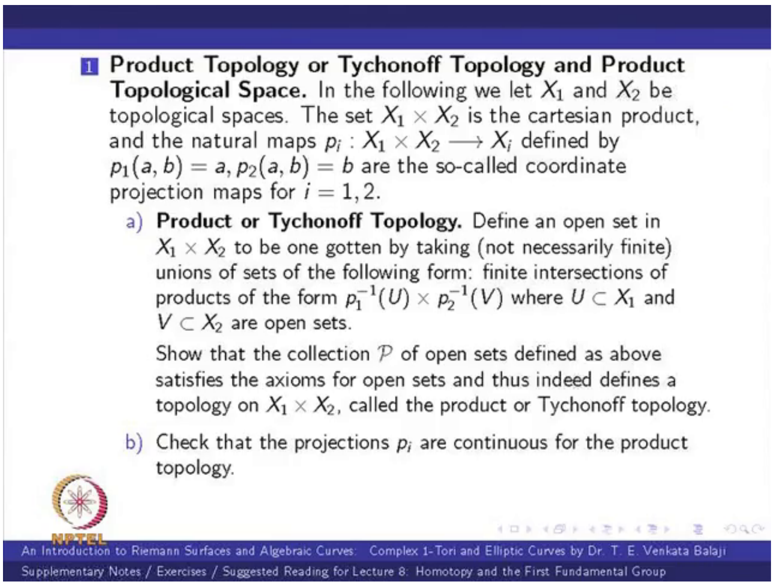

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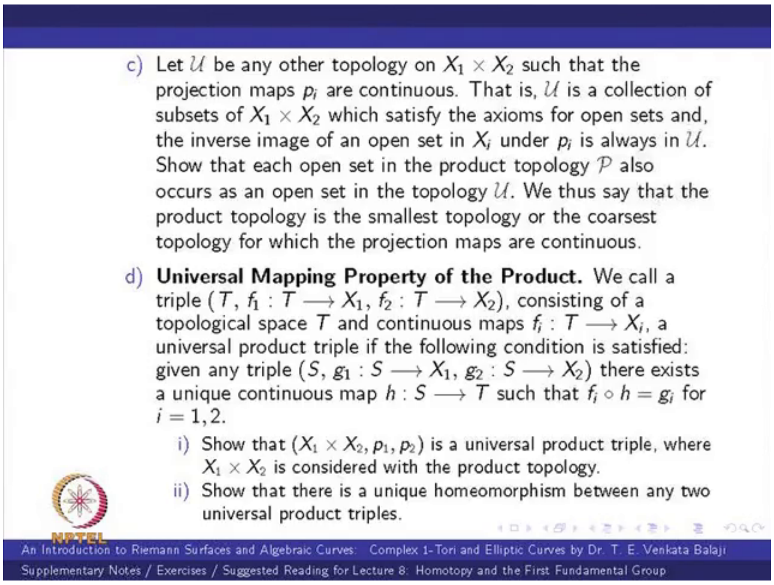

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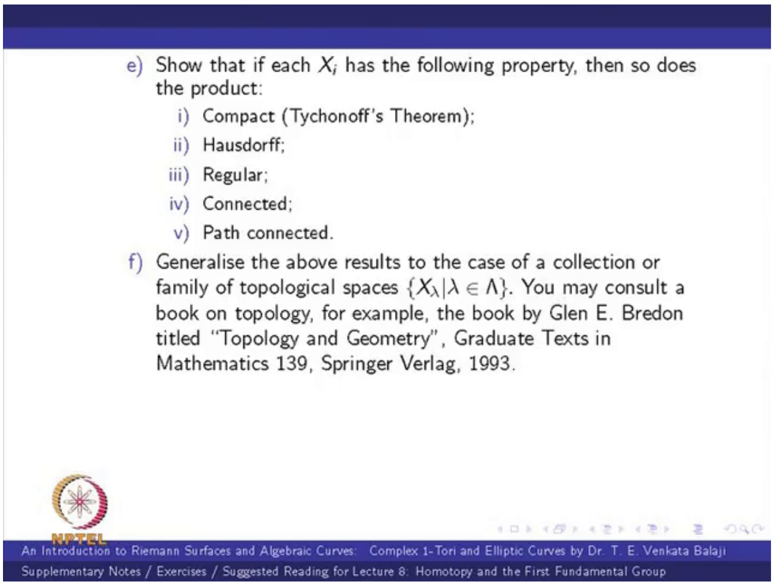

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2 Compact-Open Topology. Let X, Y be topological spaces. Let $\text{Maps}_{Cts}(X, Y)$ denote the set of continuous maps from X to Y. Given a pair (K, U) where $K \subset X$ is a compact subset and $U \subset Y$ is an open subset, let $\langle K, U \rangle$ denote the subset of $\text{Maps}_{Cts}(X, Y)$ of continuous maps that take K into U. Let open sets in $\text{Maps}_{Cts}(X, Y)$ be defined as those gotten by taking (not necessarily finite) unions of finite intersections of subsets of the form $\langle K, U \rangle$ for various K and U. Show that the collection of open sets so defined satisfies the axioms for open sets. This gives a topology on $\text{Maps}_{Cts}(X, Y)$ called the Compact-Open topology.

If X is a locally compact Hausdorff space, show that $F : X \times T \longrightarrow Y$ is continuous iff the corresponding map $T \longrightarrow \text{Maps}_{Cts}(X, Y) : t \mapsto F_t = F(x, t)$ is continuous. Here we consider $X \times T$ with the product topology and $\text{Maps}_{Cts}(X, Y)$ with the compact-open topology.

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3 Homotopy of Maps as a Continuous Deformation of Maps. Let $f, g : X \longrightarrow Y$ be two continuous maps, and $F : X \times I \longrightarrow Y$ be a homotopy from f to g. Thus, if we set $F_t(x) = F(x, t)$ then $F_0 = f$ and $F_1 = g$. Recall that $I = [0, 1] \subset \mathbb{R}$. If X is locally compact Hausdorff, it follows from the exercises on the compact-open topology above that $\{F_t : X \longrightarrow Y | t \in I\}$ is a family of maps varying continuously with $t \in I$. Since continuous variations are usually called deformations, we may say that the homotopy F specifies a deformation of f into g. We usually think of it as a deformation over unit time, starting with $f = F_0$ at time $t = 0$ and ending with $g = f_1$ at $t = 1$.

In particular, if $X = I = [0, 1] \subset \mathbb{R}$, this shows that a homotopy F of the path f to the path g is a continuous variation or deformation of f into g. The members of the deformation are the paths $\{F_t | t \in I\}$.

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4 Homotopy is an Equivalence Relation.

Let $f, g, h : X \longrightarrow Y$ be maps and define the relation $f \sim g$ if there is a homotopy $F : X \times I \longrightarrow Y$ from f to g.

Show that $\sim$ is an equivalence relation. That is, prove reflexivity: $f \sim f$, $\forall f$, symmetry: $(f \sim g) \Rightarrow (g \sim f)$, $\forall f, g$ and transitivity: $(f \sim g)$ and $(g \sim h) \Rightarrow (f \sim h)$, $\forall f, g, h$.

For proving transitivity, you may need the so-called Glueing or Pasting Lemma, which says that two continuous maps defined on a pair of open (or closed) subsets whose union is the whole space, give rise to a unique continuous map on the whole space if they agree on the intersection of these two subsets.

Given homotopies F from f to g and G from g to h, you can get a homotopy from f to h by deforming first via F in half the time and follow by deforming via G. The subsets involved here are $X \times [0, 1/2]$ and $X \times [1/2, 1]$.

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5 Show that fixed-end-point (FEP) homotopy is also an equivalence relation, for a fixed choice of initial and final points.

6 Geometric Paths and Parametrisation.

Let $\gamma : I \longrightarrow X$ be a path. When we write γ as a function $\gamma(t), t \in I$ we call t a parameter. We call the image $\gamma(I) \subset X$ as the geometric path in X defined by γ.

The same geometric path could have different parametrisations: for example,

$\gamma_1(t) = e^{(2\pi ti)}$ for $0 \leq t \leq 1$,

$\gamma_2(t) = e^{(4\pi ti/3)}$ for $0 \leq t \leq 1/2$;

$\quad = e^{(2\pi i(4t-1)/3)}$ for $1/2 \leq t \leq 1$, and

$\gamma_3(t) = e^{(-2\pi it)}$ for $0 \leq t \leq 1$

are different parametrisations of the same geometric path viz. the unit circle in the plane, starting and ending at $(1, 0)$.

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Different parametrisations of the same geometric path may or may not be homotopic: show that all the above paths are homotopic in the plane if we do not insist on fixed-end-point (FEP) homotopy; however, if we do insist on FEP homotopy, γ_1 and γ_2 are homotopic, but neither is homotopic to γ_3.

Note that a path is always thought of as a map from I to X; so it has a starting or initial or beginning point, namely the image of $0 \in I$ and an ending or final or terminal point, which is the image of $1 \in I$. This means that a path is always directed. Thus for example γ_3 is not FEP homotopic to any of the other two paths since it is in the opposite direction.

Note also that homotopy depends on the space in which the paths are being considered. For example, if we take the space to be an open annulus centred at the origin with inner radius $1/2$ and outer radius $3/2$, then check that γ_3 is homotopic to neither of the other two.

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7. Verify that the concatenated path $\alpha\beta$ as defined in the lecture is indeed a continuous map.
8. If $\alpha \sim \alpha_1, \beta \sim \beta_1$ and $\alpha(1) = \beta(0)$, then show that $\alpha\beta \sim \alpha_1\beta_1$ in two steps: first show $\alpha\beta \sim \alpha_1\beta$ and then $\alpha_1\beta \sim \alpha_1\beta_1$. All the homotopies here are FEP homotopies.

   This shows that concatenation of loops at $x \in X$ defines a binary operation $[\alpha][\beta] := [\alpha\beta]$ on the first fundamental group $\Pi_1(X, x)$ of X based at x, where $[\alpha], [\beta]$ are the FEP-homotopy equivalence classes of α, β respectively.
9. Let α, β, γ be paths with $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ so that $\alpha\beta$, $\beta\gamma$, $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are defined. Show that $(\alpha\beta)\gamma \sim \alpha(\beta\gamma)$. All the homotopies involved here are FEP homotopies.

   This shows that the binary operation in the first fundamental group $\Pi_1(X, x)$ of X based at x induced by concatenation is an associative operation.

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10. For each $x \in X$, let $e_x : I \longrightarrow X$ be the constant path at x i.e., $e_x(t) = x \, \forall t \in I$. This is indeed a path, since constant maps are continuous. Show that if $\alpha : I \longrightarrow X$ is a path, then $\alpha \sim \alpha e_{\alpha(1)} \sim e_{\alpha(0)}\alpha$ (FEP-homotopies).

    Deduce from this that $[e_x]$ acts as the identity element for the binary operation on $\Pi_1(X, x)$ induced by concatenation.

11. Let $\alpha : I \longrightarrow X$ be a path. Define $\alpha^{-1} : I \longrightarrow X$ by $\alpha^{-1}(t) = \alpha(1-t)$. Check that α^{-1} is continuous, hence a path.

    Prove $\alpha\alpha^{-1} \sim e_{\alpha(0)}$ and $\alpha^{-1}\alpha \sim e_{\alpha(1)}$ (as FEP-homotopies).

    This tells us that an element $[\alpha] \in \Pi_1(X, x)$ has the element $[\alpha^{-1}]$ as its inverse element $[\alpha]^{-1}$ for the binary operation induced by concatenation. Combined with the earlier observations on associativity and existence of identity element, we may conclude that $\Pi_1(X, x)$ is a group.

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12. Find an example of a topological space for which the fundamental group is not abelian (or commutative). Hint: try a 2-torus.

13. Show that the map $\gamma_\# : \Pi_1(X, x) \longrightarrow \Pi_1(X, x')$ defined in the lecture by $[\alpha] \mapsto [\gamma^{-1}\alpha\gamma]$ is indeed well-defined and further that it is an isomorphism of groups. That is, show that this definition depends only on the FEP-homotopy equivalence class of α, that the map respects the group operations, is injective, and is surjective with a natural inverse map $\gamma_\#^{-1} := (\gamma^{-1})_\#$ defined using γ^{-1} in place of γ.

    This tells us that if X is arcwise (or pathwise) connected, then its fundamental group based at any point is isomorphic to the one based at any other point.

    Thus the isomorphism class of its fundamental group does not change with the base point.

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