An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves
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Lecture – 07
Moebius Transformations make up Fundamental Groups of Riemann Surfaces

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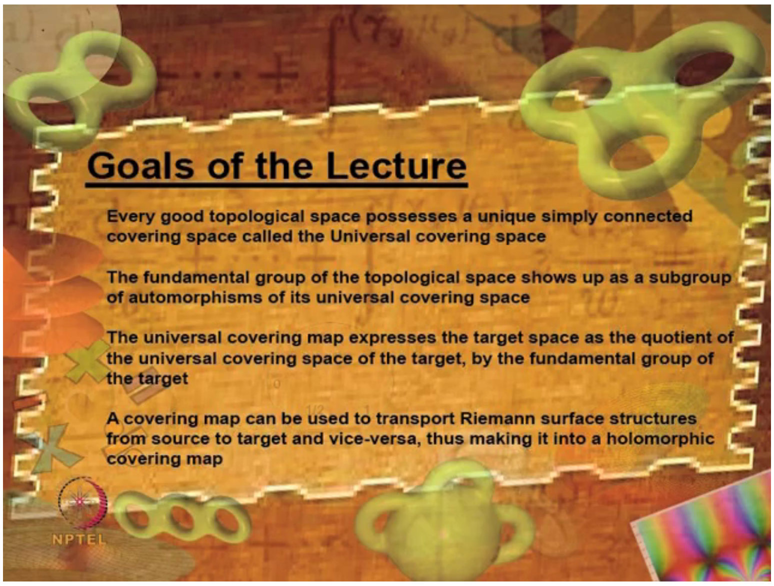

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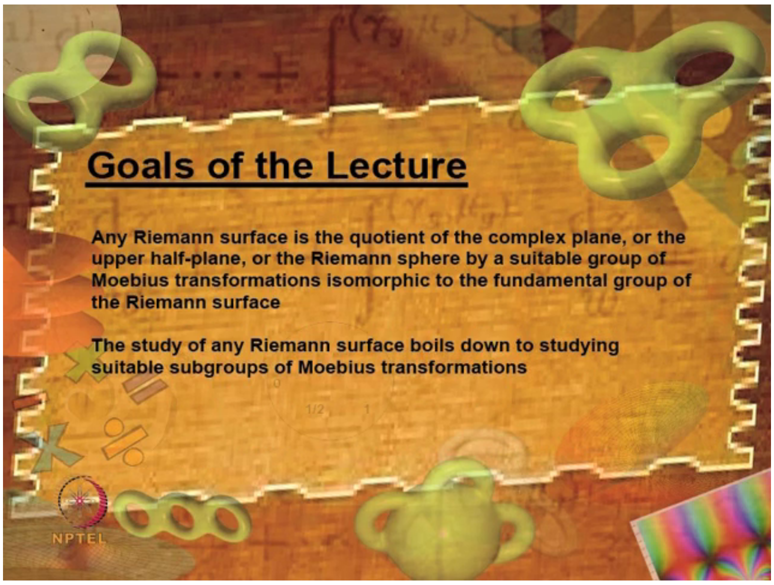

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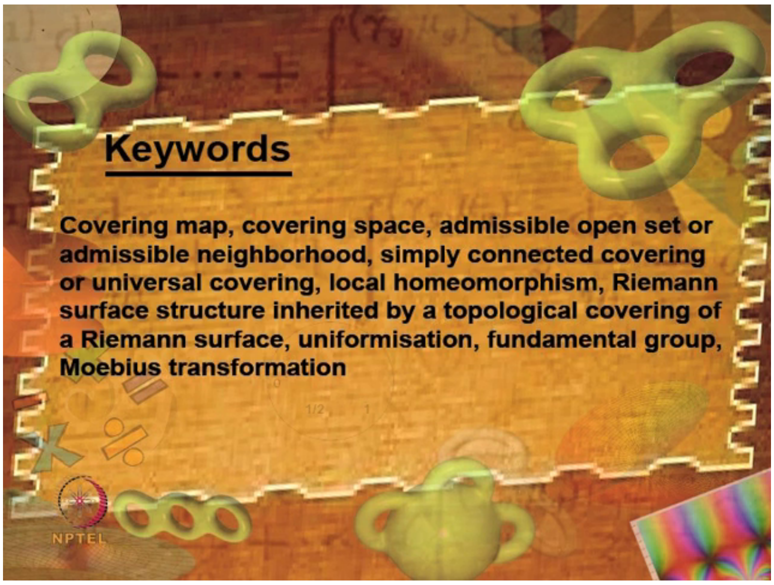

Welcome to this 7th lecture. So, let me recall what briefly I said towards the end of the last lecture. I was trying to tell you that trying to study general Riemann surface is greatly facilitated by the use of covering spaces. So, let me again recall the definition.

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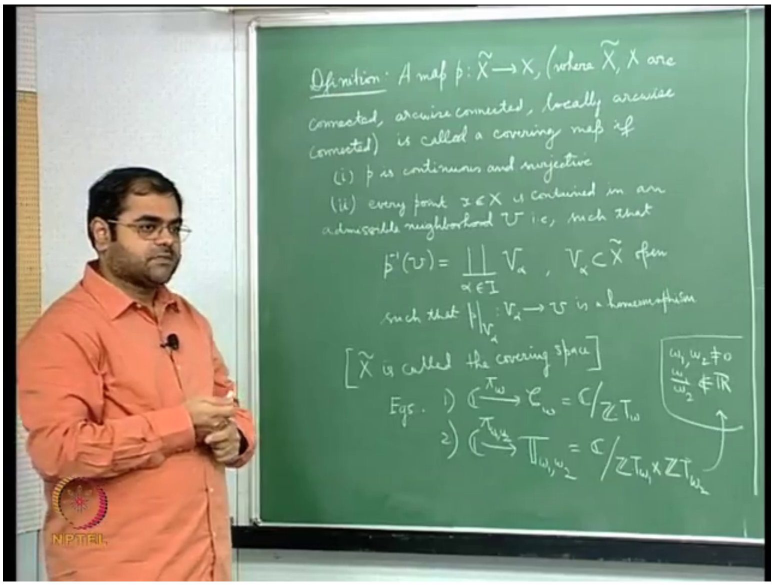

So, definition; A map p from X tilde to x. Where X tilde X or connected arc wise connected, locally arc wise connected. So, let me put this in brackets. So, these are certain technical conditions, I will explain them; is called a covering map if so, let me list the conditions number 1, p is continuous and subjective. Number 2 every point X belongs to X is contained in an admissible neighborhood u.

So, what is this admissible neighborhood u? Namely such that p inverse of u is disjoint union of certain v alphas, alpha running through an indexing set i, v alpha in X tilde open. So, the inverse image of this of course, is a continuous map. So, the inverse image of an open set contain X will of course, be an open set, but that is not that is not you know of what we want is that should break up into a disjoint union of open sets. Such that p restricted to each v alpha from v alpha to u is a homeomorphism. And in fact, I should also talk about I should also say that there is a there is a need for a basic assumption in in all my talks, that all the topological spaces I am going to considerer going to be hausdorff.

So, to be very strict I am certainly not going to work with a non hausdorff topological spaces. So, I am not going to say that very often. So, whenever I say topological space of course, I mean hausdorff topological space. So, you know what that means from basic topology; it means that pair of distinct points can be separated by disjoint open neighborhoods. So, that is the hausdorff ness condition right. So, of course, X tilde and X are hausdorff, and they connected they are arc wise connected and locally arc wise connected.

So, let me explain these terms. So, connected is of course, again the condition that the space cannot be written as a disjoint union of open subsets. Try it if you are able to try it a space as a disjoint union of open subsets, you are basically disconnecting it. And that this should not happen is the definition of connectedness. And arc wise connected arc wise connectedness is otherwise also called pathways connectedness, and this is the condition that any 2 points on the space can be joined by an arc or a path, and by an arc or a path we mean a continuous image of an interval of a closed interval on the real line.

So, if any two points of your space can be connected by a path in a continuous manner and you get that as the image of an interval under continuous map. Then you say that the spaces arc wise connected, and the space that is locally arc wise connected is a space for which every neighborhood of every point contains an arc wise connected neighborhood. So, these are all conditions that are required in the theory the proper theory of covering spaces. And of course, at the at the back of it all spaces we are dealing with or host of spaces. So, I am not going to write that down.

Now so, this definition of covering space is very, very important. And it is the key to understanding the geometry and the analysis on an arbitrary Riemann surface. So, let me say a few things. First of all, the space X tilde is called the covering space. And the map p is called a covering map all right. So, let me write that here X tilde is called the covering space and of course, as I have already said there p is a covering map. So, let me note write that again.

Now you remember why we got into this situation, regard in this situation because yesterday as I was telling you that was at this situation where we had on the one hand the cylinder, which is if you want S 1 the unit circle cross R.

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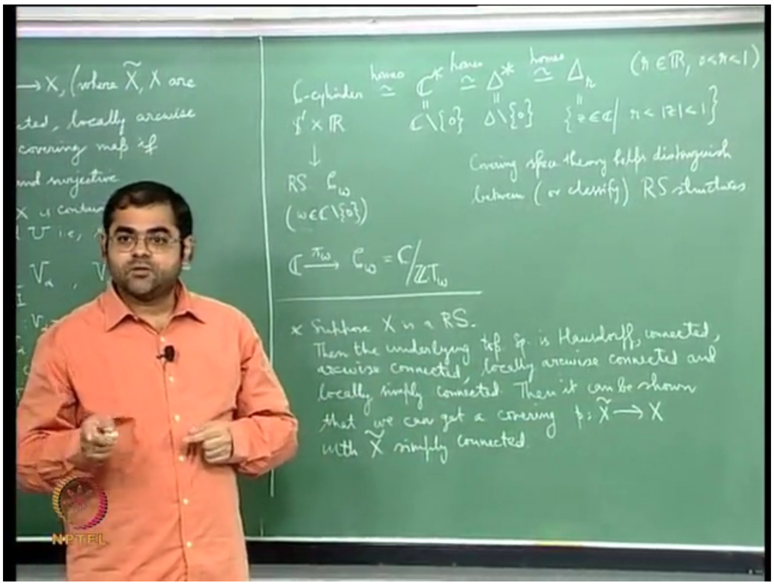

This is the cylinder the inside cylinder. And then which was which was homeomorphic to C star, which is C minus the origin complex the punctured complex plane, the punctured plane as we call it. And this is also homeomorphic to delta star, which is the unit disc the punctured unit disc unit disc minus the origin. And it is also homeo homeomorphic to any annulus of this form, and this is of course, all those complex numbers which with modulus less than 1 and greater than r. Where r is of course, r is r is a real number with 0 less than small r less than 1.

So, all these spaces are all homeomorphic. They there are homeomorphic. So, let me write may write that, but I was explaining a theorem yesterday, which said that you know actually you can get different Riemann surface structures. On the same if you call all these spaces just as one space up to homeomorphism. Then all of them give you different Riemann surface structures. Namely C star is a Riemann surface structure which is the natural Riemann surface except that you get as a as an open subset of the complex plane. Similarly, delta star also that is also an open subset of the complex plane, and also for delta r, and I told you that of course, the these are all distinct Riemann surface structures. And of course, on the cylinder there is an Riemann structure that corresponds to any complex number omega which is non- 0 complex number. So, I should say.

So, this is essentially you go modulo you take the complex plane and then you go modulo the group of translations by integer multiples of this number non-0 number omega. And then you get the cylinder topologically, but the point is that you actually have a quotient map C to C omega; which is just C modulo the group generated the group of translations by integer multiples of omega. And I told you that you can turn this this is turned into a Riemann surface in such a way. So, that this map pi omega is holomorphic. So, you get a Riemann surface structure here. And the point is that if you if you vary omega this Riemann surface structure seems to depend on omega, but if you vary omega is not going to really depend.

In other words, for different omegas, you are going to get Riemann surface structures which are all isomorphic and that by isomorphism I mean by holomorphic isomorphisms holomorphic isomorphisms. And of course, these and all these Riemann surface structures are all different. So, the Riemann surface structure see here and the Riemann surface structure here, and the Riemann surface structure there. They are all Riemann surface structures that are inherited just because these spaces are open subsets of the complex plane. So, they are naturally inherited from the complex plane, but yet the Riemann surface structures are different. They are not by holomorphic to each other.

So, I was a trying to explain to you that from you know from for example, C star to delta star or C star to delta r it is not possible to have a holomorphic map, because of essentially because of Riemann similarity theorem removal similarity theorem and liouvilles theorem. So, and it takes a little bit of work to show that if r 1 is not equal to r 2 where r 1 r 2 real numbers fractions, lying between 0 one then this delta r 1 and delta r

2 are not the same. So, there of course, let me again write this this is a set of all z in C such that r is less than mod z less than 1.

and mind you take any annulus in general if you take any open annulus the inner radius is let us say r 1, the outer radius is let us say r 2, then you can scale you can divide by the outer radius and bring it to an annulus of this form. So, actually this covers all possible you know annuali up to holomorphic isomorphism. And the fact is that these are distinct the fact that the these are all distinct is the amazing thing. So, the point is how do you distinguish between all these Riemann surface structures.

So, I told you that somehow the fact that you know you do not have a map from C star to delta star or C star to delta r, which would extend to a map from c. So, trying to get a map from C to any of these things does not happen. It does not happen for delta star r delta r whereas, it happens here. So, and what is the morals to the moral story is that once you have a map like this map has the properties of a covering map, and you get this Riemann surface structure as a quotient of C by a certain subgroup of automorphism, and these are moebius transformations. And I told you that the amazing thing is that this is what is true in general.

So, let me write that down, but before that let me let me give you 2 examples. So, the that the first one is say yeah of course, this is one example the other example is so, let me write them down examples are number 1 C to C omega; which is C mod user times translation by omega, and this is the this is the way you get a holomorphic structure Riemann surface structure on the cylinder.

The other one is of course; you get Riemann surface structure on the torus. So, the other map is C to t sub omega 1 comma omega 2, and I will write it this is pi sub omega 1, comma omega 2, and what is this? This is the complex plane modulo, the group of translations by omega 1, and are let me just put cross group of translations by omega 2. Where of course, the condition is that let me write it here omega 1 omega 2 not equal to 0. And omega 1 by omega 2 is not a real number. So, of course, these conditions are there.

This is how you get a holomorphic structure on a torus and it seems to. So, here you see it seems to depend on 2 complex parameters, but then I told you if you take the set of isomorphism classes of such structures, that set is by z 2 2 complex numbers. And In fact, it is say it becomes naturally a Riemann surface and the Riemann surface structure is just the complex plane. So that means, for each complex number I can give you a distinct holomorphic structure on the torus Riemann surface structure on the torus and conversely. But the key to all these theorems is the notion of a covering space. So, these are 2 examples of covering spaces.

And so, let me so, covering spaces. So, let me write that here covering space theory helps distinguish between, or let me say classify Riemann surface structures. So, what is it that that really happens? So, what really happens is series of things.

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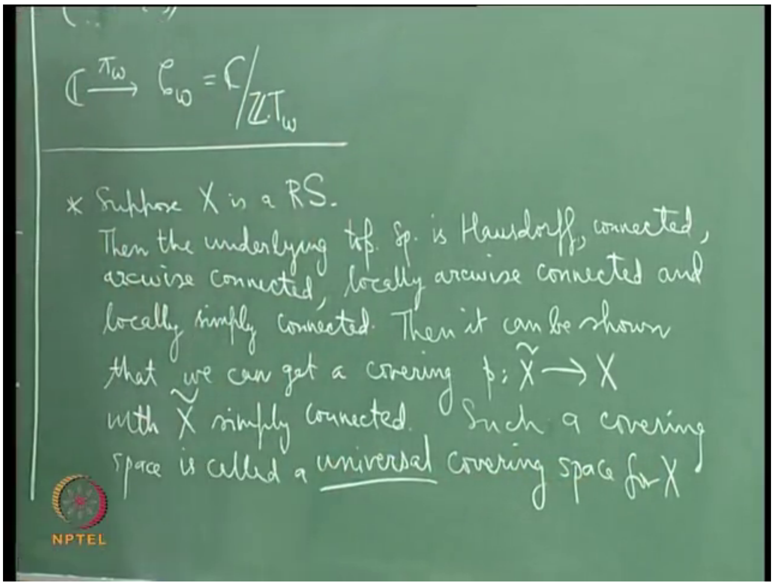

Suppose you have; suppose X is a Riemann surface, suppose I take a Riemann surface x then the underlying topological space. So, I am writing an abbreviating topological space. Like this then the underlying topological space is of course, hausdorff and I will put all these conditions connected, arc wise connected locally arc wise connected, and let me add one more line locally simply connected.

So, of course, I forgot to tell you the arc wise connected if a topological spaces are connected then it is connected. And the reason the, but it is not true the other way, there is something called the topology is sine curve which is connected, but not arc wise connected. So, when I say arc wise connected. I do not have to say connected, but nevertheless I am including it right. And the reason why it is connected if it is arc wise connected is because suppose you have an arc wise connected space, and if it is not connected then a disconnection of the space will break into 2 pieces. So, you take a point in this piece you take a point in the other piece and of course, because it is arc wise connected I can join it by an arc, but because you have broken the space into 2 pieces, this arc will also break in 2 pieces and since the arc is the image of a closed interval. It will essentially break the interval the 2 pieces. And you cannot break the since the intervals connected you cannot do that. So, that is the proof essentially.

So, then the other condition that I have put here is locally simply connected. So, this is the condition that every point has an open neighborhood which is simply connected, and the by simply connected the I mean that; any loop in that space can be continuously shrunk to a point. And you can think of a loop as an arc with the initial point the same as the final point. So, I put all these conditions and why do I need all these conditions. There is a theorem which says that whenever a topological space has all these conditions.

Then you can construct a covering for that space that is we can get the covering map like this for that space. Such that the thing on top the covering space on top is itself simply connected. So, let me write that then can be shown that we can get a covering a covering p from X tilde to X with X tilde simply connected. The space above so, it will be a covering map with the extra condition that the space above is simply connected which means that you know it has a property that any loop in that space, that is any arc in that space or path in that space which has initial point equal to the final point can be continuously shrunk to a single point.

I mean this is essentially the condition that there are no holes. So, for example, you know if I took an annulus and I took loop that is going around the annulus. I cannot shrink it to a point because there is a hole in between. So, in that sense trying to shrink a loop into a point is just trying to check whether there are any holes. So, that is the situation here for X tilde for this X tilde, and then among all the covering spaces of a topological space if you look at those coverings for which the covering space is simply connected. That space turns out to be special namely, it is uniquely determined up to unique isomorphism.

So, let me write that also such a covering space is called a universal covering space for x. So, universal covering space for X and is uniquely determined up to isomorphism.

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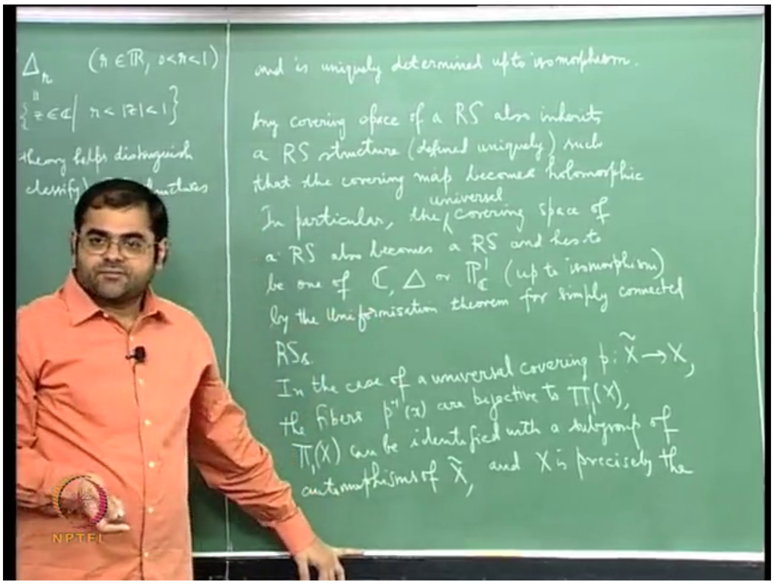

So, there is the something special about this covering space if it is simply connected. Now why does this help. So, if you first look at simply any covering space of us of a space like this and suppose this space was a Riemann surface. Then after all you see this covering map is a local homeomorphism; that means, a local homeomorphism means for every point above I can find a neighborhood such that the image of that neighborhood below is an open set, and the map restricted to that neighborhood is a homeomorphism that comes because of this condition.

So, a map like this is certainly a local homeomorphism. And what is the advantage of X being a Riemann surface if X was also a Riemann surface, then X has charts locally and because X tilde is locally homeomorphic to X. I can transport the charts I can define charts on X tilde and in this way, I can make see X tilde into a Riemann surface. So, the moral of story is; if you have a curving space for a Riemann surface, then the covering space the then the covering space itself becomes a remand surface and that Riemann surface structure is also uniquely determined up to an isomorphism.

So, let me write that statement here. The covering space, or let me say any covering space not necessarily the universal covering you take any covering space any covering space of a Riemann surface also inherits, a remains surface structure defined uniquely defined uniquely up to a unique isomorphism. Such that such that the covering map becomes holomorphic.

So, you must really understand, what is happening? See if you took the condition of a cylinder, namely this one. Or that of a torus what actually happened was; we these both of these covering spaces, but to begin with there was no there was no Riemann surface structure here or here. We were trying to put the this is the Riemann surface structure that we got because of the covering. And here again the reason was that because this is a local homeomorphism you are able to get charts in this direction right. And that is how this becomes a Riemann surface. And once this becomes a Riemann surface this projection becomes holomorphic. Similarly, that is also the case in the case of that is also the case here for a torus.

So, here what is happening is you have a covering space situation, but the top is a Riemann surface, and what is below the bottom is a quotient of the top by a certain group of automorphisms, in this case these are moebius transformations here automorphisms of c. So, the moral of the stories that this whole argument is being turned around here. Instead of so, here I am obtaining a Riemann surface as the base space, the space below of a covering, and rather I would say I have a topological space, which is the base space of a covering and the top space is already a Riemann surface in this case the top space is c.

So, I am getting a Riemann surface structure on the base space, and what is happening here is exactly the reverse. If I start with a Riemann surface, any Riemann surface, then you take any covering space of the Riemann surface. Then the covering space the top space that inherits a Riemann surface structure and this map becomes holomorphic. So, it is just the. So, what you can see is that this notion of Riemann surface can be if it is there on the top it can be pushed to the bottom, if it is there in the bottom you can push it pull it up to the top. So, that is the importance of this method ok.

So, here I have said that any covering space of a Riemann surface also inherits the Riemann surface structure such as the covering map becomes holomorphic. So, in particular if I take the universal covering space for this Riemann surface then the universal covering space will also become a Riemann surface, but now what is the point the point is that the universal covering space is simply connected. So, what you have on top now is a simply connected Riemann surface, but now there is a fundamental theorem of uniformization which says any simply connected Riemann surface has to be either of the complex plane or the open the unit disk which is the same as upper half plane or the Riemann space.

So, this tells you that any Riemann surface is obtained from these 3 simple Riemann surface surfaces namely simply connected Riemann surfaces just by going modulo a certain group of automorphisms which are going to be moebius transformations. So, that is the importance of this. So, let me write that down. So, in particular the covering space of the universal covering space of Riemann surface also becomes Riemann surface and has to be to be one of C or delta or P 1 the Riemann's here up to isomorphism, by the uniformisation theorem for simply connected Riemann surfaces. That is one point and again, let me stress on another point that I have a that. I was explaining the last lecture I was trying to tell you where the fundamental group fits into the picture.

So, I told you that in this case if you take the cylinder the fundamental group with a cylinder is z. And I will define fundamental group a little bit more formally very soon. Or if not now in probably in the next lecture, but the fundamental group of the cylinder is z, notice that that is the group that group as a set is exactly the fiber over any point that the inverse image of any point is bijective to z. And the fundamental group z is also the same as the group of translations, going modulo which, you got the Riemann surface below all right.

So, I can I can write this as I can write this a C modulo fundamental group of C the C omega. Where the fundamentals group of C omega is just z and z is isomorphic to z dot t omega, and look at this situation in this situation also. You take any point here the inverse image of that point will be a grid of points and this grid is just a translate of the vertices of the parallelograms the fundamental translates fundamental parallelogram that is formed by omega 1 and omega 2 as to coterminous edges. And what is that grid that grid is just a copy of z cross z. After all they are integer translates of a pair of vectors.

So, the inverse image of a point under this map is also a said theoretically a copy of z cross z. And that z cross z is also the fundamental group of the torus, the fundamental group of the torus is z cross z. So, I can write this as C to C modulo the fundamental group of the torus. So, what is happening here is valid in general. If you take a universal covering space what will happen the fiber is that the fiber or any point that is the inverse image of any point will be bijective with the fundamental group of the base, and the top space will have a subgroup of automorphisms, isomorphic to this fundamental group of the base.

And going modulo that the top space going mod going modulo that subgroup of automorhisms is what gives you the space below, and the covering map is just the quotient map. So, this is what happens in general. So, let me write that down here in the case of a universal covering p, from X tilde to X, the fibers p inverse X, or bijective to pi 1 of x. So, here I am using pi sub one this pi sub one is the fundamental group maybe I will I will write pi little bit more carefully. So, that you do not. So, I will put it like this there the fibers are all the fiber over every point is a set which is bijected to the fundamental group.

So, let me write those conditions that is the first condition the then the second not a condition is another consequence. The second thing is that the fundamental group of X is a subgroup of automorphisms of the space above pi of X can be pi 1 of x. So, it is pi one pi sub 1 please correct that pi sub one for the first fundamental group and normally when we talk about fundamental group we always talk about first fundamental group. So, we never say first we do not add the objective first. So, and we just say fundamental group it means first fundamental group the first fundamental group of the base space can be identified with a subgroup of automorphisms of the top space of the covering space all right.

So, what do I mean by automorhisms? If the situation is that X was just topological space for which you have this covering, then by automorphism of X tilde I mean homeomorphisms of X tilde onto itself and of course, you know the that forms a group under composition. And I am saying that you can realize the fundamental group as a subgroup of that group of automorphisms. So, the big space above and X is precisely the quotient.

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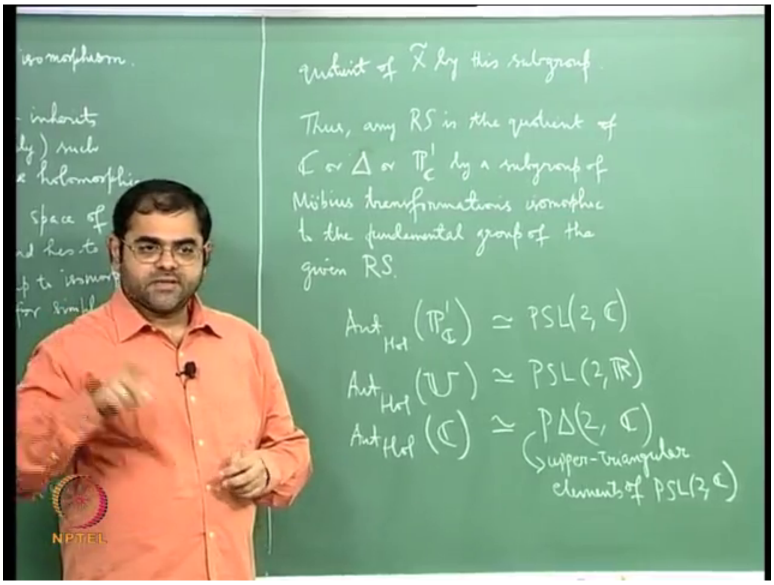

So, let me write that let me continue here; of X tilde by this sub group and this also happens if you are in a Riemann surface situation if X where a Riemann surface, then I told you that X tilde automatically inherits a Riemann surface structure such that the map p becomes holomorphic.

Then again what happens is that every fiber the inverse image of any point is always going to be bijected to the fundamental group below number 1 number 2 the fundamental group below of the of the space below can be identified to the subgroup of automorphism of X tilde, but now automorphisms of f tilde as a Riemann surface not just homeomorphisms, because X tilde has now the additional structure of a Riemann surface now automorphisms I mean self by holomorphic maps.

So, here it will be automorhisms as Riemann surface. And again, X is precisely the quotient of X tilde by the subgroup. Now if you put together all these things the moral of the story is any Riemann surface is just a quotient of the complex plane, or the unit disk which is the same is upper half plane or the Riemann sphere modulo the fundamental group of the Riemann surface by a universal covering.

So that means, that to study any Riemann surface, I need to only study the subgroup the subgroups of automorphisms of the complex plane or the unit disk or upper half plane or P 1 which all turn out to be as you know moebius transformations. So, essentially this reduces to a large extent the study of moebius transformations. And now what I am going to do next is I will try to tell you what kind of classification preliminary classification you can do for Riemann surfaces, that you can obtain by this method.

So, let me write that just let me look at it for a moment. So, thus any Riemann surface is the quotient of C or u or C or delta let me use delta or p 1 by a subgroup of moebius transformations a sub group of moebius transformations, isomorphic to the fundamental group of the given Riemann surface.

So, this is this statement is called the general uniformization theory. So, it is called uniformization because you are you are getting everything as quotient of one of these 3 basic Riemann surface. And in this regard let me note that you know the automorphisms holomorphic automorphisms of P 1 C, this is the this is the Riemann sphere these are going to be all moebius transformations. And they are given by the group PSL2Comma C and then automorphisms holomorphic automorphisms of the upper half plane is going to be isomorphic to so, let me write this down first and then I will explain, automorphic homomorphism of C is going to be psp delta 2 comma c. So, this is upper triangular elements of PSL2C.

So, let us let me explain this this notation for those who may not be familiar with it. So, you know a general moebius transformation these are the form a is plus b by C is it plus t. Now you can associate of course, the condition is ad minus bc is 0, by that of course, I mean the map is exactly ez going to aez plus b on the numerator divided by cez plus d on the denominator where abcd are complex numbers. Ad minus bc not equal to 0 now to that moebius transformation you associate you have the matrix a b c d. Now if you associate it by this matrix this association does not give you a unique matrix, because you know instead of abcd if I put lambda a lambda b lambda C lambda d or lambda is any non0 number. Then the resulting moebius transformation is still the same because it will be lambda a, you know ez plus lambda b divided by lambda cez plus lambda d. And the lambda will just cancel out.

So, the moral of the story is that you can go modulo scalars you can go modulo scalars and the one way of doing it is by assuming that the determinant is one. So, if you assume determinant is one then you can identify the moebius transformations with the special linear group. These are all matrices the determinant one SL2C. And even then, you do not get a unique representative. Because you know for if I take is that going to aez plus b by cz plus d with ad minus bc equal to 1 which is the condition for SL2C, then I have also the representative minus a minus b minus C minus d. Because that is also going to give me aez plus v by cez plus d and the determinant is still one.

So, I still have 2 choices; that means, you know I have to still go modulo the subgroup given by plus or minus the identity matrix. And it is that quotient which is written as PSL. So, PSL is SL modulo the subgroup plus or minus identity map plus or minus identity map is a is a finite subgroup just with 2 elements and going modulo, that is what you is what gives you PSL2C and you can check that the I will written upper half plane, because that is holomorphic to the unit disk because a moebius transformation you can you can always find you can map any disk open disk to any half plane you know.

So, it is in the context of uniformization one always tries to work with the upper half plane and the automorphisms the upper half plane will be PSL 2. So, here you are only looking at real quotients. You are looking at real quotients. And the reason why you get real quotients is because you see if you have moebius transformation that has to map the upper half plane to the upper half plane by continuity it has to map the boundary to the boundary. So, it has to be a map it has to fix r because a boundary is the real line for the upper half plane in the complex plane the real axis. So, if it has to fix the real axis that is the condition that makes the quotients forces the quotients to be real.

So, that is how you get this as a subgroup of this and you can check that if you want automorphisms of the complex plane; that means, you know the point at infinity has to go back to the point at infinity right. And that condition we will tell you that the subgroup of this which will do that will be upper triangular elements of PSL. So, the moral of the story is that you have to study certain subgroups of these groups, which occur which have the possibility of occurring as the fundamental group of array of the topological space underlying a Riemann surface. And then that Riemann surface is gotten by going modulo that subgroup, from the covering space which comes if you are going to look at this the covering spaces C, this corresponds to the covering space delta or u and this corresponds to the universal covering space P 1. So, the no so this corresponds to C this corresponds to delta and of course, this corresponds to P 1.

So, the moral of the story is that you are look you have to study certain moebius transformations properties of moebius transformations. And it happens that once you do that just assuming this basic uniformization theorem for simply connected surfaces. You can get a lot of information about classification. So, that is what I will try to do and of course, I should tell you that trying to prove the basic uniformization theorem is will involve further techniques that will for example, involve techniques from analysis. And we will we will do that later in one of the later courses. I mean one of the later lectures. So, I will stop here.

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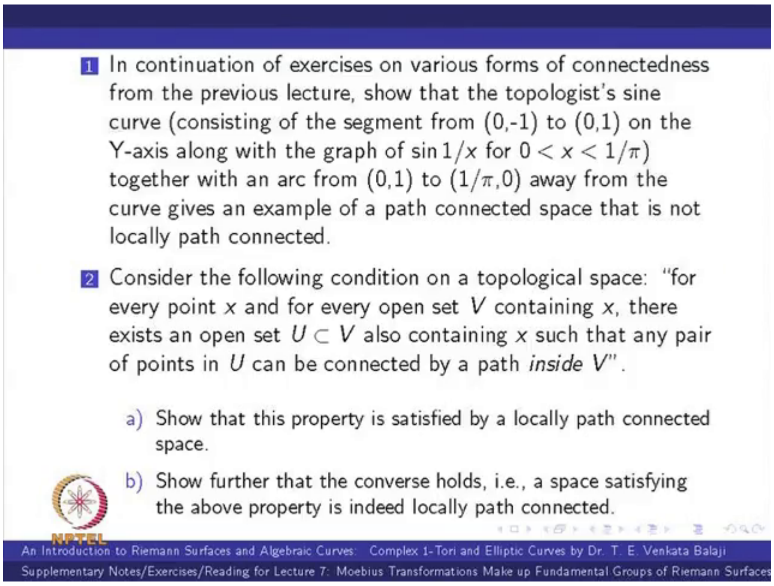

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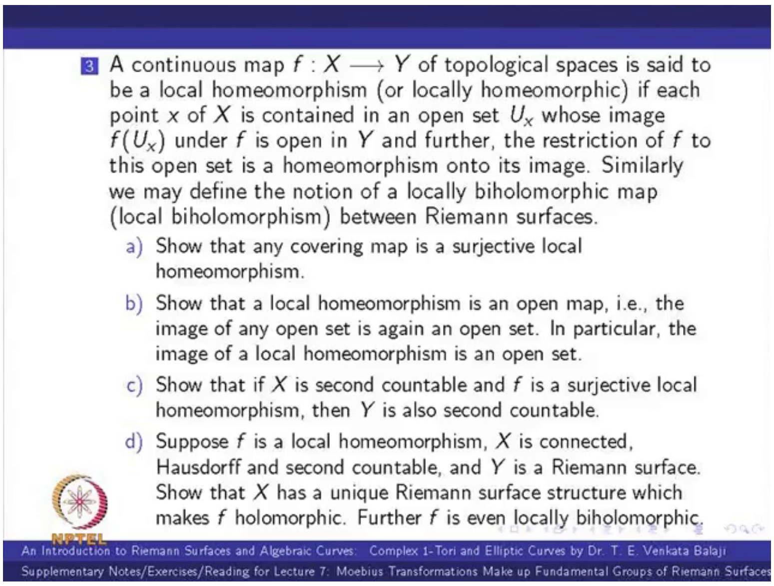

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4. Consider a topological covering

$$p : \widetilde{X} \longrightarrow X.$$

Suppose that $\widetilde{X}$ is second countable. Recall that the source and target of any covering map are supposed to be connected, path connected, locally path connected and Hausdorff. Suppose X further has the structure of a Riemann surface. Show that $\widetilde{X}$ inherits a Riemann surface structure from X which makes p holomorphic, and moreover that this is the unique Riemann surface structure for which the identity map on $\widetilde{X}$ is holomorphic. In fact, p is even locally biholomorphic. We thus say that p becomes a holomorphic covering in a unique way.

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5. (A converse to the statement above.) Consider a topological universal covering

$$p : \widetilde{X} \longrightarrow X.$$

Suppose $\widetilde{X}$ further has the structure of a Riemann surface and that the subgroup of topological automorphisms of $\widetilde{X}$ that identifies with the fundamental group of X is actually a subgroup of biholomorphic automorphisms. Then X inherits a Riemann surface structure from $\widetilde{X}$ which makes p holomorphic and locally biholomorphic, and moreover this is the unique Riemann surface structure for which the identity map on X is holomorphic. We shall prove this statement later when we discuss going modulo a subgroup of biholomorphic automorphisms that act freely or properly discontinuously.

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6. Compute the groups of holomorphic automorphisms (holomorphic self-maps which are isomorphisms) for the complex plane, the upper half-plane and the Riemann sphere. Verify that you get the groups mentioned at the end of the lecture.
7. Show that you can always find a Moebius transformation that maps any given (open) half-plane into any given (open) disc (of finite positive radius).

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