An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture - 06
Riemann Surface Structures on Cylinders and Tori via Covering Spaces

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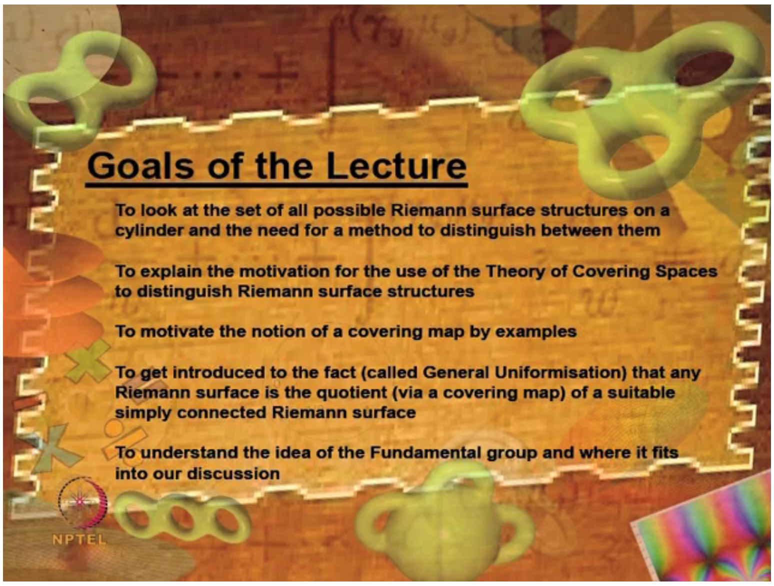

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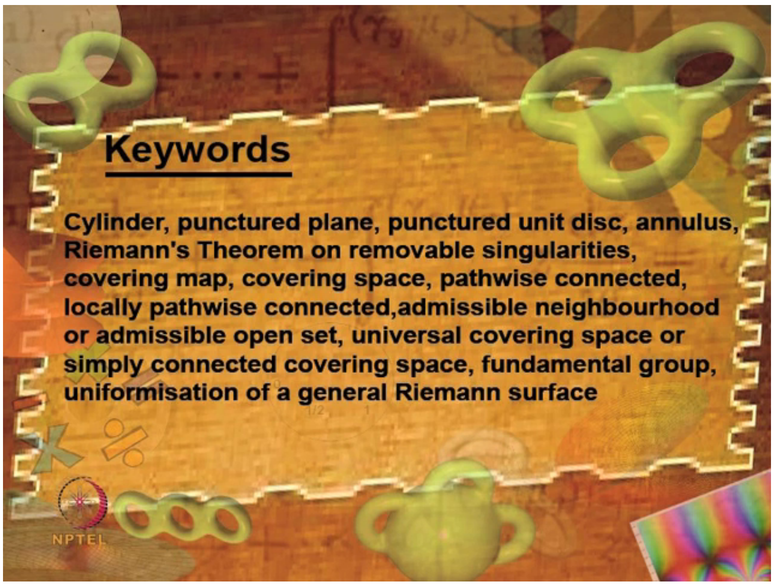

Ok welcome to this 6th lecture. In order to motivate what I am going to do the next few lectures. Let us look at a particular example. So, I take the cylinder.

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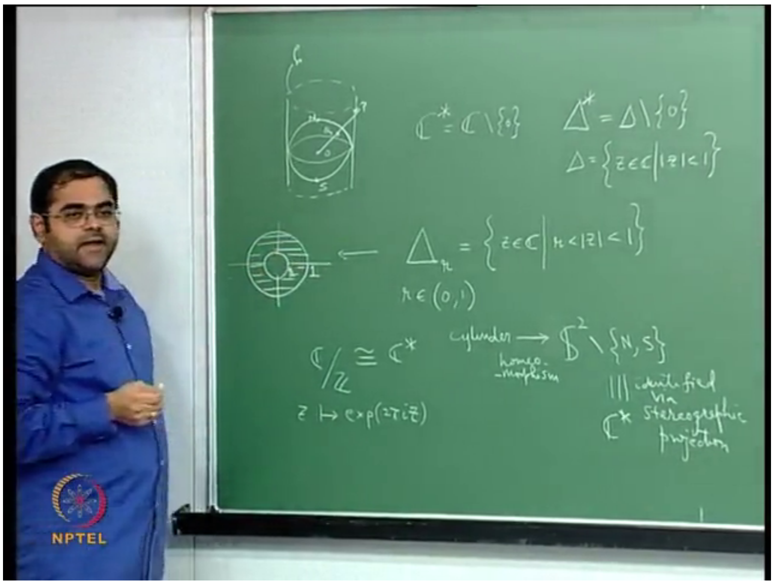

And on the other hand, I also take C star, which is the set of complex numbers accepting 0, set of non zero complex numbers. And then I can also look at delta star, which is the unit disc minus the origin. So, this is the punctured unit disc. Where of course, delta is unique disc namely it is a set of all complex numbers, such that mod z is less than 1. And then of course, I can also consider an annulus delta r.

So, this is the set of all complex number z, such that the modulus of z is less than 1, and it is greater than r. And here r is a real number, positive real number positive fraction. It is an element of 0 1 right. So, I have basically the cylinder, I have the punctured plane, I have the punctured unit disc and I have the annulus here, and you know the point I am trying to say is that if you are going to look at these things as these objects topological spaces, then they are all the same they are all homeomorphic.

For example, I can easily tell you how you can show that the cylinder and C star are homeomorphic. In fact, in the last lecture I told you that if you try to look at various a structures of a riemann surface on a cylinder, then I told you that if you take the set of isomorphism classes of riemann surface structures on a cylinder; such that you know the though and I am looking at riemann surface structures that have been gotten by choosing a non-0 complex number, and looking at the you know sub group of automorphisms of the whole complex plane given by translations by that complex number. Which was a subgroup that was isomorphic to z, and then I said you go c mod z you go c mod the subgroup, then you get c mod z which is essentially a cylinder. And I was also trying to tell you that the various riemann surface structures that you get in this way, if you look at the isomorphism classes; set up isomorphism classes of such riemann surfaces. Then there is only one.

And I told you that the standard representative is C star. And in order to show that C star is c mod z, I told you that is because of this because of the following c mod z is isomorphic to C star; that is by considering the at the exponential map. So, you just send z to e power 2 pi iz, and this mapping will have for it is kernel the integers. And therefore, the source modulo the kernel thought of as an additive group is isomorphic to C star thought of as a multiplicative group, and this is the map.

So, essentially, I am just saying that c mod z is C star and of course, you know c mod z is also cylinder, because that is the way we got the riemann surface structure on the cylinder.

So, in fact, one way of one other way of looking at an a homeomorphism between C star and the cylinder is as follows, as one of my students pointed out. So, it is what you do is that you think of notice that if p 1.

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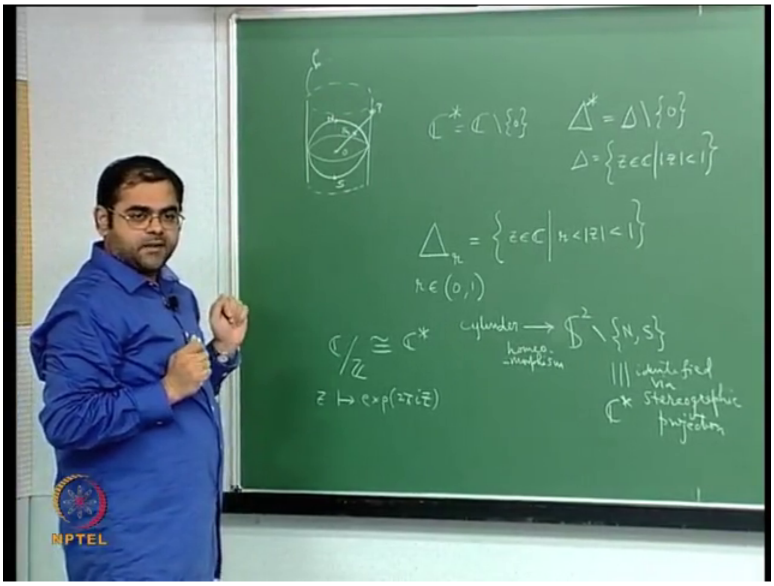

So, let me not use p 1, let s 2 be the real sphere. And so, you think of the real sphere as sitting here. So, this is the think of this is unit circle, on the xy plane. The axis of the cylinder is along the z axis, but do not confuse that z with a complex number z, and you have the unit sphere here. And what you do is you have the origin here.

Now, to every point on the cylinder, you join the point to the to the origin by a line. And it will meet the sphere at a point. And call that point of Q. In this way you will get a map from you will get a map from the cylinder to the sphere. And only point is that you will not get the north pole, because essentially there is no point on the cylinder which one joined to the origin is going to give as an north forth, and the same thing is going to happen to the south pole as well.

So, if I call the north pole as n and the south pole as s. So, you get a map on the cylinder minus you know the north pole and the south pole. And you can check that this is a homeomorphism. And this is another way of saying the cylinder is actually C star. And the reason is because, you know the sphere minus the north pole by the stereographic projection is just the complex plane. And the complex plane minus you know, the this in under basis stereo out stereographic projection the south pole corresponds to the origin. So, if you remove the origin from the complex plane this is just C star.

So, let me let me put identify via a stereographic projection with C star. So, this is the direct way of trying to understand to trying to realize the cylinder and C star or homeomorphic. Now let me tell you about following theorem.

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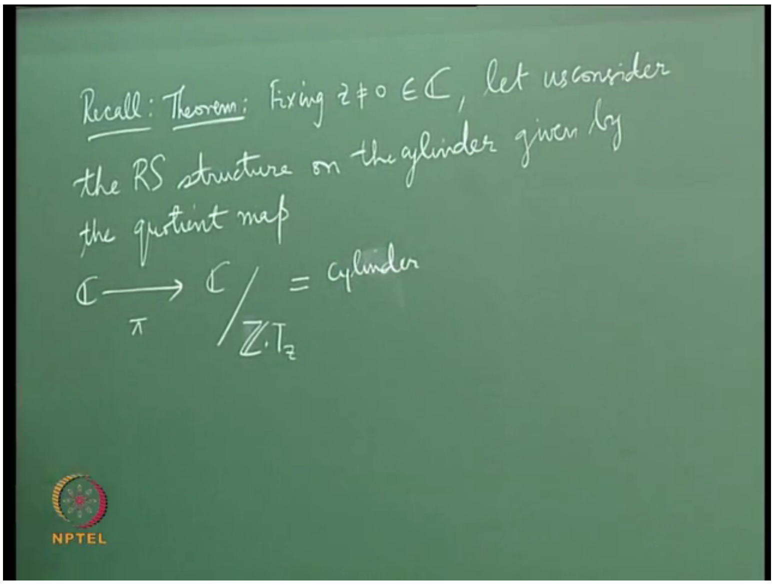

So, I will recall this theorem, theorem fixing is z not equal to 0. Let me say in a non zero complex number.

Let as consider the 3 months the riemann surface structure on the cylinder given by the given by the quotient map c modulo n. So, let me use z dot Tz not Tz which is a cylinder, in cylinder and I have. So, this is the quotient map which sends every. So, what it will do is; if you want a let me call this as or let me call the variable here rather let me call this variable as ohms. So, as omega and let me call this T sub omega.

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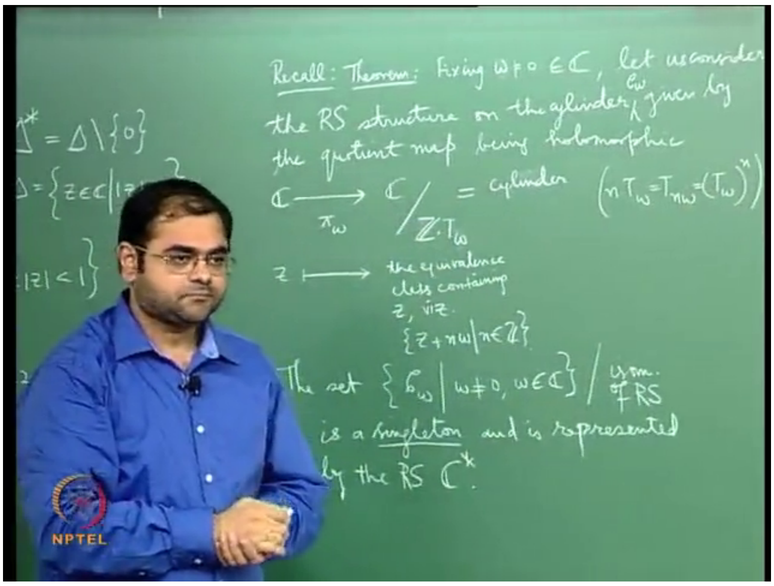

So, you have the map. So, this is this is the natural quotient map, and what is the quotient map? You send every complex number 2 it is equivalence class under the under the action by the group of translations by integer multiples of omega. So, what is this map? This is just you sent z to the equivalence class containing z namely which is the equivalence class is the set the set of all ez plus n times omega where n is an integer.

So, these are just translate of z by omega, and integer multiples of omega. And this is this is one equivalence class which contains z, and this is the unique equivalence class which contains z and this is the map. And when I say fixing omega not equal to 0, what we mean is that in in writing down this map, maybe I should even put pi sub omega. Because it depends on this choice of omega. The only condition on omega is at omega is non zero. And what do I mean by consider the riemann surface structure on the cylinder given by the quotient map. What I mean is you make this into riemann surface, the way you make into riemann surface is by actually getting and for a point on the cylinder, you get a map in the other direction, which is a homeomorphism. Namely the method was you take a point here, and then you take a take a pre image for example, take z. So, a point here will and be an equivalence class, that will come from a point z.

So, you take a pre-image, take a small enough disc and make sure that the areas of the disc or the diameter of the disc is very small when compared to the modulus of this complex number. And then you take the image of that disc under this map. This mapping is an open mapping, and as a result you will get a neighborhood of that point, which is isomorphic to this disc which is that which is a disc around the pre-image, and this gives you a local coordinate chart. And this is how you get the riemann surface structure. And once you get the riemann surface structure this map becomes a holomorphic map.

So, this is what we saw last time. So, let me complete that sentence fixing omega naught equal to 0 and c. Let us consider the riemann surface section on the cylinder given by the quotient map, being holomorphic as. So now the question is if I wary omega in principle I can get various Riemann surface structures on the cylinder which would depend on omega.

And what is the, but of course, all these structures will correspond to a quotient map like this, all right? That is the only a common thing, but as you change omega, you are getting different riemann surfaces. In principle, but what is it that the theorem says the theorem says, no matter what you no matter how you change omega, if you consider them up to isomorphism you get only 1. And that one riemann surface is going to be the riemann surface structure on C star. See mind you C star is just an open subset of c. And any open subset of riemann surface is again a riemann surface. Because the only condition for something for surface to be a riemann surface, an open subset of a surface is still a surface it is still 2 dimensional. And it is still it is still a riemann surface because at every point in that surface I can still find local coordinate shots.

All I have to do is take a coordinate chart with respect to the full surface, and then intersect it with this open set, I will get a smaller coordinate chart. So, every open subset of a riemann surfaces is again a riemann surface. In particular C star is also a riemann surface naturally being an open subset of c. So, C star in inherits the riemann surface structure from c natural riemann surface structure from c. And what I am saying is that this riemann surface structure on C star which is the sphere minus these 2 points by this identification, if I transfer it to the cylinder by this homeomorphism, I will get a riemann surface structure on the cylinder. And I am saying it is that riemann surface, which is a representative for the set of isomorphism classes of riemann surface is arising in this way which I say is a single term.

So, let me write that down, the set of all let me call this as well let me do something I will call the cylinder as a script c, and let me call this let me call the riemann surface structure on the cylinder as c sub omega. Because in general you might expect it to depend on omega. Let me called a c sub omega. So, you take the set c sub omega, where omega naught equal to 0, omega is the complex number, take this set. And then you go modulo the equivalence relation which is isomorphism of riemann surfaces.

So, mod isomorphism of riemann surfaces. Then this is a singleton. This is singleton. This is a singleton, and is identified and is represented by the riemann surface C stock. So, this is the theorem that I wrote down last time right, one of the theorem that I wrote down last time last time. Before I consider the case of a torus I consider the case with cylinder. Now I want to and now I want to state another theorem.

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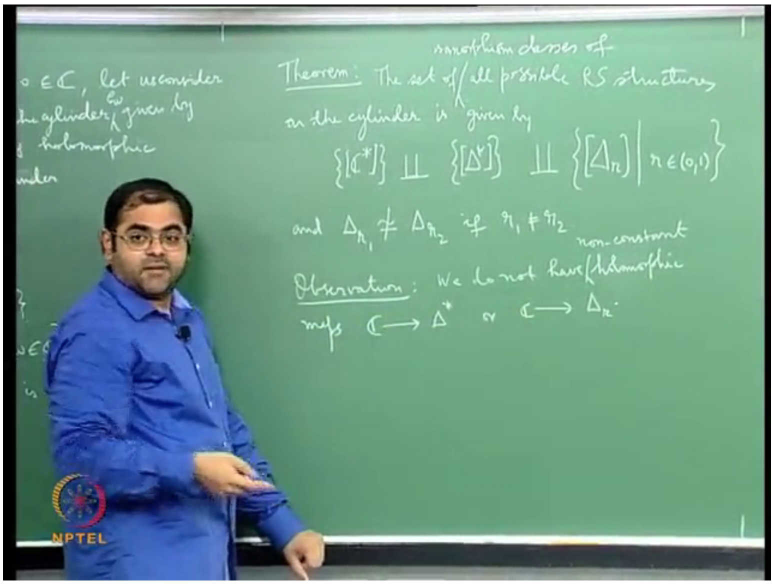

So, here is the theorem.

So now let me look at all possible riemann surface structures on the cylinder, take the cylinder. Let me look at all possible riemann surface structures on the cylinder. Notice that when I do this the cylinder is also C star from topologically cylind cylinder is also delta star. And there is there are also the annular, they are all from a mafia. And the point I want to make is that on delta star there is an actual riemann surface structure. Because delta star the unit disc for the punctured unit disc that is unit disc minus the origin is also an open subset of c. So, it also has a riemann surface structure, but notice that and similarly the an annulus like this. So, this is this corresponds to an annulus like this. So, you know it is it is the unit disc minus the origin. So, I put a dot there. So, this is no. So, that is delta star. So, this that was delta. So, it raw it here are let me just draw this as it is the let me draw it like this.

So, this is this is r and this is 1. So, this is my this is my annulus. And this is also a riemann surface naturally because I told every open subset of riemann surface is also a riemann surface. So, this is an open subset of the complex plane, mind you the boundary circles are not included it is the is interior you do you remove the boundary surface. The boundary circles will be included only if I put also less than or equal to the inner circle easily included only if I put less than or equal to here and the outer circle is included only if I put less than or equal to there. So, these boundary circles are not there. It is the interior of that is an open annulus and that has a riemann surface structure.

Now, the amazing thing is that these are all these are all homeomorphic topologically they are all homeomorphic cylinder. So, they should all give you riemann surface structures of the cylinder, but notice that C star C star and delta star or for that matter C star and delta r can never be by holomorphic. Why because you see if C star and delta star or C star and delta r is by holomorphic; that means, you have you have a by holomorphic mapping from C star to delta star are from C star to delta r, but it is impossible to have even a holomorphic map from C star to delta r or delta star delta r or delta star. Why because once you have a holomorphic map from C star delta r or delta star, the these a holomorphic map, which in a neighborhood of the origin is in a deleted neighborhood of the origin is bounded. Because the image is going to go into delta star or delta r they are bounded.

So, you have a holomorphic map which is in a deleted neighborhood of the point namely the origin it is bounded map, and reimanns theorem under removable singularities will tell you that that will extend to holomorphic map from c. And if it extends to holomorphic map from c to either delta star or delta r you are going to get an entire function which is bounded. Because it has extended from C star to c it is going to become an entire function, and then since it is image is going to lie in delta star or delta r it is going to be a boundary entire function, and liouvilles theorem is now going to tell you that it has to be a constant.

So, what this tells you is that is impossible to find a holomorphic map from C star to delta r to do it to delta r or to delta star. So, the moral of the story is; delta star and delta r are certainly going to give you rent riemann surface structures, which are different from their natural riemann surface structure on C star. So, you are going to get some other non-isomorphic riemann surface structures on the cylinder. And then in fact, the amazing thing is that no riemann surface structure of the form that is there on delta r can be by holomorphic to the riemann surface structure on delta star. And in fact, you know. In fact, delta if you take delta r 1 and delta r 2 for different r's, they are all non-by holomorphic.

So, the amazing thing is you get a whole family of a riemann surface structures possibility of riemann surface structures on the on the cylinder. So, so that is the that is the next level of complication that you see in the theory. And somehow the idea is how do you how do you analyze the situation. And the point is you analyze the situation by using some more techniques namely you use the notion of what is called a fundamental group, and the associated covering space and the techniques of fundamental group and covering spaces helps you 2 distinct distinguish these a riemann surface structures.

So, let me write it down here. So, the set of all possible riemann surface structures. In fact, I should say isomorphism classes. They said of isomorphism classes of all possible riemann surface structures on the cylinder is; so, is given by. So, that is one structure on the cylinder which is the same as C star. So, what I will do is I will put C star here, and I will put a square bracket saying that I am looking at all those Riemann surface structures on the cylinder which are by holomorphic that is holomorphically isomorphic to C star.

So, that is what the square bracket means it means, take the equivalence class under the equivalence relation which is given by 2 things 2 riemann surfaces being equivalent if there is a holomorphic isomorphism with an inverse which is also a holomorphic as small of course, when I say holomorphic isomorphism from inverse automatically holomorphic and isom and it is an isomorphism. And so, this is one class. Then I will put this here. So, this means disjoint union.

So, you put disjoint union to say that something here does not occur anywhere else. So, the other possibility is delta star, and again take the equivalence class. And then what I will get is this is the disjoint union of all these delta r, such that r is real fraction real positive fraction. And delta r 1 is not by holomorphic isomorphic to delta r 2 if r 1 is not equal to r 2.

So, you see when you look at all possible riemann surface structures on a cylinder, you notice that you are getting actually 3 types. 2 of them are just singletons and, but the third one is a continuous family depending on one real parameter. This real parameter is a real number chosen in 0 1, but of course, I could scale 0 1 to be the whole real line because there is not a problem. So, essentially what happens is that if you look at isomorphism classes of riemann surface structures and cylinder you get 3 families. These 2 are just trivial, but the third one depends on one parameter.

So, this is the theorem. Now the question is how is it that one can go about proving such theorem. So, eventually we are we are we are going to prove all these theorems, but one needs the motivation for introducing certain more and more complicated tools. And we should have some motivation otherwise there is no point in just giving definitions after definitions. So, the key to all this is what is called the idea of a of a covering space. And the notion of a covering space is also connected with the notion of the fundamental group. So, these are 2 notions that I have to explain. So, the observation; so, this is the observation we do not have by holomorphic maps I should not say even by holomorphic could be not will be do not have for even holomorphic maps, from C star c 2 delta star or c 2 delta r.

So, this is this reiterates what I was trying to say a you cannot have a holomorphic map from c 2 delta star. You cannot have holomorphic map from c to any delta r. Because if it is you know you will get a boundary entire function. And of course, I should say when I say holomorphic maps I mean non-trivial maps. I am not worried about the constant map. So, I should say we do not have let me say non-constant. That is very important. We do not have non-constant holomorphic maps from c to delta star or from c to delta r. But we do have a holomorphic map from c 2 for example, C star or any you know riemann surface structure which comes in this way. And so, what is very special is this map the existence of this map from c to this, and that is a non-trivial map mind you it is a surjective map. And that map expresses is this this riemann surface as a quotient of the complex plane.

So, in some sense this is leading you to believe, that you know delta star cannot be a holomorphic quotient of c. Delta r cannot be a holomorphic quotient of c, but C star is. And every cylinder of a every riemann surface structure c sub omega they have got here gotten here is of that type. And that should be the distinguishing feature. So, it is very important to look at maps such as this, and these maps are called covering space maps.

So, let us let us so, let me write that down.

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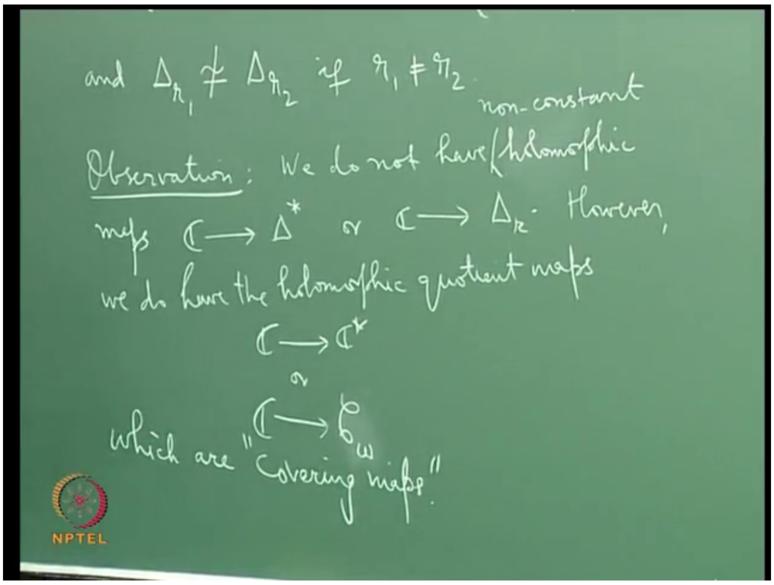

However, we do have the holomorphic map quotient map C to C want C star or C to C. So, that was a script c sub calligraphic c sub omega which are which are covering maps. So, so what is so, what is this what is the speciality of these covering maps. So, let me explain that. So, let us look at this map from c to c script c sub double omega, which is just c modulo the going modulo the translations by integral multiples of a single complex number omega.

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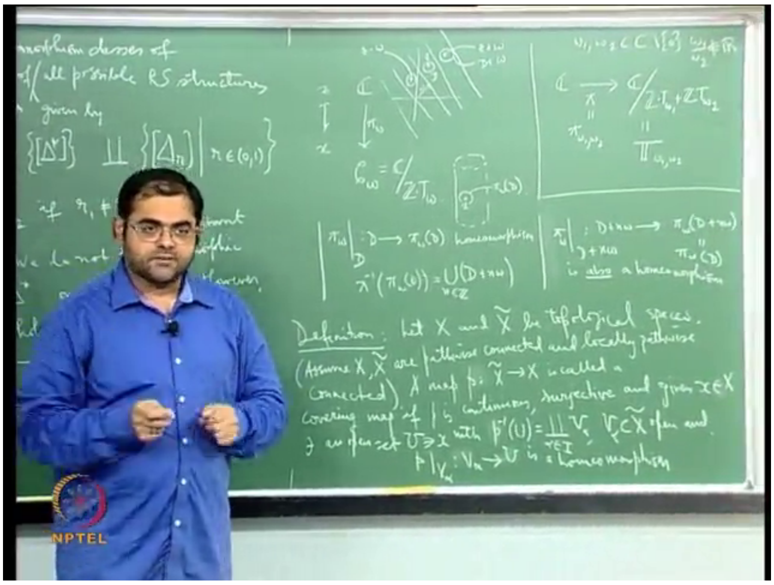

But by the way I should remark that n times t omega, which is it which is t n omega is actually if you want write it in multiplicative notation it is t omega to the n. If you think of it as composition, but additive notation is n times t omega.

So, anyway so, look at this map. So, if you look at this map, and you look at this map and look at the way in which we got the riemann surface structure. Let us go back and look at the way we got the riemann surface structure. How did we get it? We need the following team. So, here was so, here was the cylinder. So, here was a complex plane. And you know omega was this vector, and you know we drew these strips, and all these trips were off length mode omega, and you know omega was translation by omega mapped each of these trips into the other. And we went modulo the translations. And the effect a of that was we have to take a single strip and just identify the edges giving us a cylinder.

So, you got the cylinder here. So, let me write here a given a point on the cylinder x, what we did was we fixed we fixed a point said, such that z goes to x under this map. And we took a small enough neighborhood of z d. Small enough disc surrounding z call that as dn, we took the image of that D under this map this is pi sub omega.

So, this image here is pi sub omega of d. And we noticed that because I had chosen the disc D small enough the map, the projection map restricted to that disc is a homeomorphism unto pi omega of d. So, pi omega restricted to D from D to pi omega of D is a homeomorphism. And in fact, we took the inverse of this homeomorphism to give us a local coordinate chart at the point x, which is which if you want is the is the center of pi omega of d. Or which is the point of pi omega of T. And this how we got we got the coordinators. And the other thing that we noticed is that if you take I inverse if you take the inverse image of this under the whole map. What you would get is just translates the union of all translates of D with centers corresponding to the translates of z.

So, I inverse of pi omega of D is just a union of a let me say Un over n belonging to z D plus translation D plus n omega. So, were of course, by D plus n omega, I mean the set of all complex numbers of the form ez plus n omega were z velocity. And mind you this union because I chose t to be sufficiently small, this union is a disjoint union. So, what is happening is that this is z and this is D maybe if I draw another strip here. Then I will get the point here, this point will be ez plus omega, and this disc will be just D plus omega.

And similarly, if I take the point here, this point would be is z minus omega, and you know this disc would be D minus omega. And so, in this way you get a unit of discs. They are all disjoint. And this is essentially the standard example of what a covering spaces.

So, I will let me make a definition. Let x and x tilde the topological spaces. Assume x tilde r path ways connected and locally pathways connected. So, this is what I am writing in bracket is a technicality. And the importance of that technicality, when it when it when it has to be stress I stress it, but for the moment just assume this technicality do not worry about it for the moment. So, there are 2 topological spaces. A map pi or let me call it has p from x tilde to x is called a covering map if p is continuous subjective, and given x in x given a point of x small x of capital x, you are able to produce a an open set surrounding x, such that the inverse image of that open set under p. So, I am thinking of this as x tilde this is x and I am thinking of this as p. And you give me a point in x small x in capital x, I should be able to find an open set such that the inverse image of this open set is a disjoint union of open sets. And the project the map p restricted to each of these open sets is a homomorphism.

So. In fact, what I should tell you here also. So, so let me also add this here, let me add that condition also here. That is another remark if pi restricted to D plus let me take some n times omega from D plus n omega to pi omega. So, this is pi sub omega of D plus n omega which is the same as pi omega of D is also homeomorphism. In fact, I took pi from this disc D into it is image here, but I could as well taken any translate. Because after all of them are going to be identify.

So, I want the same kind of thing to happen in a in current space situation. So, let me write that down. So, given x belong to belong at x there exists an open set u containing x with pi inverse or p inverse of u p inverse of u is disjoint union over some indexing set alpha v alpha v alpha in the space above the open, and pi or rather p restricted to v alpha from v alpha to u homeomorphism, is a homeomorphism.

So, the standard situation is that we have x tilde above you have x below. We have a map which is surjective continuous; given a point below, I am should be able to find a neighborhood u such that the inverse image is a disjoint union of open sets about such that when you map when you restrict this map is prove this map to each of those open sets what you get is a homeomorphism with the open set below. Such an open set is called an admissible open set it is called an admissible neighborhood for the point x and we say that is a covering space is every point has an admissible open set.

And so, this is the standard covering space situation, right. And the other example that you can look at is that of the torus. So, in the case of the torus, what we did was we fixed 2 complex numbers. And we went we took the complex plane and went modulo, the translations by integer multiples of these 2 complex numbers of course, we made the condition that the ratio of these 2 complex numbers is not real. Because we wanted them as vectors to be linearly independent on the plane.

So, look at that situation. So, we have omega let me call that as omega one omega 2, in in c well of course, non-0 and with omega 1 by omega 2 is not a real number. And then you have this map c to c mod you take a the group which is which consists of translation by integer multiples of omega 1 plus translation by integer multiples of omega 2.

So, this is group is isomorphic to z cross z, and then this was this gives this gave you on. So, there was the quotient is a torus, the quotient is a torus, and he gave you a riemann surface structure on the torus. And let us call that riemann surface structure as omega 1 comma omega 2. And in this situation also if you call this map as pi of course, you know you should be this is pi, which depends on omega 1 and omega 2.

Then you can check that this is also a covering map this is also covering space map, in exactly the same way. In exactly the same way, the only thing is that in this case the cylinder case the plane was divided into you know strips of infinite length, translations of a single strip. But there the plane would be divided into a grid of parallelograms. And essentially all these parallelograms can be identified 2 one parallelogram, and then the only thing you will have to do is identify the opposite edges of the parallelogram and that will give you a torus.

So, you can check that this is also a covering space. So, the moral of the story is that the trying to get the riemann surface structure on a torus, or trying to get riemann surface structure on cylinder in this way as a quotient of the complex plane. Already involves a situation where there is natural occurrence of this idea of a covering space. And that is and that is the motivation for looking at covering spaces in general. So, the most general theorem that that we have is that take any riemann surface, just take any riemann surface, then you can think of it as the quotient by a suitable covering space, and to make this covering space very special namely to make to choose one special covering space among all covering spaces that could be different covering spaces. In this context let me tell you that this map factors to first going modulo you know just going modulo translations by integral multiples of omega 1 alone. That would give you a cylinder. And then there is there is going to be another quotient.

So, I can do this in any order I want and what it will tell you is that this covering space. Diagram can break down into two more covering spaces with the intermediate one being cylinder. So that means, that you have 2 covering spaces. For this and why is the covering space c special, and why is it different from the covering space that corresponds to a cylinder if the point is it c is simply connected, where is a cylinder is not. And it happens that among all the covering spaces, if you look at covering spaces which are simply connected, you get a very special type of covering space, that is called universal covering space, and then it is an amazing theorem that any riemann surface can be obtained as a quotient of a universal covering space, which is a simply connected cover. And the story does not stop there, what happens is suppose you believe that theorem.

So, given a riemann surface, I have a covering space for it. And that covering space is simply connected. Now because of the covering space property, locally neighborhoods there are homeomorphic to neighbor boo neighborhoods here. So, what I can do is; I can transport the riemann surface structure on the riemann surface to the covering space itself. And that will make the covering space into a riemann surface. And so, what I would get is; I would get a riemann surface which is simply connected, but then my basic uniformisation theorem says, any simply connected riemann surface has to either be the complex plane, or the unit disc which is the same as upper half plane, or the riemann sphere.

So, what is the upshot of all this, the upshot of all this is take any riemann surface, I have to obtain it is possible for me to obtain it as a quotient of either the riemann sphere, or the complex plane or the unit disc or the upper half plane. So, I am I am able to get a hold on the riemann surfaces by being able to get them as quotients of these 3 basic riemann surfaces. It is just simply connected. And that is that is what helps. And 2 to carry out this process therefore, you need to get into a study of covering spaces. So, that is what I am going to do in the forthcoming lectures. And the other thing that I have to tell you is where is it that the fundamental group is coming to the picture.

So, I let I will I will have I will define what the fundamental group is in a very formal way in the next lecture, but roughly let me tell you the fundamental group is trying to look at trying to fix a point on a topological space and trying to look at loops star starting at that point you know continuous path starting at that point and ending at that point. And you declare 2 such loops to be one the same if one can be continuously deformed to the other. So, you do this and you can compose loops, and in this way, you can form a group which is called the fundamental group.

And this fundamental group shows up in the in the case when this covering is a universal covering. Namely the fibers of this map, when x tilde is universal covering of x, mind you I told you that x tilde is called universal covering it is simply connected. And then by putting this condition of simply connectedness on x tilde that makes x tilde unique in a certain way, which I will explain it singles x tilde out as a special covering space called the universal curving space. And then what happens if this is the special universal covering space for x, then the fundamental group of x shows up as the fibers of this map fibers of a map a just inverse images of single points.

So, if you watch carefully in this map, the inverse image of a point is just a copy of z it is all translates of. So, the inverse image the point x is all translates of z by omega integer translate subset by omega. And the set of all integer translates of z by omega is can be treated as z. Similarly, you take a point here. The inverse image of a point here, will be a grid. It will be it will be all integer translate translates by integer multiples of these 2 non zero complex numbers. So, the inverse image of this point is a set of dots going in a direction perpendicular in an action is equal direction of omega which is perpendicular to the direction of the strips. And that the inverse image of a point will give you a grid. It will be actually the grid of parallelograms, if you take the vertices of the parallelograms, it is just that grid translated by a certain vector. And what is that, what is the grid of points each r it is again z direct from z.

So, in this case the inverse image of a point is just can be identified at the copy of z. In this case the inverse image of a point can be identified as the copy of z direct from z, but what I what is the z what is z directions z the fundamental group of the cylinder which is the object below is z and that is what is showing up as a fiber as the inverse image of a point. The fundamental group of the torus is z direction is z cross z, and that is what is showing up as a fiber over a point.

So, what will happen is; when you take the universal covering space, the fundamental group of the base will show up as a fibers. Not as a group at least as a set. So, in some sense the universal covering space is gotten by putting as many copies of the fundamental group of x as there are points in x. Putting together of course, in a topological kind of way. So, the moral of the story is the fundamental group enters into the picture and that is in and that is the reason why the fundamental groups also have to be studied in this connection that. So, that is one point. Then the other point. Notice that the fundamental group in this case which is z is also equal to this group. It is the group of translations by which we have gone modulo. Look at this case, the fundamental group is z direction z or z cross c. That is again the same as isomorphic to this group, which is the group of translations that we have gone modulo.

So, the moral of the story is; not only does the fundamental group show up said theoretical has the fiber over any point, it is also the group of automorphisms of the space above going modulo which you get this space. So, if p from x tilde to x is a universal covering space. Then x is actually a quotient of x tilde by a certain group of automorphisms of x tilde, and that group of automorphisms of x tilde is nothing but the fundamental group of x. So, the fundamental group shows up in 2 ways. It shows up as a fiber which is inverse image of any point any single point. And it also shows up as the group of automorphisms of the covering space the space on the top, modulo of which if you go you get the space below. And that is exactly the quotient situation that gives you that helped us to give you know a riemann surface structure on the cylinder or on the torus.

So, the moral of the story is therefore, if you understand the notion of covering space, and the notion of a fundamental I mean plan the notion of universal covering space. Then you have a key to trying to reduce the study of riemann surface structures, just to studying such maps which are covering space maps. And that is the importance.

So, and that is in fact, that is a technique by which most of the theorems of this type are proved. In fact, all the theorems that I have told so far except for the very fundamental theorem, that you know the only simply connected riemann surfaces are those which are either isomorphic to the you know, a non-in the non-case it is either the whole plane. Or the unit disc which is upper half plane. And the compact case it is just the riemann sphere. But this is the most fundamental theorem, proving this requires further techniques of analysis for example, but if you assume this theorem, and you assume this stuff about covering spaces. Then more or less all other theorems can be proved, which is what I am going to do in the next few lectures.

So, all this is just a pep talk for you to you know await the definition and you know the intricacies that involve covering spaces, and fundamental groups in the forthcoming lectures right. So, I will stop here.

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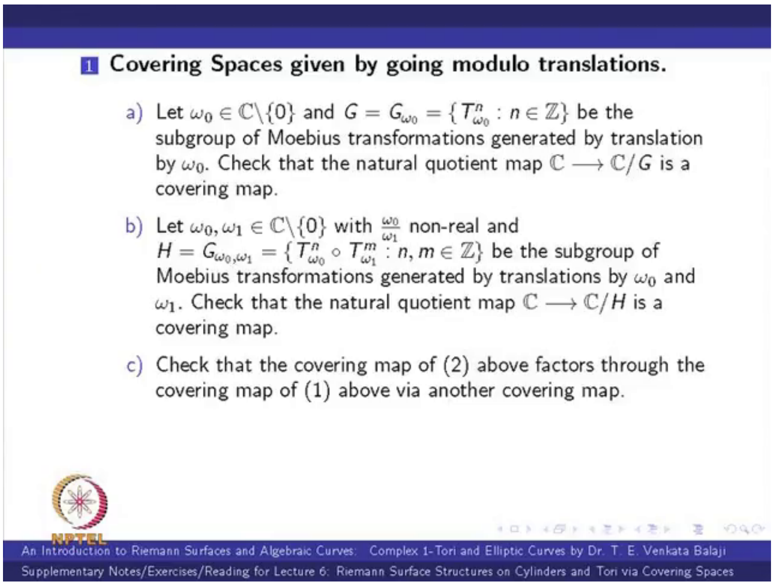

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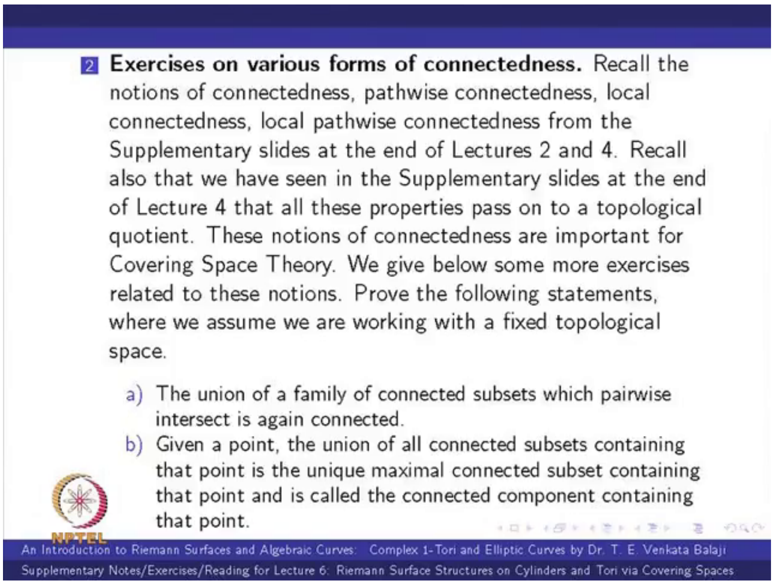

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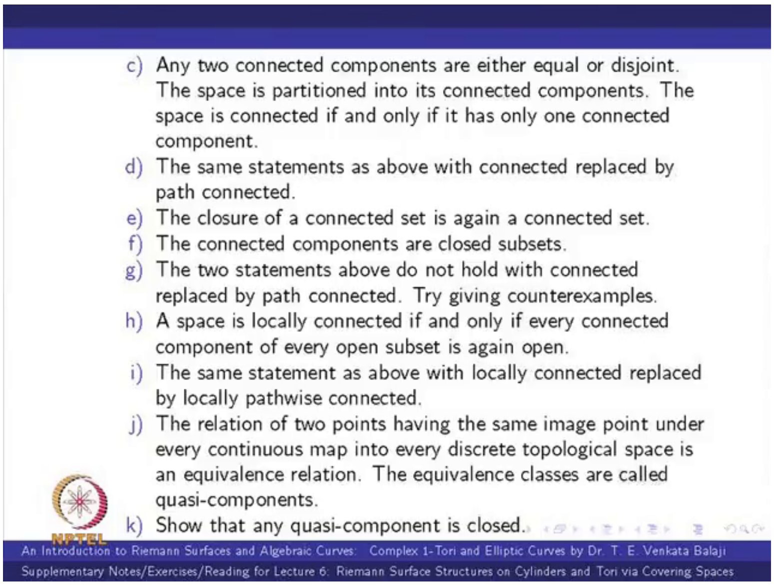

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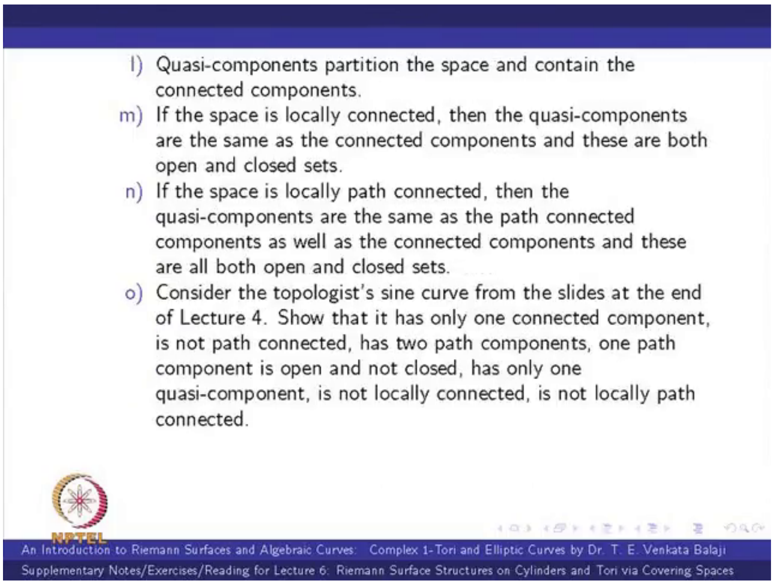

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