An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic curves

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Lecture - 45B
Complex Tori are the same as Elliptic Algebraic Projective Curves

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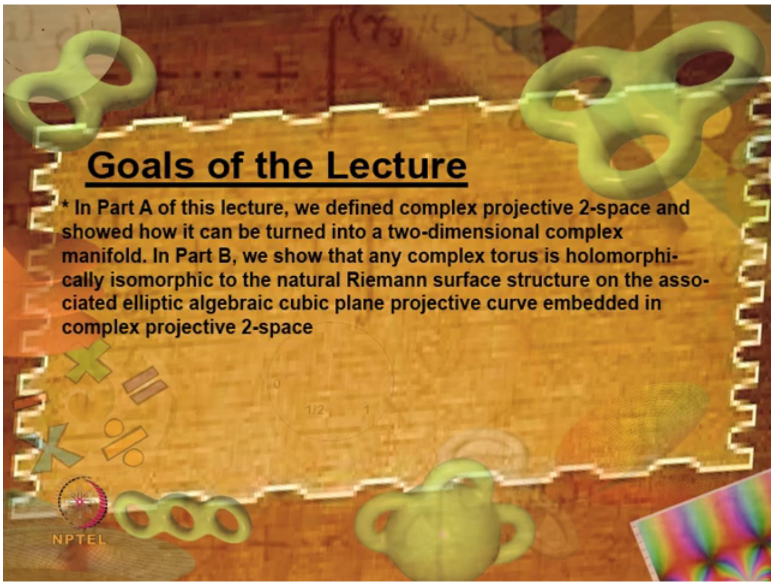

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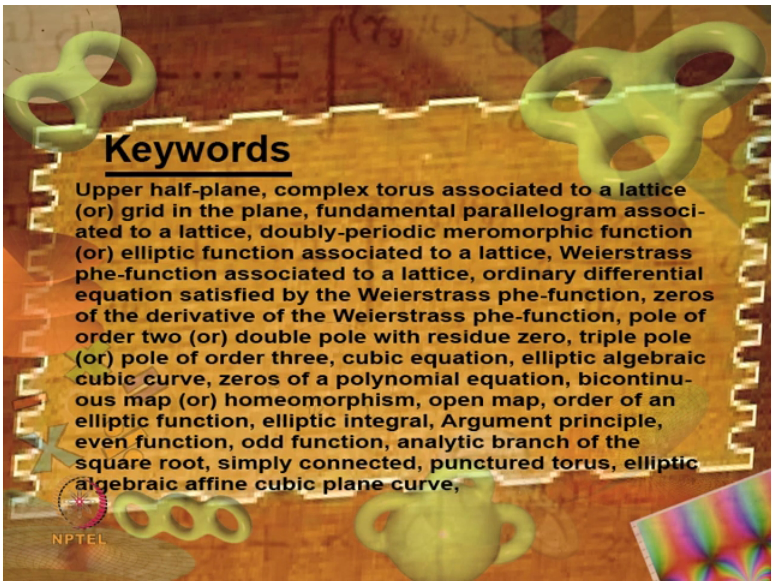

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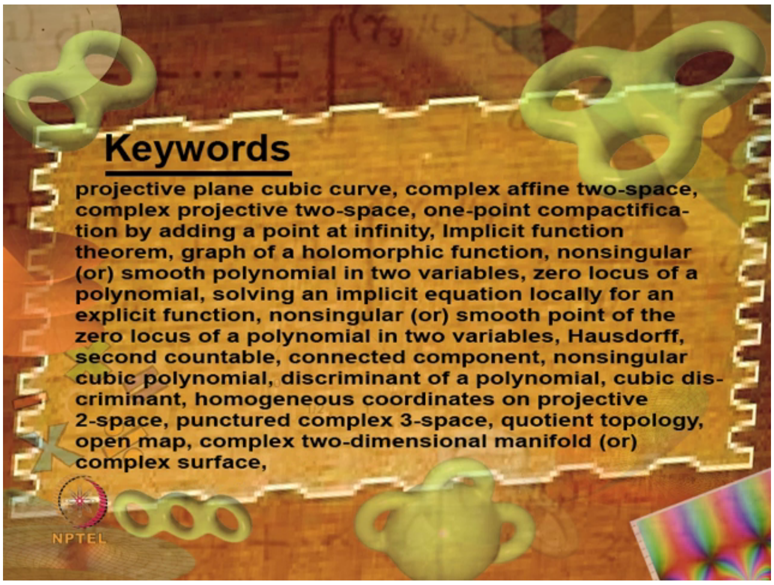

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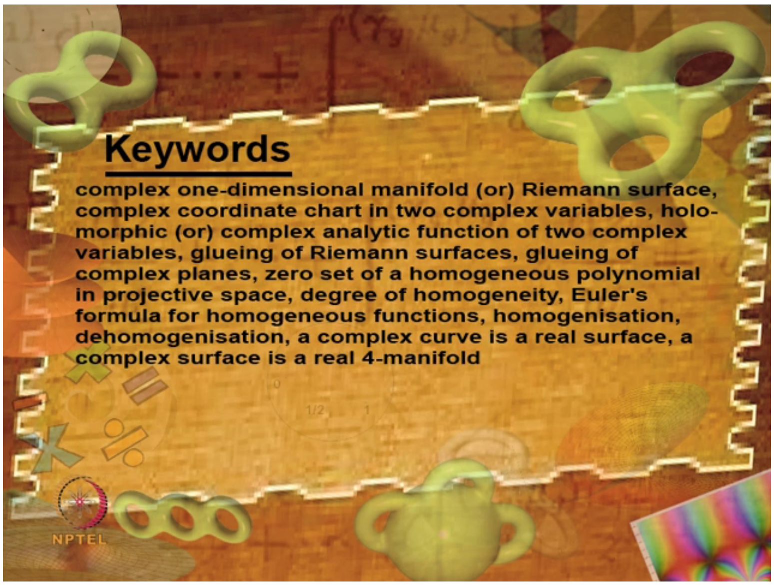

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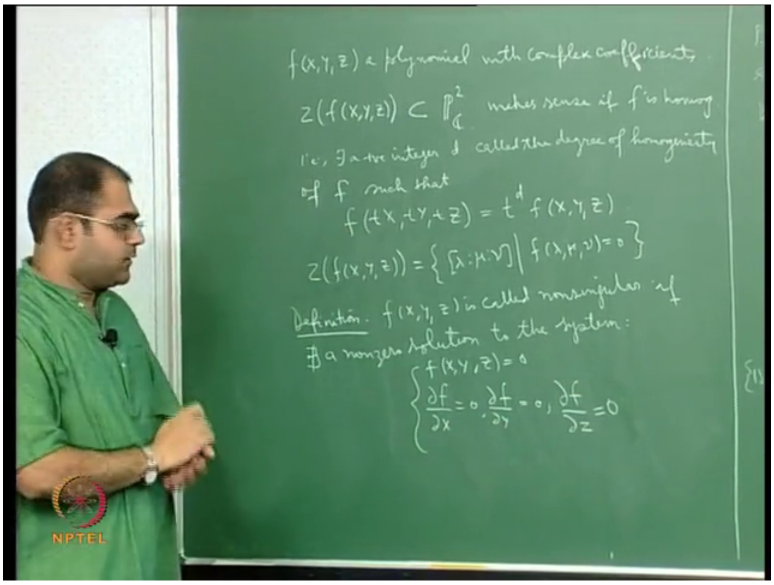

We will start with. So, if I take x comma y comma z a polynomial with complex coefficients. I will take a polynomial complex coefficients.

Then of course, the first thing I would like to do is to look at the zero set of this polynomial in projective space. I would like to talk about zeros of this polynomial projective space, but the point is that, I have to put a condition on the polynomial for this to make sense. I have to assume with the polynomial actually a homogeneous polynomial. So, this make sense, if f is homogeneous. What does that mean? It means that exist a positive integer d, called the degree of homogeneity of f; such that.

Well if I put f of t x, t y, t z I multiply each of the variables by t then what I get is, t power d f of x comma y comma z. So, why do I need this condition, because it is only, then that I can define z of x f of x comma y comma z, I can define it to be the set of all those points with homogeneous coordinates lambda mu nu; such that f of lambda comma mu comma nu is 0. You see if f of lambda comma mu comma nu is 0, then if I put this point can also be represented by t lambda t mu t nu, then it should happen that f of t lambda t mu t nu should also be 0, but that it will be if it is homogeneous, because I can pull the t out with the power d, where d is degree of homogeneity. Therefore, the zero set of f homogeneous polynomial make sense for homogeneous polynomial in three variables in projective space. So, basically what I am trying to say is, that if a polynomial vanishes had see after all this is supposed to be a line in c 3 passing through the origin and through this point.

And all I want is that, if the polynomial vanishes at one point on the line, I want it to vanish everywhere on the line and for that the polynomial has to be homogeneous only, then it will go down to something meaningful, in P 2 from a fine punctured a fine 3 space. So, well I have this zero set and the question is, you see P 2 is two dimensional, because we just now saw that P 2 is a two dimensional complex my manifold and then they, and you are looking at the zero locus of one equation. So, you should expect the zero locus to be one dimensionless. We expect, we should expect this to be one dimensional and if everything goes well, it should be a. again this should also have a 1 dimensional complex structure and you could expect this to be; therefore, a Riemann surface.

So I will have to tell you what are the conditions that you have to put on small f the so, called non singularity conditions, that will ensure that z of the, that the zero locus actually a Riemann surface sitting insideP, the P 2 the projective space. So, what I am going to do is, I am just going to tell you something about yes. So, I will make a definition and give a problem, which is a very simple exercise. So, I make the definition f of x comma y comma z is called nonsingular if that does not exist a nonzero solution to the system.

So, the system is f of x comma y comma z is equal to 0 dou f by dou x is 0,dou f by dou y is 0 dou f by dou z is 0. So, what you must understand is, since is a homogeneous polynomial. Of course, 0 will be a solution to the system, but when we, but 0 does not correspond of pointing projective space. So, we insist that there should not be a nonzero solution to this system. Now the point is that you know basically the way we got the complex structure on projective space was by going modulo one of the homogeneous coordinates to reduce it to a two variable situation.

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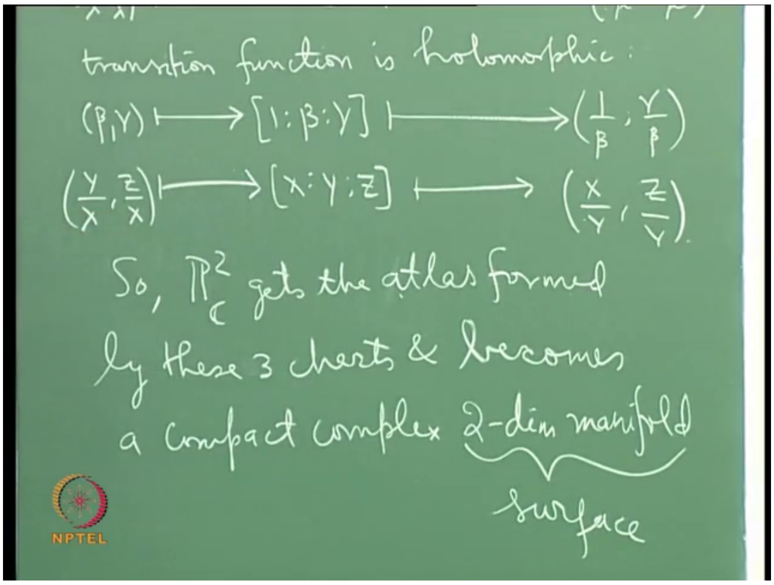

So, essentially what we did was [vocalized - noise]. For example, if x is not, if you are, if your nu not then you divided by x. So, something that depends on the three variables is reduce to something, that is, that depends on two variables. So, essentially what we are going to do is, I am going to relate this to the two variable. No singularity assumption which ensures that zero set of the two variable, the corresponding two variable polynomial is a Riemann surface, which we proved in the previous lecture ok.

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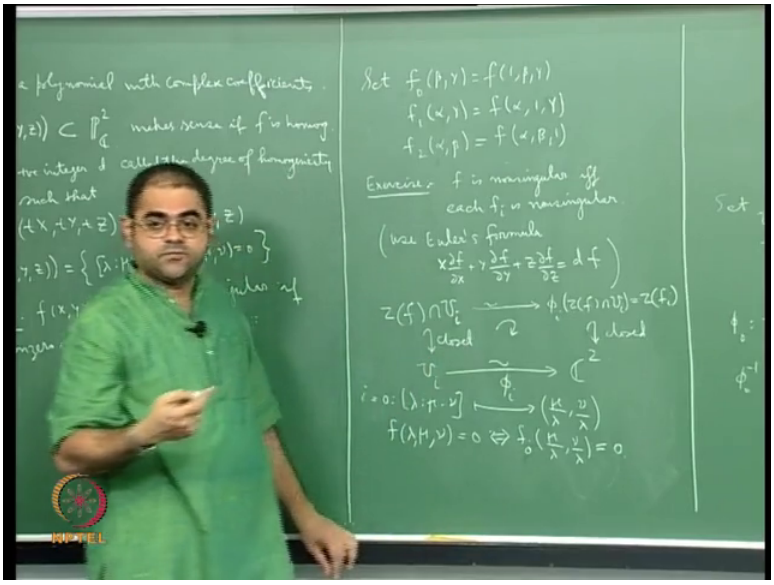

You will just said f naught to be. So, f you just said f naught alpha f naught beta comma gamma to be f of 1 comma beta comma gamma and you said f 1 of b alpha comma gamma to be f of alpha comma 1 comma gamma, and you set f 2 of alpha comma beta to be f of alpha comma beta comma 1. So, well I could have written it in terms in variable cycle of return f naught of y comma z is f of 1 comma y comma z; that is you put x equal to 1. You get the polynomial in the variables y and z. If you put y equal to 1, you get the polynomial f 1 which is a polynomial x and z, and if you put z is equal to 1, you will get the polynomial in x and y then it is a simple exercise.

That f is nonsingular, if and only if each f I is nonsingular. This is a simple exercise, and what it uses is the very elementary eulers formula that you know. If you multiply, the partial divertive is, by the corresponding variables and add them you get the degree times the function. So, use Euler's formula sigma or rather x dou f by dou x plus y dou f by dou y plus z dou f by dou z is equal to d f, where d is degree of d, is of course, the degree of homogeneity of f. So, it is a simple exercise to check this, and now if you grant this, if f is that, if f is nonsingular then f naught f 1 f 2 are nonsingular. Therefore, you see what it was, what this will tell you, is that the zero locus, the zero locus of these three loci will all be naturally Riemann surfaces. So, let me make that more cleared. So, what you do is, you take 0, you take 0 of f, and if you intersect into with u naught this is a closed sub set. So, if i intra, let me intra sect u i right. More generally this is a close sub set of a of u i, but then phi i is a biholomorphic identification of u i with c 2.

So what will happen is that, if you look at the image of phi i, the image of this under phi i z of f intersection u i. What you will just get is just is, z of f i inside which is a closed subset inside c 2, because after all the map is, you are sending lambda mu nu to. So, you know I am simply sending lambda. So, you know. So, if I take i could 0 for simplicity, then I am just going to send lambda mu nu to the map, is going to send lambda mu to, mu nu to mu by lambda, nu by lambda.

And then if f of lambda mu nu is 0, then f of 1 mu by lambda nu by lambda is going to be 0 and; that means, this lies if the zero set of f naught. So, f of lambda mu nu equal to 0 implies, and if and only if f i, or in this case f naught of a nu by lambda nu by lambda is 0.

So, the moral of the story is that the, but you see phi i are, they are complex coordinate charts. Therefore, see once you have, if you remember the ((Refer Time:11:35)) that you know you locally give charts once here compatible, you get Riemann. I mean you get a complex manifold structure, once you decided that you gotten the complex manifold structure, then each of the charts become biholomorphic. So, this is, u i is biholomorphic to c 2. Therefore, this close subset is biholomorphic to this subset, but this close subset f i being if f is nonsingular, then by this exercise f i is nonsingular. Therefore, this is a Riemann surface and therefore, by this. So, this is biholomorphic to Riemann surface; therefore, this itself becomes the Reimann surface. So, the moral of the story is that, if f is nonsingular then each of these intersections of the zero locus of f, with these three open sets that cover projective space, they are individually Riemann surfaces.

Now, I will write one more step to show these Riemann surface structure is actually coincide; that is the agree and therefore, you get a natural Riemann surface structure on the zero locus of f provided f is nonsingular. So, how, by do that also pretty easy.

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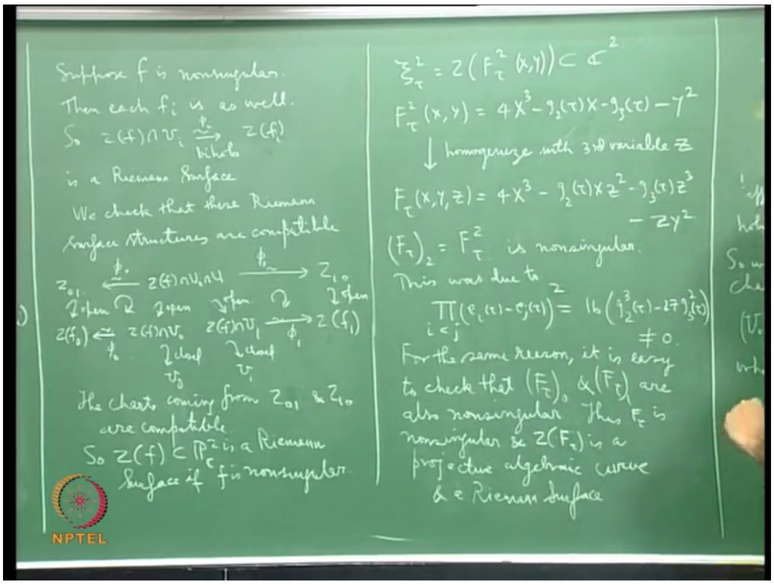

Suppose f is nonsingular then each f i is as well. So, z f i z f intersection u i which is biholomorphic to z of f i via phi i is a Riemann surface. This is something that we saw yesterday. Basically the fact that f i is nonsingular, will allow us to conclude by the implicit function theorem that f i looks that the zero set of f i looks like a graph of a holomorphic function, then you know if you have a graph of holomorphic function the first projection, the projection on to the independent variable will give you a complex coordinate chart. Therefore, each of these seeing are Riemann surface. We check that these Riemann surface structures are compatible; namely you see u I u 1, not u 1 u aught 2 are an open cover to phi 2. Therefore, z f intra section u naught come a z z of intra section u 1 comma z of intra section u 2 forms an open cover of z f. So, u have an open cover of z f by three Riemann surfaces, if you check that all intersection is Riemann surface structures is compatible, then z f itself becomes a Riemann surface. So, you see you have. So, let me take z f intersection u 0 intersection u 1 this is sitting inside z f intersection u 0 and you know this has been identified with z of z of f 0.

And that is by phi 0, and you know this also sitting inside z f intersection u 1, which has been identified as a Riemann surface by phi 1, because it is biholomorphic to z of f 1. Now what I can do is, I can again go by phi 0 here, and take it is image here, and call this is z 0 1, which is, this is open, and this is also open right, this is open and of course, you know. So, this is open, this is open, and of course, this is open in a closed.

So, you know actually this is further, this is further closed inside u naught, and this is further closed inside u 1, but the fact is these two are open, this is also an open set, and if I take the image by ((Refer Time:15:56)) sum of image by phi i, I will call that as z 1 0, which is also open here and these diagrams commute ok.

So, now I will have to tell you that this Riemann surface structure, and this Riemann surface structure, when it transport it to this, they agree. So, what that, does it that I will have, I mean basically I do not want is, that is this a Riemann surface structure, because is an open subset of a Riemann surface, open surface of Riemann surface, automatically becomes Riemann surface, because you can always restricts the charts to a given, you can restrict an atlas to a given open set is no problem.

So, this because Riemann surface opens open subset of this Riemann surface, that becomes an open subset that Riemann surface, but then these two are, you know holomorphic isomorphisms. So, you are, you will get two structures of Riemann surface on this, and I do not want them to contracting each other. So, basically what I will have to do is, I will take a, I will take a coordinate chart here, I will take a coordinate chart there, and show that they agree ok.

But you see. So, I will, I am just going to explain this, you see what is a coordinate chart on this Riemann surface. A coordinate chart on this Riemann surface is basically you knew f not is nonsingular. Therefore, you see the coordinate chart, if you remember is given by either you know projection on the. It is projection of one of the two variables if the partial derivative with respect their second variable is non vanishing, then the coordinate chart is given on the first variable, and the second variable holomorphic function on the first variable.

And if the partial derivability f not with respect to the second variable, first variable is nonzero, then the coordinate chart is given by projecting on a second variable ,and the first variable is a holomorphic function, the second variable and the similar story works here. So, basically what will happen is, when I look at the effect of a chart here and a chart there, I will get the same kind of, where I will get the same kind of equation, I will get the same kind of function transition function.

When I compare those two charts and the point is that, if you know if the coordinate is given by beta, then beta, if it the coordinate given by beta which is projection on the first variable, then you know in the implicit function theorem says a gamma is a holomorphic function of beta. Therefore, one by beta and gamma by beta are also holomorphic functions of beta. Therefore, this is holomorphic if the coordinate is given by the projection on to the second variable, which is gamma then inflict function will theorem, will tell you that the first variable is a holomorphic function b beta is a holomorphic function of gamma.

Therefore it will be gamma going to 1 by beta gamma by beta, and that is again a holomorphic function, and you see you can divide by beta, because beta is not equal to zero. So, therefore, that argument I am not writing it down here, that argument tells you that these the, these Riemann surface structures are actually compatible ok.

So, if you take a chart here, and then compose it with this, to get a chart here. If you get a, take a chart here and compose with this to get a chart here, if the chart intra sect then the transition functions are holomorphic. So, there is. So, let me repeat the important point is that, the charts here and there are given by projections on to the independent variable with depend with the second variable being of holomorphic function of the independent variable, and with the partial dire verity with respect to the partial direverity with respect the dependent variable being nonzero ok.

So, the charts coming from phi naught from z f not z. So, let me write let z 0 1 and z z is 1 0 are compatible. So, I have just done this for u not a nu 1, but it is a same story, if you do it with any other pair u i intra section u j and ah. So, the moral of the story is that z of f becomes Riemann surface.

So, z of f in P 2 c is a Riemann surface, if f is nonsingular; that is a small technicality here. If you insist that Riemann surface is actually should, actually be connected, then you know you will have to prove that this is connected, but let us not worry about it, at the worst it is a fact that is connected. What you must always remember, is that you can always giving a Riemann surface structure, is trying, is essentially giving Riemann surface structure on each connected component.

So, it that really does, not that need not matters. So, seriously now I need to tell you how I can go about doing what I initially wanted to do. So, you see. So, the first thing I want to state, is that you see the ah. So, basically you know our e tau 2 was a 0 set of f 2 tau of x comma y, aware inside c 2, and you know it was basically f 2 tau of x comma y was just 4 x cubed minus g 2 tau x minus g 3 tau minus y squared. And now what you have do is, I cook up three variable puddle of homogeneous polynomial from this, by homogenizing it by adding an extra variable z.

So, that, and to homogenize it I just make sure that all e, each theorem has total power 3, which is the highest power here. What I have to do is, I homogenize, you homogenize with third variable z. what you will get is, you get f tau of f comma y comma z, and that is 4 x cubed, is already degree 3. I do not touch it. This is x. So, it is degree 1. So, I want degree 3. So, I add the, multiplied by z square. So, I will get g 2 tau x z squared. And this g 3 tau is constant. So, I have to multiplied by z cube. So, minus g 3 tau into z cube minus, and this is degree 2. So, I multiplied by z.

So, this is my three variable homogeneous polynomial, and then you see immediately that if I calculate f tau sub 2, I simply get my f 2 tau; that means, to calculate f sub f tau sub 2. I am supposed to put the third variable. I have supposed to put 1. So, if I put z is equal to 1, I end up getting f 2 tau. And of course, this is already nonsingular. This is nonsingular. As we saw in the lecture before, because of the fact that the three zeros of a phi prime even tau e 2 tau and e 3 tau are distinct ok.

So, and actually. So, let me write that, and it; that is a same reason for which if you check that f tau subzero, and f tau sub one are also a nonsingular. It is a direct exercise to check that. So, this was due to conduct discriminant of the cubic on the right side of the differential equation, satisfied by the excessive function. What I less than j e i tau minus e j tau, the whole squared is i. I guess it was 16 times g 2 tau minus 27 g 3 tau squared which is not 0.

And if you check carefully, it is for the, exactly the same reason that you will see that f tau sub 0, and f tau sub 1 are also nonsingular for the same reason. It is easy to check that f tau sub 0 and f tau sub 1 are also nonsingular. Therefore, by that exercise, that f i is been nonsingular, will tell you that f is nonsingular. What this will tell you, is that f tau is nonsingular. Therefore, it will tell you that the zero locus of f tau is a Riemann surface, and it is it. And I forgot to tell you that this Riemann surface, this Riemann suface is called projective algebraic curve associated to f.

It is called projective algebraic curve. It is called projective curve, because it is, it lives in projective space. Thus f tau is nonsingular and z of f tau z of f tau is a projective algebraic curve. It is a projective algebraic curve, and a and a Riemann surface. So, it is a Riemann surface now, and if you notice the way we have defined it, if you intrasect with u 2, you get exactly e 2 tau.

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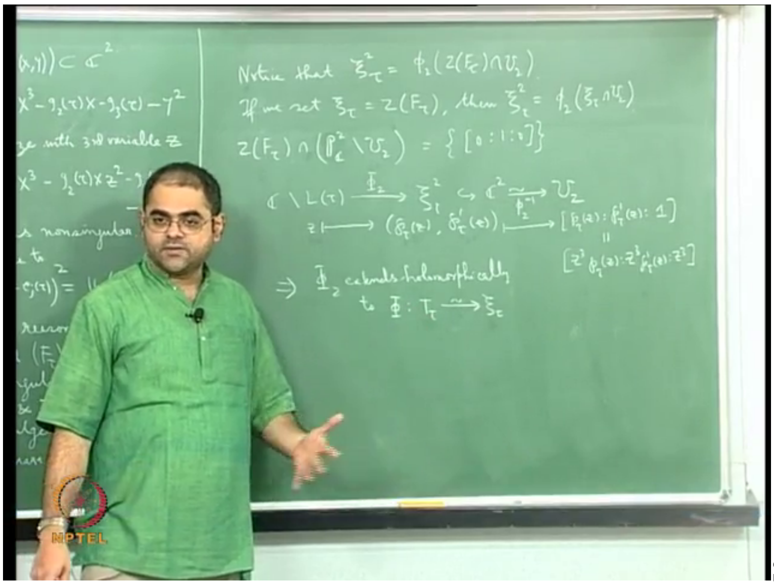

So, let me read that notice that f, your e 2, e 2 tau is simply is going to be phi 2 of z of f tau intersection u 2, because phi 2 actually identifies u 2 with c 2. And under that identification, the zero locus intersection with u 2 is identified with zero locus that we saw, and this was, we see, this was the. So, what is happening is that this affine. So, you know, if we call, if we set e tau to be just z of f tau, then you see e 2 tau is just phi 2 of e tau intersection u 2.

And phi 2 is of course, a biholomorphic map. So, what this tells you is, that the projective algebraic curve to projective algebraic elliptic curve, associated with tau on this, on this open set, where the third coordinate is not zero, exactly looks like the affine algebraic elliptic curve associated to tau. What you have done is, you have just projectivized it, and I will show you that you have done the projectivization, which is you should, you must think of it as a compactification by simply adding a single point. So, you see, you know if I take z of f tau, and intersect it with. If I intersect with u 2 compliments; namely the compliment of u 2 in the projective space. So, I am going to intersect it with. So, I am going to look at points on this; such that the third homogeneous coordinate is actually 0.

Ok; that means, I am going to look at homogeneous coordinates x colon y colon z; such that z is 0, but you see if z is 0, and then all these terms wipe out, I get only 4 x cube, and that has to be 0. So, that you will mean that x is also 0, but you are in projective space. Therefore, the y coordinate cannot be 0. So, you will get 0 some non. So, the homogeneous coordinate becomes zero, some nonzero value again zero, but that is equivalant to 0 1 0. So, essentially the intersection of this, is a single point, it is simply 0 1 0. the. So, called the point at infinity, this is called the point at infinity of affine curve, and what you have done is, you have added this point at infinity, to this affine curve, it is a one point compactification, you get the projective curve.

So, this is how the affine curve, the affine elliptic algebraic curve associated to tau is compactified by adding a single point into a projective algebraic curve. And now the only thing I have to show that the torus, the complex torus associated to tau is biholomorphic to this Riemann surface structure on the projective elliptic algebraic curve; that is very pretty, that is very easy. You see I started with c minus the lattice, and I define this map phi is sub 2 to the to e 2 tau. The map was just sending z to phi of phi tau of z comma phi prime tau of z. And now what I am going to do, is, I am going to think of this as sitting inside. Well c c 2 which is identified by phi 2 immerse by with u 2 alright and.

The fact is that, what you do is, of course, I cannot define it at any of the lattice points, because I cannot put z equal to a lattice point, because both of them have force, but what I can, but what you remember, is that. Well this goes all the way to by our mapping, that phi 2 is, it goes to phi of phi tau of z colon phi prime of z colon 1, this is what it goes to. Now the problem is, I cannot put z 0. For example, of course, if I prove that I can extend it to 0, then I have done, because it will I, how to extend it, holomorphic into 0, but it is very simple, I cannot, that add it to it, but the fact is, you know this has a pole of order 2. This has a pole of order 3.

So, at 0. So, I can multiply. So, here I can multiply throughout by z cubed. So, what I do is, if I think of this z cubed phi tau of z colon z cubed phi prime tau of z colon z cube, which is which it is. So, by the definition of projective space, then I can really, I can plug in z equal to 0, because when I plug in z equal to 0 phi tau, only has a order of 2. So, this will evaluate to 0 phi prime tau has a pole of order three. So, this is evaluate to nonzero number, and this is of course, going to, could 0. So, at z equal to 0. This literally going to, go to 0 1 0, which is the one. This is the one extra point you have added compactify. So, this will tell you that phi 2 extends holomorphically, 2 phi from the torus defined by tau to the elliptic projective algebraic elliptic curve, associated with tau, and you see. Now you have holomorphic map, which is bijective, and infact; therefore, it is automatically biholomorphic map. If you want t tau is compact. This is a compact space, and this is a ((Refer Time:33:32)) of space, and you have a bijective continuous map. So, it is. So, the extension is of course, continuous, but the way we extended is actually a holomorphic ok

Therefore, the moral of the story is, that given a tau; the complex one dimensional torus t sub tau is naturally a isomorphic to the projective algebraic elliptic curve associated to tau, which arises out of the homogenization of the differential equation, satisfied by the vestas phi function associated tau. So, this is the, really the beautiful convex stone of the subject that you start with something; that is f; that is of analytic nature in the Riemann surface, which has been obtained by locally glowing complex coordinates, and you end up that showing, that it is actually the zeros of an algebraic curve.

So, this is the beginning of very rich and classical theory, which has extensions to today and. So, this is of. So, this generalizes as a theorem which says that, if you take a compact Riemann surface. If you take a Riemann surface, you put the extra condition; that is a compact that is actually algebraic. So, it is given by algebraic, it is given by the zero locus of algebraic equations ok.

And then of course, I should also tell you that, I have proved that every complex one dimensional torus is a projective elliptic curve. Conversely you can define a projective elliptic curve by, you know taking an affine elliptic curve, which is of the form y sqared equal to a cubic in x with descriminant of the cubic nonzero, and then you can again projectivise it in a same way. Projectivising the elliptic curve will be the same as adding a point at infinity, and that would algebratly mean homogenizing the equation with the third variable.

Ah and looking at, it is 0 is in projective two space, and then it is. Then you can actually show that for every projective algebraic elliptic curve, it is isomophic to carvellex one dimensional torus. Therefore, there is no difference between complex one dimension tori i and projective algebraic elliptive curves, and that is why tori are called as elliptic curves. So, I will stop with that.