**An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves**

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**Lecture - 45A**
**Complex Projective 2-Space as a Compact Complex Manifold of Dimension Two**

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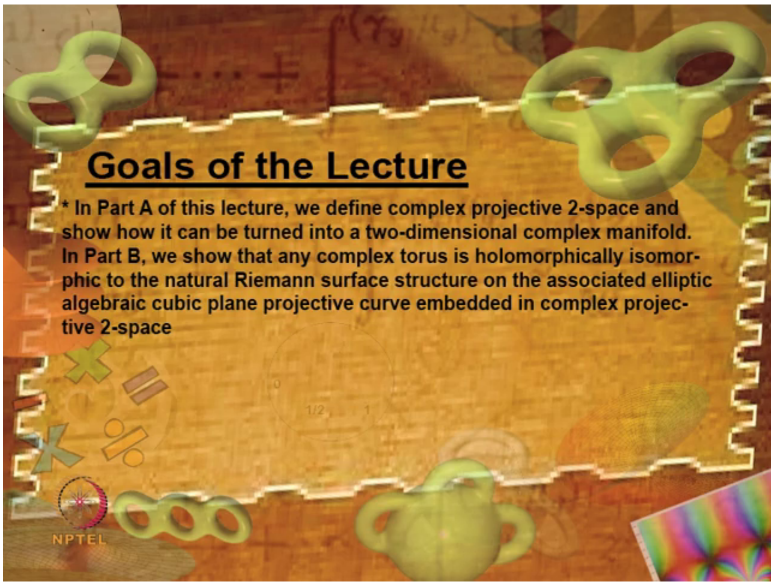


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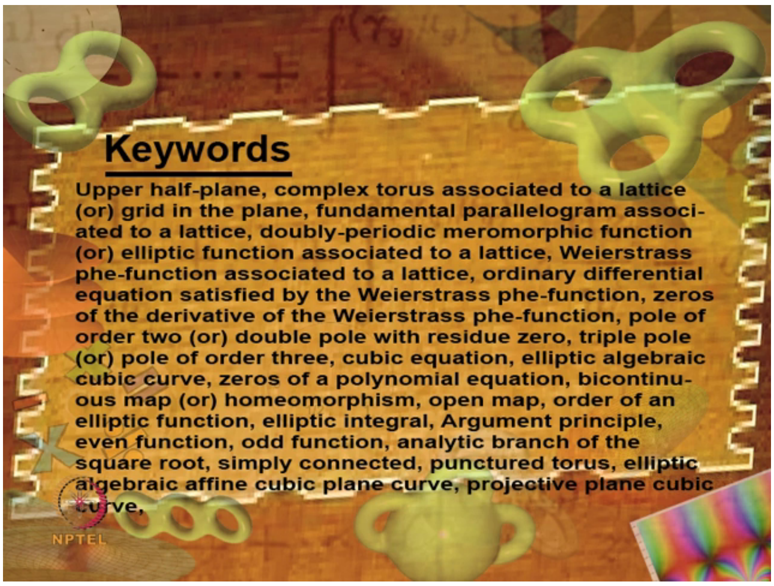

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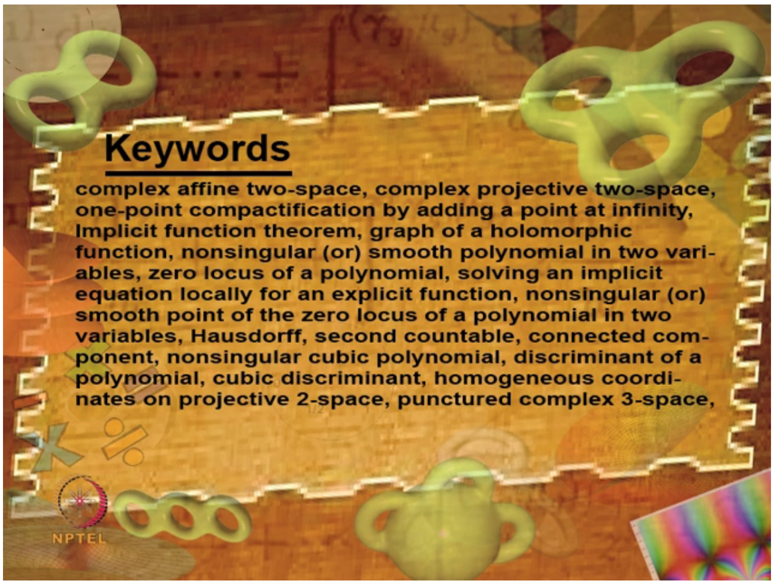


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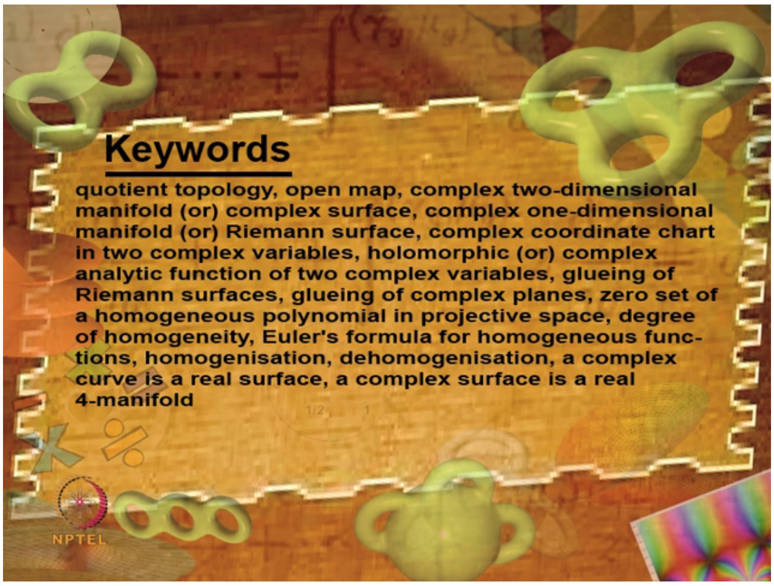


Well, let me recall what we did in the previous lecture.

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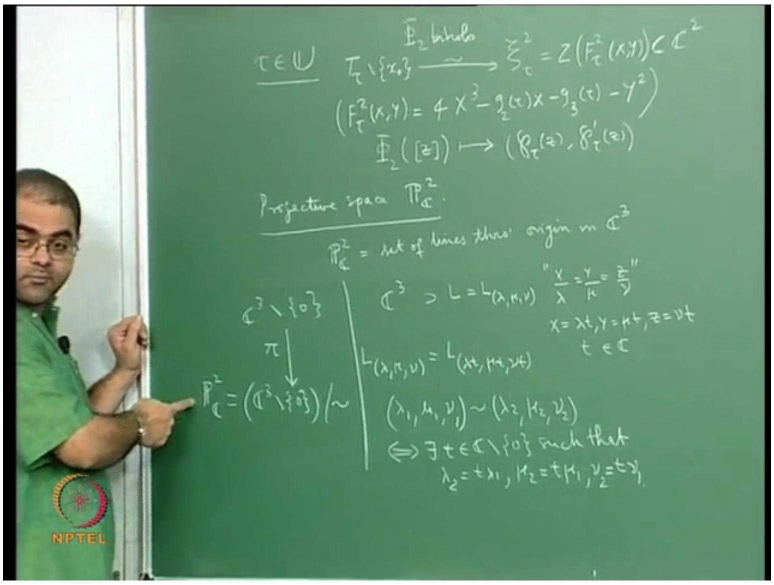


So, we saw that given tau in the upper half plane, you are able to get a bi-holomorphic map namely a holomorphic isomorphism of the punctured torus define by tau. So, it is a torus minus of point and this point is actually the image of the lattice define by tau under the natural map from the complex plane to the torus which is the quotient map. So, this is the punctured torus, the torus define the tau minus of point and this is bi-holomorphic to the so called affine algebraic elliptic curve associated to tau which I called as e sub tau 2 and this e sub tau 2 was the 0 set of the polynomial F 2 tau X,Y inside C 2 C cross C Cartesian product of C with itself and where F 2 tau of X comma Y was simply 4 X cube minus g 2 tau into X minus g 3 tau minus Y squared and in fact, this bi-holomorphic map was gotten by in fact sending.

So, phi 2 sent the point z I am putting square bracket z to be the element in the torus which corresponds to z in the complex plane under the quotient map. So, this is the equivalent class of z and that is being sent. And of course, since I am the punctured torus z is not a lattice point and I am just sending this to the point to the pair of complex numbers given by p tau of z comma p prime tau of z. And this the image lands inside the 0 set of this polynomial because of the first order degree 3, first order degree 2, differential equation satisfied by the p function.

We also proved that this e 2 tau is actually naturally a Riemann surface structure because it is by the implicit function theorem. It is locally a graph and the coordinate projections will serve as complex coordinates they will give you charts and that these chart agree and the fact that you can apply the implicit function theorem came from the fact that this polynomial is non singular. Namely, there is no common solution to the system consisting of this equation being 0 and also the first partial with respect to X and Y also being 0 simultaneously.

So, there is no point on this locus, there is no point on this elliptic algebraic affine curve at which both partial derivatives of this polynomial with respective first partials with respective X and Y both vanish on that point. That was the non singularity condition. That was required to apply the implicit function theorem and the implicit function theorem told you that locally this locus is graph of a holomorphic function and therefore; it acquires the structure of Riemann surface. And, then we check that all these local Riemann surface structure glue together that is agree. In other words the projections on to the 2 coordinates here actually give charts which are compatible whenever they intersect.

So, now what to do today, I am going to tell you how to extend this map to full torus on this side and on the other side I am going to extend it from this affine elliptical algebraic curve to the so called projective elliptic algebraic curve, associated tau and you see and how that is obtained is you see the by adding x naught as a one point compactificationto this punctured torus we get the torus. And similarly, there also you are just going to add one extra pointso called point at infinity as it is called in a algebra geometry to get the projective algebraic curve, but then after you add it no longer lives in C 2, it lives in projective space it lives in projective 2 space.

So, I will have to explain what this projective 2 space. So, I will begin by trying to explain what is projective spaces. So, projective space P 2 C by definition P 2 C is just the set of lines through the origin in C 3. So, you look at C 3 which is 3 dimensional complex space and you look at the lines through the origin and you the set of lines is called the projective space which call the projective 2 space.

So, by definition a point of P 2 C is a line through the origin in C 3. Now let us try to understand what that means, you see if you give me C 3 and if you give me a line through the origin C 3. Then you know this line can be written as L sub lambda mu nu where the lambda mu nu are the so called you may call direction ratios of the line. So, you know general equation of the line would be something like you know you would normally write in coordinate geometry. You will write it as X by lambda is equal to Y by mu is equal to z by nu means this is how one would write it the equation of line passing through the origin with direction ratios lambda mu nu.

And of course, the only problem in writing like this is one of the lambdas or mu's or one of the lambda mu nu could be 0. So, it is not legitimate to put that in the denominator a batter way of writing it is; you can also write it a parametrically as X equal to lambda t Y equal to mu t Z equal to nu t where t is a complex number. You can write in this parametric form and now you see that you also see so what you do is given a line like this you send it to.

So, this corresponds to a point in P 2 C and then you notice that L see L sub lambda mu nu is also equal to L sub lambda t mu t nu t. And of course, I want really a line through the origins. So, all lambda mu and nu cannot vanish at the same time. So, I want at list one of them to be nonzero only I will get it really a line through the origin and of course, if lambda mu and nu are all simultaneously 0. I simply get the origin I would not get a line. And you see if t is a complex number, you see that the line joining the origin to this point is a same as a line origin joining the origin to this point and therefore, you see that if you define this equivalence between triples in this way then the quotient will naturally be the set of lines.

So, in other words what you do is that we have C 3 minus the origin. So, I am simply writing a single 0 for the point with all 3 coordinate 0 and what you do is you have this projection pi into the quotient which is C 3 minus the origin modulo an equivalence relation. And what is the equivalence relation? The equivalence relation is lam lambda one mu one nu one is equivalent to lambda 2 mu 2 nu 2 if and only if there exist a complex number they exist a complex number t such that to t not equal to 0 such that that these 3 are just multiples of these by t: that is lambda 2 is equal to t lambda 1 mu 2 is equal to t mu 1 nu 2 is equal to t times nu 1.

So, if you go modulo this equivalence relation. What you get here? Is actually going to be it can be identify with p 2 c set of lines. In other words if you give me a point lambda mu nu, here it goes to the line L sub lambda mu nu which is thought as a point here all. And of course, L sub lambda mu nu and L sub t lambda t mu t nu will be the same there on the same line. So, they give the same point here. So, in this way we are able to think of projective 2 space as a quotient of this open set which is the punctured 3 space by this equivalence relation. And of course, we would like to topology and geometry on it. So, what we would do is a first natural thing that we would try to put the quotient topology on P 2 C. So, that pi becomes continuous map.

So, we will do that.

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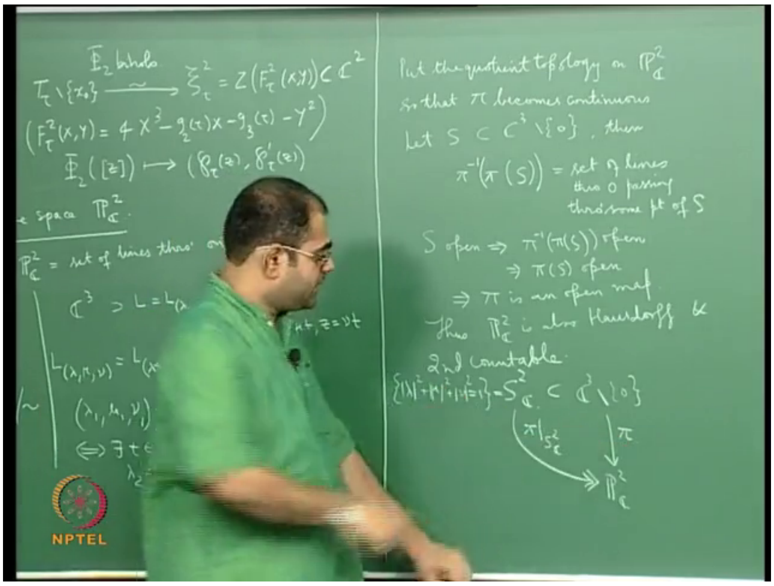


Put the quotient topology on P 2 C. So, that pi becomes continuous and this is very nice quotient. In fact, I just want to tell you that this quotient map is actually an open map. Yes, because you see if you take subset S in the punctured 3 space, the 3 space minus origin and then if you project it down by pi and then you take the inverse image. What you will get is precise the set of all lines to the origin which pass through some point of S this is exactly set of lines through 0 passing through some point of S this is what you will get and in particular if s is an open set already, then this set of lines through the origin passing through points of S will give you a kind of cone generated by S and that will also be an open set.

So, it is very clear that S open will imply that pi inverse pi of S is open, but then this will tell you that pi of S is open; because a set in the quotient namely in the projective space is set to be open in the quotient topology if and if only it is inverse images open. So, the moral the story is that this will tell you that pi is an open map and since it is an open map. What you can immediately conclude? Is that since C 3 is both housed off an second countable. So, this sub space is also housed off a second countable and since is an open map the quotient is also housed off an second countable. So, thus P 2 C is also housed off and second countable.

So, basically it is housed off because if you give me 2 points here: I can find these 2 points distinct points above in the lines that corresponding to these 2 points below and since that is housed off. I can separate them by disjoint open neighborhoods and then if I take the images below. I will get disjoint open neighborhoods below and then as for second countablity I just choose a countable base here and then take it is image. I will get a countable base because pi is of course, surjective open map of because this is a quotient it is just a set of equivalent classes.

So, it is housed off second countable space, then the other thing that one as to notice is that this is actually. In fact, compact as a topological space you see if you take in C 3 minus the origin you take so, called units fear which is S2 C this is precisely the set of all those points with the property that mod lambda squared plus mod mu squared plus mod nu squared is equal to one this is the set of all points on the unit sphere and of course, you know this is a close and bounded set. So, it is compact set and it lives here and so. In fact, I should put sub of this it lives here and then you know if actually from C 3 minus 0, if you take the projection pi to P 2. You actually see that if you restrict the projection pi restricted to S 2 c this is also going to be surjective.

Therefore, it is very clear that you have this is a continuous map, you take a continuous map and restrict to a subspace this also continuous it is surjective this is compact therefore, that is compact.

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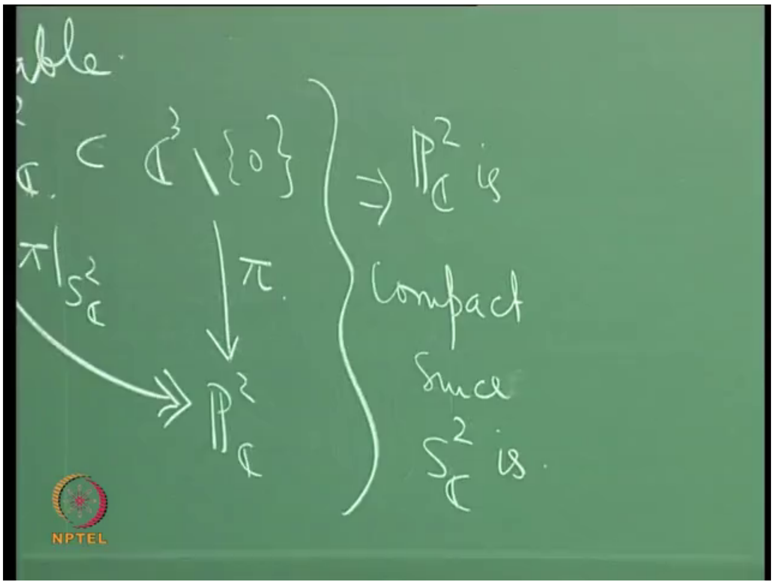


So, what this will tell you is that P 2 the projective space is compact since S 2 is. So, this is a compact. So, we have got a compact housed off second countable space then I need to introduce to you so called homogenous coordinates on P 2. So, let me just explain it. So, you see you have C 3 minus 0.

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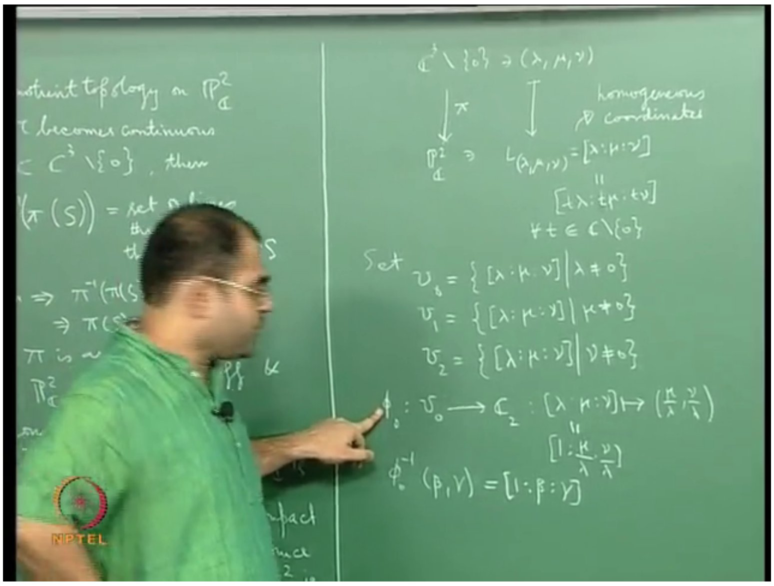


And you know if you take a point lambda mu nu it goes to well under the map pi it goes to a point of P 2 C and the point that it goes to is actually the line passing through lambda mu nu and as you seen this line lambda mu passing through lambda mu nu and the origin in C 3 actually depends only on the multiples of these 3 numbers lambda mu and nu on a common multiple of this numbers by the equals. So, what we do is. We simply denote this as we want to think of it as a point and we denote it by these coordinates call the homogenous coordinates. So, you put a colon lambda colon mu colon nu to just to say that is ratios that matter and then once you chose to write it like this then the these 3 are called the homogeneous coordinates and mind you that if I multiply each homogeneous coordinate by the same nonzero complex number t then I still get the same point.

So, this is also equal to t lambda t mu t nu for all t nonzero complex number. So, this are called homogeneous coordinates. So, let me write that down. So, now, what I am going to now do is you see I am going to show how you can turn projective 2 space into what is called a complex 2 dimensional manifold? So, let me recall; what was a Riemann surface. A Riemann surface is basically a complex one dimensional manifold by that it is a basically it is you know it is housed off second countable topological space if you want connected such that there are locally there are complex coordinate charts they are all charts into the complex plane they will give you single variable coordinates at each point neighborhood of each point and this charts should be compatible.

So, that they give an atlas and then the Riemann surface structure is define. In fact, you normally take the Riemann surface structure to be the one that is define by a maximal atlas. But there is just a technicality, but now what you can do is what are we did? Therefore, one variable you could also do it for 2 complex variables and in this case. So, it means that you are trying to take topological space with this which is housed off second countable and for each point you are trying to get hold of an open set which looks like an open set in C 2.

Now that means, it is homeomorphic to an open set C 2 and then you transport the complex structure on C 2 to this on that open set of C 2. And then you again check that all these 2 variable complex coordinate charts they all agreed with one another wherever the interest. And this is what gives raise to what is called a 2 dimensional complex manifold. So, what I am going to now try to tell you is that P 2 C is the 2 dimensional compact complex manifolds.

So, in this case there are essentially 3 charts and we say these charts are given on the open sets where the first variable, the first homogeneous coordinates, the second homogeneous coordinate and the third homogeneous coordinate respectively do not vanish. So, what will do is we will set u 0 to be the set of all lambda mu nu such that lambda is not zero the first coordinate homogeneous coordinate is not zero. So, you must understand that this is well defined, because if you have the first variable nonzero then any multiple of then this point is remain the same, if I multiply throughout by a nonzero complex number and for that nonzero complex number also I mean for that representation also the first variable continuous to be nonzero. So, similarly we define u 1 to be the second homogeneous coordinate on non-vanishing and we define u 2 to be the third homogeneous coordinate not vanishing now. What I want to tell you is that I am going to explain how u naught u 1 u 2 are actually homeomorphic to copy of c 2 itself?

So, for example, let me define phi naught from u naught to C 2 by well if I take a point in u naught is of the formula lambda mu nu lambda is to mu is to nu this are the homogeneous coordinates and you see the point it is a nu naught. So, lambda is not zero. So, I can multiply throughout by 1 by lambda and I will get one. So, this point is the same as 1 colon mu by lambda colon nu by lambda. And well, I send it is simply I have to give you an order pair of complex number I simply send it mu by lambda nu by lambda. So, this is my map and you can check very clearly that phi naught inverse exists and phi naught inverse simply takes beta comma gamma to one beta gamma to the point with homogenous coordinate is one beta gamma.

So, if lambda was one and this was beta and gamma; I will simply get beta and gamma. So, you see phi naught is a bijective map. Now it is very easy to check that phi naught is homeomorphism. So, how does one check that? So, one does the following thing well you see you have C 3 minus.

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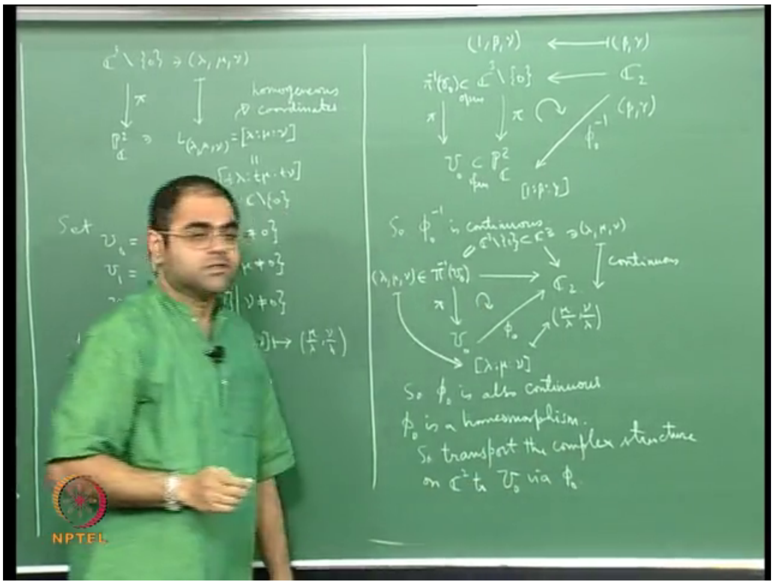


The origin, the punctured 3 piece and then you have to projection pi on to this projects on to the projective space and then here you have the open set u naught. Mind you, u naught is an open set because it is inverse image above is all those points which are contained in the punctured affine space and there is actually a legitimate coordinate and coordinate vanishing is locus. A close set it is compliment where the coordinate does not vanish is an open set.

So, the inverse image of u naught pi inverse u naught, is actually an open set and pi is pi restricted to that will carry it on to u naught this is also open in the projective space. And now you see I can define a map from. So, let us say yes. So, you see what I do is from C 2 I define a map by sending. I will simply send beta comma gamma to one comma beta comma gamma which is the point here this point actually lies here and it goes down to this map.

So, if this map actually goes down to a map like this and the fact is that this map actually lies in pi inverse of u naught and if I compose this map actually lies in u naught. So, the map is just beta comma gamma going to one colon beta comma gamma which is just the image of this. And of course, this is a continuous map because I am just using the coordinates to give this map. In fact, it is a holomorphic map. See a map in several variables in several complex variables is said to be holomorphic; if it is continuous and it is in each variable separately a holomorphic function.

So, you see this is individually there are 3 variables and first variables is consent, the second variable it is beta and gamma which are any way holomorphic, because they are coordinates. So, this is a holomorphic map it is continuous and this of course, continuous. Therefore, this is continuous, but what is this is just phi naught inverse. So, this tells you that phi naught inverse is continuous and so, phi naught inverse is continuous and you will also see that phi naught itself is continuous. So, I have u naught to C 2 which sends lambda mu nu to mu by lambda nu by lambda. So, what I have above is pi inverse of u naught and then I have this map and this map is just. So, pi inverse u naught sit in sits insides C 3 minus the origin which sits which in turns it sits insides C 3 and then this map from C 3 to C 2 is just.

So, you know if I take a lambda mu nu above which goes to this under the map pi then what I do is I simply take lambda mu nu as a triple in c 3 and simply send lambda mu nu. So, I will have to divide the lambda. So, I will just send it to mu by lambda mu nu by lambda and this is of course, you see this is since lambda is not zero division by lambda is not a problem and then this is therefore a continuous map. So, if actually 3, I have 3 coordinates and I am dividing by a coordinate which is not zero. And that is contain that continuous to be continuous. So, you see therefore, the moral of the story is that. So, this should tell you this is continuous and this diagram commutes. So, this is our affine phi naught. So, phi naught is also continuous.

In fact, to be more clear I start with an open set here. I take it is if I take the inverse image here, it is same as taking the inverse image here at projecting it down, but if I take the inverse image here. I will first take the inverse image here which is going to be open. And, but it will lie here and then if I project it down since pi is an open map. I will get a open set here that is how the inverse image under phi naught of an open set here.

And therefore, by the topological definition what a continuous map is phi naught is a continuous map. So, of short this story is that phi naught is a homomorphism. So, what you do is you use phi naught, you use u naught comma phi naught as a complex chart coordinate chart, but now it is in 2 variables because you are identify with C 2. So, there are 2 complex variables. So, transport the complex structure on C 2 to u naught we are phi naught.

So, what does this mean is exactly the philosophy is a same as we define a Riemann surface. So, you know basically what do you mean by seen transport the complex structure on C 2 to u naught. We are phi naught it means that well you know. So, we are trying to do complex analysis on u naught which is homeomorphic where phi naught to C2

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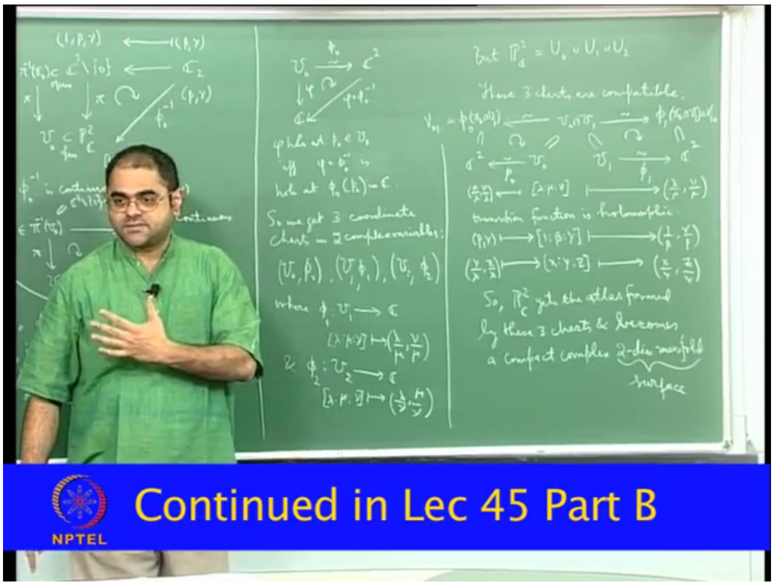


So, what you do is that every point in u naught will get 2 complex coordinates because it is image here has 2 complex coordinates and then if you give me a function here. A function here is said to be holomorphic at a point, if a function is define at a point it said to be holomorphic. If you take the inverse of this map and then follow it by that function you get a function from C 2. And then you know how to decide when a function defined on C 2 or an open subset of C 2 is holomorphic and it is holomorphic if it is continuous and it is holomorphic in each variable. Therefore, you can make since of what a holomorphic function means on u naught. So, let me write that down.

So, if I write this as phi from C phi holomorphic at, let me say p naught in u naught if and only if I apply p naught inverse of phi naught of p. So, only phi naught inverse composed with phi is holomorphic at phi naught of p naught in C. So, this is a natural way in which you look at this composition this is a homeomorphism and this is a function. So, the composite function is just apply phi naught inverse then apply you are given function and then to decide that given function is holomorphic at a point. It is enough to check that by the definition this composition should be holomorphic at the image of that point.

So, this is how you can give a so, called complex structure mu naught. So, you can talk about holomorphic functions on mu naught and in particular. So, mu naught into any other space which has such a complex structure now the point is therefore. So, we get. So, in the same way we get 3 coordinate. So, 3 coordinate charts in 2 complex variables is each in 2 complex variables. So, this is one is u naught comma phi naught the second one will be u 1 comma phi 1 and the third will be u 2 comma phi 2.

So, in exactly the way I have define phi naught, I can also define phi 1 on u 1 and phi 2 on u 2 in a very natural way and in fact, how do I define it you know? So, for example, where phi 1 is define from u 1 to C by you know if we take if you give me a point of u 1 it lambda any point in projective space is homogeneous coordinate like this it has homogeneous coordinate like this and u 1 is a place where mu does not vanish. So, I can divide by mu and I will simply send it to lambda by mu comma nu by mu.

So, this is what phi 1 is and similarly how do you define phi 2 will be from u 2 to C. It will be the homomorphism which is given by sending lambda mu nu 2 well now it is u 2. So, nu is not zero. So, I can divide throughout by nu and I will simply get lambda by nu mu by nu. So, we get 3 coordinate charts and now notice, that P 2 is actually union of all these 3 open sets it is U naught union u 1 union U 2, because it give me any point on P 2 it has if you write it in homogeneous coordinate has lambda colon mu colon nu then you know that all lambda mu nu cannot be 0.

So, at least one is not zero; that means, that point is going to lying either U naught or u 1 or U 2 it lying U naught if the first homogeneous coordinate is not zero and the U 1 and U 2 respectively the second and third coordinate homogeneous coordinates are not zero. So, P 2 is covered by these 3 open sets each of which looks like C 2 and on each of this there is a chart. And now what I am going to tell you is that on the intersection these charts they are compatible.

So, in particular if we have function which is define on U naught intersection u 1 deciding whether it is holomorphic using the chart phi naught or using the chart phi 1 will not create any problems. So, this is a compatibility that allows you to define when function is holomorphic in a neighborhood which is contain in a intersection of these 2 open sets. So, what I am going to do is now just try to explain, how that compatibility comes? So, these 3 charts or compatible why are the compatible?

So, for example, let me look at phi the chart u naught comma phi naught and u naught comma phi naught and u 1 comma phi 1 and show how they are compatible. So, if I take u naught intersection u 1 then, this is contained in u naught and u naught to phi naught is your homeomorphism which is a chart into C 2 and what it does is that it sends a map it is send a point with homogeneous coordinate lambda mu nu in nu naught. So, it is a mu naught. So, lambda is not zero. So, I am simply going to send it to mu by lambda nu by lambda.

On the other hand; I am going to get a homeomorphism of this with phi naught of u naught intersection u 1, which I if you want I call this as v 0 one and on the other hand u naught intersection u 1 is also sitting inside u 1 as an open set and you have from here the map phi sub one which goes into c 2 and how does it go I simply send. So, I lambda mu nu if it is also in u 1 then mu is not zero. So, I can divided by mu and I am simply going to send it to well. I divided by mu I get lambda by mu nu by mu this is what I am going to send it to. So, and therefore, I am going to get this map. So, you see this is contained inside this and here I have phi 1 of u naught intersection u 1 which I will call as v one 0 which is contained inside this diagram commutes. So, the transition function is simply. So, the transition function is well.

So, if I start with beta comma gamma. I will first go to 1 comma beta comma gamma and that will further go to under this, I will have to divide by beta. So, it will go to 1 by beta comma gamma by beta and you must understand that; since notice that beta is nonzero because this is suppose to be u 1 as well it suppose to be here. So, beta is nonzero. So, then the transition function is just beta comma going gamma going to one by beta comma gamma by beta and that is a holomorphic function because in the first variables it is 1 by beta and beta is not zero. Therefore, it is holomorphic in the second variable, it is ratio of 2 coordinates. So, essentially this transition function is holomorphic. It is obviously continuous an it is holomorphic in each variable: therefore, it is holomorphic.

So, this tells you that charts on u naught and u 1 are compatible. Now you can do this with any other pair of charts and check they are all compatible and therefore, the moral of story is that you get Riemann surface. I mean you get complex manifolds 2 so, called complex surface structure on the projective space.

So, before I continue let me say something let me write in some in the homogeneous coordinates in using the rotation of homogeneous coordinates. So, what I am doing is. So, here you see this will correspond to what this will go to or what this will come from uniquely? Will be just y by x, z by x and what this will go to will be I divide by y? So, I will get x by y. z by y. So, essentially we say that the transition function are just given by you know one set of ratios of the homogeneous coordinates going to some other set of ratios of homogeneous coordinates the ratio taken by the same homogeneous coordinate and that is why the transition functions are holomorphic. So, P 2 C gets the atlas from by these 2 formed by these 3 charts and becomes what is called a compact complex 2 dimensional manifold and the 2 dimensional a complex 2 dimensional manifold is always called a surface.

So, a P 2 C projective 2 space becomes a compact complex surface. So, in this regard let me make a small clarification. So, you see when we talk about Riemann surface. The Riemann surface is basically a surface over the reals, but if you think of it as the complex manifold, it is complex one dimensional. So, as a complex manifold it is like a curve it is one dimensional. And so, Riemann surface the word surface in as an adjective in Riemann surface is with respect to the reals and here you have a compact complex when you have a complex 2 dimensional manifold it is called a surface over the complex numbers. But of course, if you think of it is a real manifold then it will be 4 dimensional. So, that is it we P 2 C becomes a compact complex 2 dimensional manifold now what I have to do next is.

So, the moral of the story is that I can do complex analysis on P 2 C because on any open subset of p 2 c, I have charts that tell me clearly whether a given function is holomorphic or not. And, I can do my complex analysis on this now. I told you that the purpose of this whole discussion was to tell you that this affine elliptic algebraic curve associated to tau can be you know compact if by adding a single point and the compactification is will live in the projective space. In fact, for that what I will have to do is that I will have to basically look at projective space. And I will have to tell you when a polynomial in 3 variables if you can take the 0 locus of that in projective space that naturally becomes a Riemann surface.