An Introduction to Riemann Surfaces and Algebraic Curves:
Complex 1-dimensional Tori and Elliptic Curves
Prof. Thriuvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 44
The Natural Riemann Surface Structure on an Algebraic Affine Nonsingular Plane Curve

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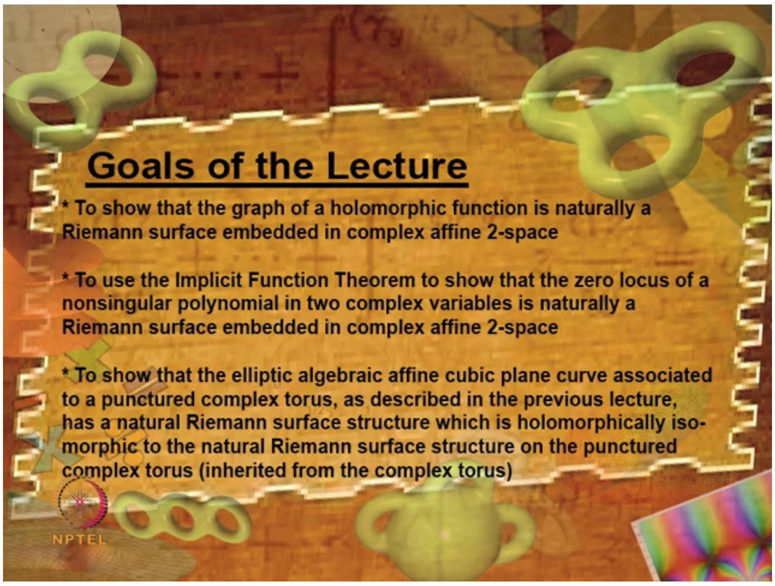

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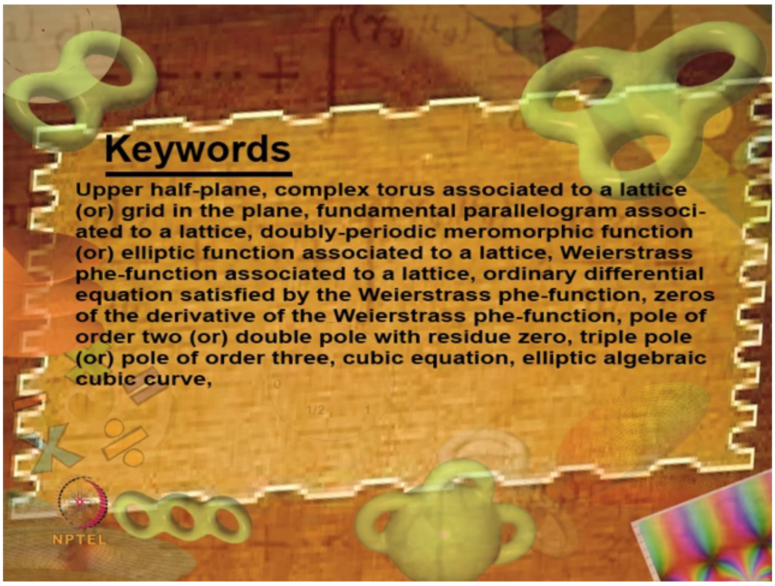

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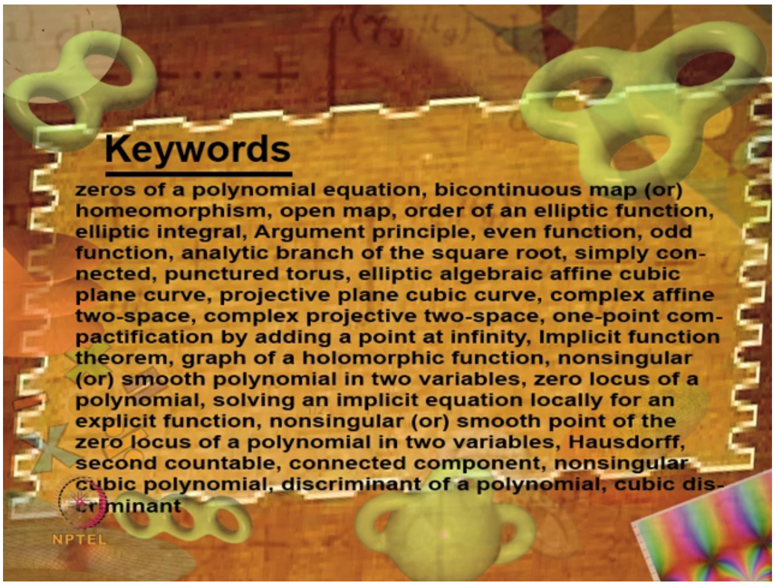

All right. So, let me recall, we are trying to show that every complex torus is actually an algebraic elliptical curve. So, let me continue with what I was doing in the previous lecture.

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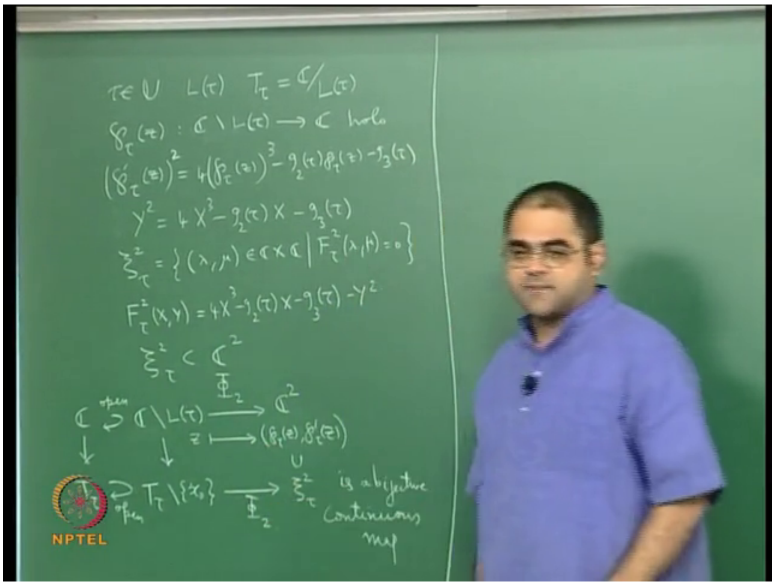

So, if you recall we started with tau in the upper half plane and the then of course, we had the lattice L of tau and we also have the torus T sub tau, which is just the complex plane modulo this lattice and associated with tau is the Weierstrass P function P tau of z and this P function is defined in the complement of the lattice or the complex plane and it is a holomorphic function and at the point of the lattice it has poles. These are double pole at each of these points of order it is pole of order 2 and with some of residue 0. And phi tau satisfies differential equation, first order differential equation; P prime tau of z whole squared is equal to 4 times P tau of z cube minus 3 minus g 2 tau or P tau of z minus g 3 tau. In fact, what we did was in the last lecture we notice that this can be treated as you can look at this as a polynomial in 2 variables.

So, you look at the polynomial Y squared is equal to 4 X cube minus g 2 tau X minus g 3 tau and then what you do is you look at the locus of zeros of the polynomial in xi 2. So, we look at this locus xi 2 sub tau, this is a set of all lambda comma mu in xi 2, C cross C such that lambda mu satisfy this equation, this polynomial equation when I plug in X lambda and Y mu.

So, well, what I do is, I will just define F 2 tau X, Y to be this polynomial, namely 4 X cube minus g 2 tau, X minus g 3 tau minus Y squared and then if lambda mu satisfies this polynomial with lambda substitute for X and mu substitute for Y if and only if lambda mu is 0 of this polynomial. So, i will just write this as F 2 tau of lambda comma mu is 0. So, this xi 2 tau is called the affine elliptic algebraic curve associated to tau. So, it lives inside the xi 2 which C cross C and what we did yesterday was we proved that you can define a map phi sub 2.

So, let me make that statement more precisely. So, on the complex plane minus the lattice points from there to C 2 there is a you can define following map z going to P tau of z comma P prime sub tau u of z and I call this map as P sub 2 and this map is of course, invariant under L tau, because P tau and P prime tau are invariant under L tau. They are w periodic function, elliptic functions and therefore, this map goes down to map of the torus minus the point x naught. The point x naught being the image of the lattice, all the point of lattice go to single point because they form a single orbit, they have single equivalence class and I call that point as x naught. And this map is just restriction of the natural projection from the complex plane to the torus and of course, this is an open set and this also an open set, here.

I have just deleted a point and so, what happens is that inside the C 2 we have xi 2 tau seating inside this affine algebraic elliptic curve, is seating inside C 2 and by definition the map goes into C 2 because the these 2 P and P prime satisfies this polynomial due to this differential equation. Therefore, you get map which also by abusive notation I call it as phi sub 2 and what we proved yesterday was that phi sub 2 is a bijective continuous map. So, let me write that here is a bijective continuous map.

Now, what I am planning to do today is, I want to tell you more about this map. So, I want to explain how there is a natural structure of Riemann surface on xi on this E 2 tau and I will also explain how this map becomes when E 2 tau is given that natural Riemann surface structure. I will explain why this map becomes a bi holomorphic map. So, therefore, the aim of the lecture will be to tell you that there is natural Riemann surface structure on this such that this map phi 2 is actually a bi holomorphic map, holomorphic isomorphism.

So, indeed the, upshot of the whole story is that this differential equation, that first order degree 3 differential equation that P satisfies gives rise to an algebraic equation a cubic equation and that cubic equation defines its 0 locus in C 2 defines an elliptic algebraic curve and this elliptic algebraic curve is naturally a Riemann surface and this identification of the torus minus this point, this open subset of the torus, with that fine elliptic curve is actually a holomorphic isomerism. So, that is what I am going to do in this lecture and let me also tell you what I want to do later on. Later on, I will explain how this affine elliptic curve can be compactified by adding a single point at infinity, so called point at infinity as a discalled algebraic geometry.

And then you can send x naught to that point and this can be done in such a way so that even after adding that extra point that one point compactification that also becomes a naturally Riemann surface and this map then the extended map then become isomorphism and then we call the corresponding curve as the projective algebraic elliptic curve associated to the tau, to that torus. The fact is that curve will actually live in projective 2 space so that, will require me to explain to you what projective 2 spaces. So, anyway, let's get along with what we need to do in this lecture namely to show that xi 2 sub tau E 2 sub tau is naturally Riemann surface.

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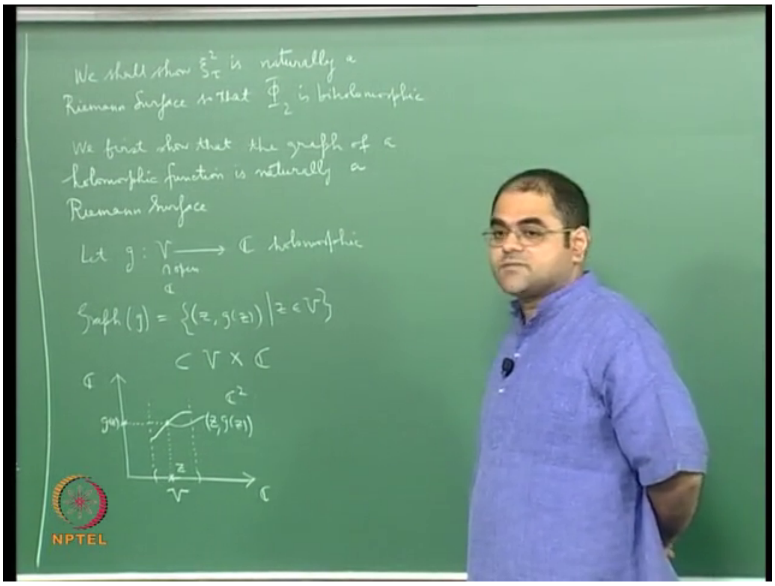

So, we shall show E 2 sub tau is naturally Riemann surface. So, that phi sub 2 is bi holomorphic. So, the point is that somehow more generally the question is that if you give me algebraic equation like this in 2 variables then I can look at set of zeros of that equation. That will be the set of zeros will be a sub set of C 2 and the question is under what circumstances can I naturally make it into Riemann surface. So, this will be done using the implicit function theorem for complex variables along with the fact that the graph of a holomorphic function is automatically a Riemann surface. So, that is the starting point.

So, what we will do is, we first show that the graph of a holomorphic function of a holomorphic function is naturally a Riemann surface. The point is that once you prove that the graph holomorphic function is naturally a Riemann surface then what you can do is that you know given a polynomial equation in 2 variables like this given a 2 variable polynomial you know using the implicit function theorem I can locally get in a explicit function of the second variable in terms of the first variable and then the fact is that this and in fact the implicit function theorem will tell you that locally this will look like a graph and locally since every graph is already a Riemann surface, you will get on the 0 locus of polynomial like this locally a Riemann surface structure and then you will have to show that the there is compatibility between the charts and once you do that you get a globally Riemann surface structure on this. But, this will not happen for any polynomial. The polynomial has to satisfy certain hypothesis of so called hypothesis of non singularity which are the hypothesis that I needed for the implicit function theorem which I will make clear.

So, start with g from V to C where holomorphic that is complex analytic and V is an open subset of the complex plane. So, you will look at the function g define on an open subset V of the complex plane and taking complex values. What is the graph of g? The graph of g is, well, it is the set of all points z comma g z where z belongs to V. This is the graph of this function. The graph is a subset of V cross C. So, pictorially though you know it is hard to visualize the graph of a complex value function complex variable because essentially it will be 4 real dimensions, but nevertheless we do it in a very suggestive kind of way.

We map, we put V here or rather we put C here and we put C here and think of this is C 2 and then we think of V, the open set, as patch here and the then you know we draw the graph. So, this is just like you draw the graph of a real valued function in first quadrant, if it has positive values. So, you know you end up drawing something like this and every point, given a point z here then I get this point which is this point is z comma g z and this, so this project down to z and that project down to g z.

So, this is a subset of you can see it is a subset of C cross C, all right. And the point is how do you make this graph into naturally into holomorphic I mean how do you make it naturally into Riemann surface. See, the fact is that you need homomorphism of this graph with an open substitute of the complex plane and that is obviously, given by the first projection.

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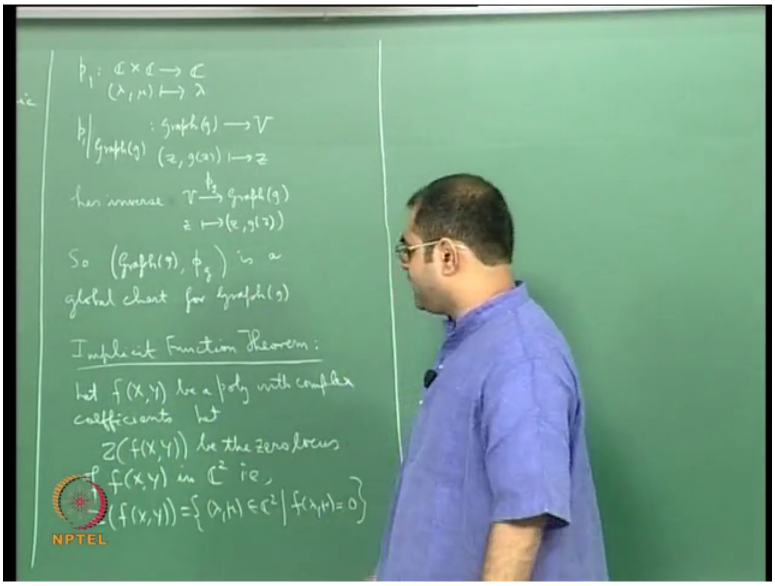


So, what you do is, you this is what you do. You define you consider P 2 I mean rather P 1 from C cross C to C that will just send any lambda comma mu to lambda this is the first projection and then what you do is you restrict this first projection to the graph. You restrict the first projection to the graph of g that will go from the graph of g to V. So, now, what you will have to notice is that you will have to notice that this is actually homeomorphism. In fact, this is you can see obviously, that you know see you are just sending z comma g z back to z and you know it's obviously, surjective because given any z, I have the corresponding point of the graph z comma g z and whose image will go back to z, so, it is surjective.

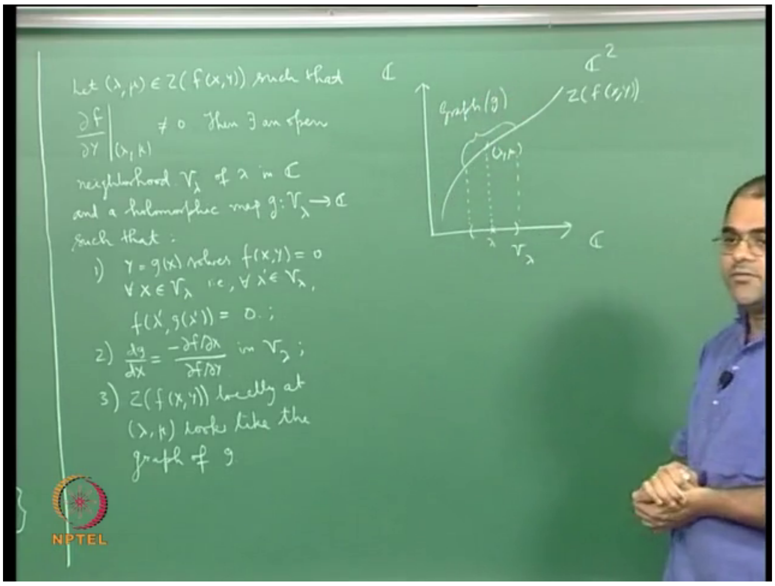
And, it is obviously, injective because if z 1 comma g z 1 and z 2 comma g z 2 go to the same z 1 equal to z 2 then if z 1 equal to z 2 then g z 1 equal to g z 2, trivially. Therefore, this is a bijective map and also this map has an inverse; namely, you just send has inverse from V to graph of g, this is a natural map. You just send z to z comma g z. This is the inverse map. So, let me call this inverse as phi sub g. So, if I take V comma phi sub g I should rather take yes. So, I should take. So, if I take graph of g comma phi sub g is a chart; is a global chart for graph g and in fact, since there is only one chart there is no compatibility to be checked. Therefore, this makes the graph of g into a Riemann surface. You just identify graph of g with V which is the domain of the function g all right.

So, this single chart will do converted into Riemann surface. And of course, you know there are other little points for a Riemann surface first of all topological space needs to be if you insist you want it to be connected then you had better assume that this V is connected subset even if not then you will have to you will always will be able to break down V into connected components and each component to be a will be converted into Riemann surface. Then the other thing is you also want to be housed of second controversial which is obvious because you see because of this homeomorphism graph g is homeomorphic to V and subset of the complex plane is of course, housed of in second countable.

So, if you stick by the definition that Riemann surface structure is a complex artless that is a system of coordinate chart which are compatible with each other on manifold which is housed of second countable and connected then all this conditions are satisfied, of course, connectedness you will have you assume for V. So, in any case this is the way in which you can make the graph of function into a Riemann surface in. And then now, so, once we do this for the graph of a holomorphic function then we can go down and do it for function of 2 variable. So, I will state the implicit function theorem. This is exactly the complex analog of the implicit function theorem that you would have in real variables and the philosophy of the proof is perhaps nearly the same.

So, let f of capital small f of X comma Y be a polynomial with complex coefficients and of course, when I write X comma Y you must always beware X and Y are complex variable, is not real variable. Probably it would have been better if I use z and w, but never mind. Please remember that capital X and capital Y are complex variables. We are already in the complex set up. So, our scalar is always complex numbers. Let Z of f comma Z of f of X comma Y be the 0 locus of f of X comma Y in C 2.

Look at the 0 locus. So, coming back to think of it is good I didn't use z and w because now I want Z to denote the set of 0. So that is what it means. It means Z of f of X comma Y is the set of all lambda comma mu in C 2 such that f of lambda comma mu is 0. This is the 0 locus and the point is the implicit function theorem says that you can solve the equation f of X comma Y equal to 0, can be solved locally as Y is equal to g of X locally; provided, there is the condition on the derivative just as in the real implicit function theorem.



So, let me see this. Let lambda comma mu be a point in the 0 locus such that if I differentiate partially f with respect to Y, second variable and evaluate it at lambda comma mu the resulting value is non-zero. So, basically if you, what is the implicit function theorem it says if you have a if we are trying to solve f of X comma Y equal to 0 which is an implicit relationship between X and Y then you can solve for it as Y is equal to g X at a point where the partial derivative of f with respect to Y does not vanish. This is the implicit function theorem. So, that is a condition I have put here alright.

So, what it says is, let me write that. Then, there exists an open neighborhood, U lambda or let me call it as V lambda of lambda in C and a holomorphic map g from V lambda to C such that, following things happen the first thing is g solves f of X comma Y for all X in V lambda. So, let me write that g Y is equal to g X solves f of X comma Y equal to 0 for all X in V lambda, that is, for every lambda prime in V lambda f of lambda prime comma g lambda prime is 0 this is the. So, this is the implicit equation is solved by an explicit equation in a neighborhood of the variable, in a neighborhood of a point where the partial derivative with respect to the explicit dependent variable is non-zero.

The second condition is of course, what will happen is that the g is holomorphic, so, what is its derivative? The derivative can be d g by d X is actually minus dou f by dou X divided by dou f by dou Y this is this is derivative in V lambda. So, this is the identity that you get the you take the total derivative of this if you take the total derivative of this and think treat Y as a function of X then you will automatically get this, but then to bring it to this form you must have that the derivative dou f by dou Y is non-zero. So, you must you must remember that small f is a polynomial. So, dou f by dou Y is also polynomial, it's always continuous. And if continuous function doesn't vanish at a point you can always choose a neighborhood where it's never going to vanish. So, that is what allows me to divide by dou f by dou Y.

And, the third thing is most important z, the 0 set of f of X comma Y locally at the point lambda comma mu looks like the graph of g. So, in other words, the first point is for every lambda prime V lambda we have f of lambda prime comma g lambda prime is 0, that is one thing; the other thing is because lambda if you take a point lambda prime comma mu prime in the 0 set of f X, Y then it must happen that mu prime is actually g of lambda prime. That is what it says. So, in other words if I take; let me draw a diagram to explain this more graphically.

So, you see you have this is C 2, C cross C and what is happening is that you have this 0 setup f of X comma Y and then you are choosing a point lambda comma mu with the property that the partial derivative of small f with respect to Y at that point does not vanish; then what happens is that you are able to, then you project down here, you get this point lambda and then you are able to find a V lambda; V lambda is a open neighborhood of lambda and on V lambda there is a holomorphic map, if you draw the graph of this map, namely, you take the set of all lambda prime comma g lambda prime here then that is exactly this graph. So, you know if I just extend it like this and if I extend it like this then this guy is exactly graph of g.

So, in particular it means that you know if you take any lambda prime comma mu prime such that lambda prime is in V lambda the mu prime has to be g of lambda prime. So, it is locally graph of that function and if you grant. So, I am not going to giving a proof of this because this is quite standard you can look it up and, but the point is that I want to say that once you have the implicit function theorem, the immediate corollary is that if you give me if you look at the 0 set of a polynomial then locally already it is a graph of a holomorphic function. Therefore, locally already a Riemann surface because I already explain to you how very easily the graph of a holomorphic function is a Riemann surface. So, what is the upshot of this result is the movement of polynomial satisfies this condition. Then, in a neighborhood of that point it is a Riemann surface. You can make neighborhood of that point on the 0 locus Riemann surface.

The question is what happens if it does not satisfy this condition? Then it might satisfy the same condition, similar condition with respect to the first variable. And then you can use projection out of second variable and you know instead of writing Y as a function of X, you will able to write X as a function of Y, if dou f by dou X at a given point is non-zero. So, all these put together the point is that if you have a polynomial which is so called nonsingular namely which satisfy the condition that one of the partial derivatives of the polynomial never vanishes at each point, then it is very clear that the graph, the set of 0 of that polynomial is naturally locally a Riemann surface. The only thing that one has to check there is globally Riemann surface is to check that all these local Riemann surface structure agree, you will have to just check that all these charts that you got by the graph construction there all compatible once you check that then it is very clear that the 0 set will be Riemann surface.

So, and of course, this 0 set being it is a close subset of C 2, so, its automatically housed off and second countable. Any sub space of housed off space and house and any subspace of a second countable space is a second countable. So, those conditions are automatic of course, you might want it to be connected and the fact is that you will. So, if you put this non singularity condition then you are all then automatically mean it is connected. In fact, more generally you could have assume or rather to be on the safer side one sufficient condition you can put on f then it is irreducible as a polynomial. So, that is a technicality that one need not worry about at this stage, but at the worst if you take a polynomial who is first partial derivative simultaneously do not vanish at any given point then it becomes a Riemann surface.

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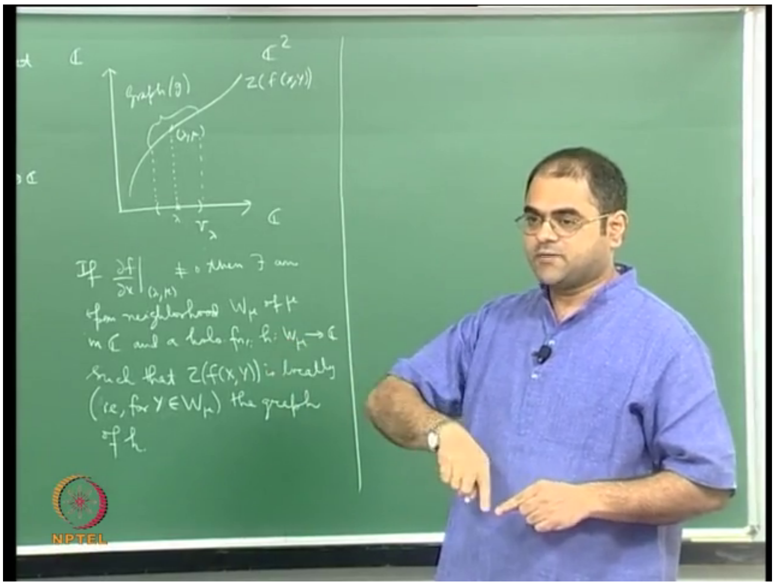

So, let me write the rest of it. So, if dou f by dou X at lambda comma mu is not equal to 0 then there exists an open neighborhood, W mu of mu in C and a holomorphic function h from W mu to C such that Z f of f X comma Y is locally for that is for X in for Y in w mu the graph of h. So, this is condition on the other variable.

So, you know you have to rewrite. I am not writing everything down I am just stating the case when the first partial with respect to the first variable is non-zero, that was the case when the first partial with respect to the second variable is not 0. Then, you are then you can write the second variable as the function of first variable in this case you will be able to write the first function, the first variable X as a function of second variable. So, X will be h of Y. So, what is the upshot of this? Upshot this is that the if you give me a polynomial which satisfy one of these 2 conditions at each point of it 0 locus, then there are Riemann surface structures locally which come out of graph construction.

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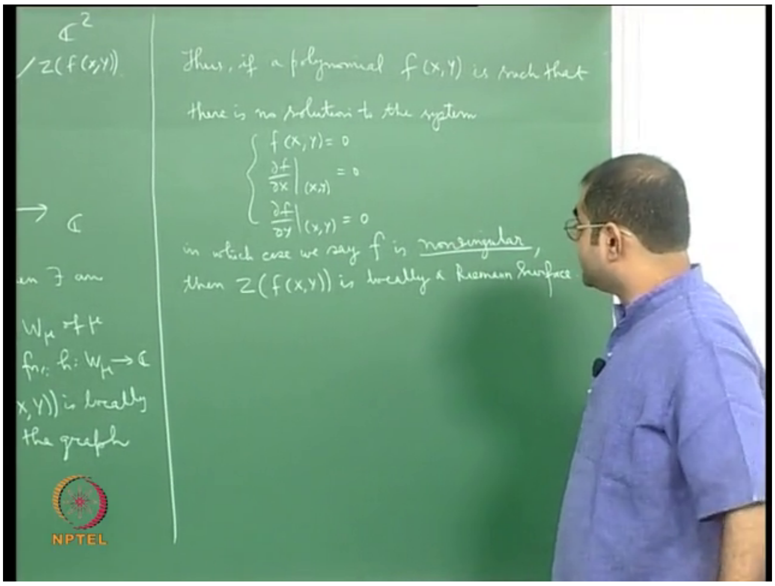

So, let me write that now. Thus, if a polynomial f X Y is such that there is no point, there is no solution to the system f of X comma Y equal to 0, dou f by dou X at X comma Y is equal to 0, dou f by dou Y at X comma Y is equal to 0, in which case we say f is non singular, then the 0 set of f of X comma Y is locally a Riemann surface.

So, the upshot of the thing is that the 0 set of this polynomial which is non singular polynomial; non singularity means that you should not able to find the point 0 locus at which both first partial derivatives vanish, then f is such a polynomial is called non singular and then the 0 locus is locally Riemann surface. Now, I only need to verify that. To verify a Riemann surface I have to only tell you that all this Riemann surface structure locally they all glue together, I mean they all agree; namely, I just what do I mean by saying it locally Riemann surface locally I am getting charts which are given by the projection of because locally these are graphs of holomorphic functions and then I take projections on to the independent variable then I get the charts.

So, to show that all these together make this into a Riemann surface I will have to only check that they charts are compatible. So, if I do that then for a non singular polynomial, I will have that it is automatically a Riemann surface. Then you know what I am going to do, I am going to check that this particular polynomial that I having this cubic satisfy and this polynomial here. I am going to show that this satisfies a non singularly hypothesis and therefore, that will tell you that this E 2 tau is in fact, a Riemann surface alright.

So, and we will also the way in which we have done it. We will also see this map is naturally holomorphic, because what will actually happen is, that locally the coordinate chart I just given by projection and the first projection is going to be P which is holomorphic, the second projection is going to be P prime which is holomorphic; therefore, this map becomes holomorphic, is a holomorphic bijective map. So, it is a holomorphic isomorphism and we are done.

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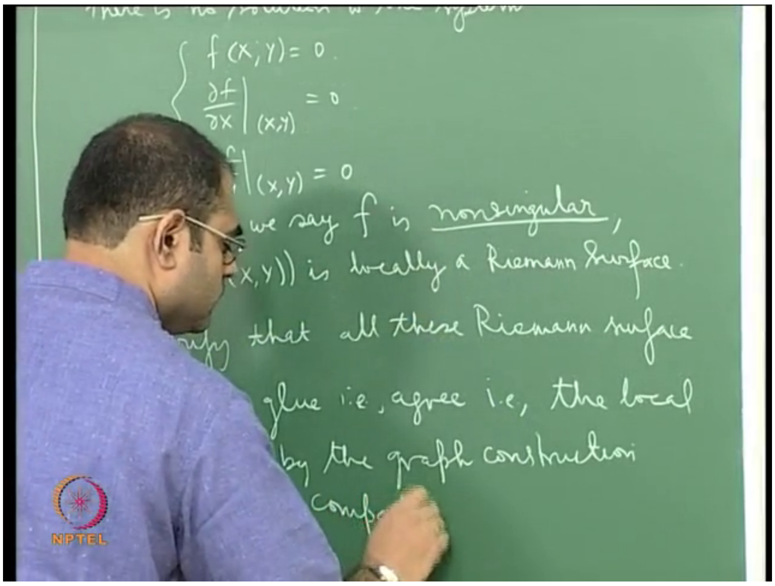

So, let me go ahead and try to show you why all this Riemann surface structure locally they all give same Riemann surface structure on globally. So, we verify that all these Riemann surface of structures glue, that is, agree, that is, the charts given by the graph construction the local charts given by the graph construction are mutually compatible. So, basically there are 3 cases; the first case is when the local Riemann surface structures comes from both projections are on the first variable.

Basically, I will have to take 2 local Riemann surface structures and show that at the intersection the charts the transition function are holomorphic that is what I have to show. So, it might happen that the Riemann surface structure on these 2 overlapping pieces open pieces of the 0 locus they both come from first projection or they both come from the second projection or one may come from the first projection and the other may come this from the second projection.

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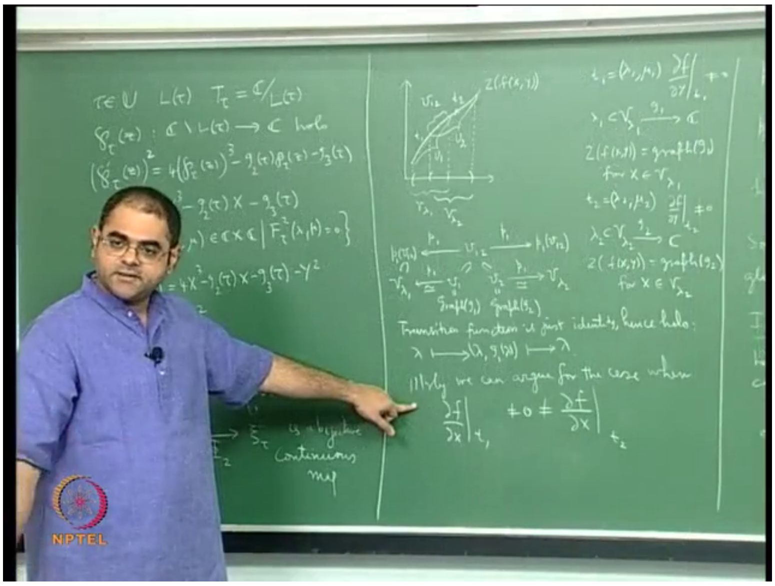

So, there are 3 cases. So, let me draw diagram for each of this. So, you know, this is first case namely. So, here this is my 0 locus and the well basically I have point here let me call this is P 1, oh god, no. Let me call this as something let me call this is as t 1, let me call this is as t 2 and t 1 is lambda 1 comma mu 1 and dou f by dou Y at t 1 not 0. So, what happens is that you find a neighborhood lambda of lambda 1 in C which is given by V sub lambda 1 and you find a function g 1, holomorphic function from g 1 to C such that the graph, Z of f X comma Y is equal to graph of g 1 for X in V sub lambda 1. So, in this case, basically I draw a diagram it is going to look like this.

So, this is my V 1, V 3 sub lambda 1 and then similarly let me assume that, this portion that I have drawn here, from here to here the Z of X comma Y is the graph of g 1 and then you also have the same conditions for t 2. So, dou f, t 2 is again lambda 2 comma mu 2 and dou f by dou Y at t 2 is non-zero; then you get and open neighborhood V sub lambda 2 of lambda 2 and a function g 2, holomorphic function g 2, defined on that with values in C such that Z of f of X comma Y is actually graph of g 2 for X in V lambda 2.

So, what happens is that, this V lambda 2, so this is V lambda 1 and well V lambda 2 could be something like this. So, let me not write here. So, here is V lambda 2. This lambda 2, this projection is lambda 2. So, from here to here this is V sub lambda 2 alright and so what are the, how do I check that these 2 charts are compatible. So, you see basically I have this open set which is the image of V lambda 1 and then I have this open set which is the image of the holomorphic of V lambda 2 under the holomorphic map and this is the intersection. So, this intersection I can call this intersecting set as W 12 rather let me call it as U 12 and what is the situation is I have to look at transaction function. So, I have from U 12 on the one hand I have. So, there something that I have to correct here, see the when I write a chart I the first member is an open set on the Riemann topological space on which I want to get a complex coordinate, the second one should be defined on that. So, actually this should not be phi sub g, this should actually be P 1. So, please correct this should be P 1. So, that is a global chart.

So, you see, you know, I have P 1, that is one P 1 which goes from U 12 is thought of a sitting inside well you know if I call this whole thing as U 1 and if I call this whole open set as U 2. So, this is if I think of it sitting inside U 1, then my and U 1 is actually graph of g 1 and this is from U 1, it will go to V sub lambda 1 this is this is just P 1 and this is this is a holomorphic isomerism, homeomorphism and so, I will go into P 1 of U 12. So, this is which sits inside this. So, it will be this region here I mean it will be this open set and you also have the coordinate coming from U 2 via g 2. So, you have also U 2 which is graph of g 2 and you have well, again it is P 1, so, this is P 1 which is again an isomorphism with V sub lambda 2.

So, this will go to t 1 of U 12 again. Now, you know if I follow this up, the transition function is the composition of these 2. If I do that I actually I get the identity map see because if I start with a lambda; transition function, what is the transition function? The transition function will be just lambda going to from here to here you will go to lambda comma g 1 of lambda, but then you see and then if you project it again the first variable I simply get lambda. So, you can see this is transmission function is just identity map. It is just the identity map on this intersection and the fact these 2 open set V lambda 1, V lambda 2 intersect is because U 1 and U 2 intersection and I am actually trying to check the compatibility on U 12 which is intersection of U 1 and U 2.

So, the transition function is just identity. It is just identity map, hence holomorphic. So, this settles the case that the charts, the local charts that you get compatibility when both are graphs of function of first variable. Yes, you can I would not be writing it down, but if I do it with the assumption that both partial derivatives with respect to X are not vanishing, then I will have to take projection with respect to second variable. So, effectively the argument is you just take a mirror image of this. You take a mirror image about the diagonal line, that is kind of diagram you will get and essentially in that case also you will see that the transition function will continued to be identity, so, it is holomorphic.

So, the only case that I will have to worry about is when one of the partial derivative with respect to X and non vanishing and the other is with respect to Y is non vanishing and in that case you will see that it will be one of those functions whose graph is being considered. So, in that case also it will also be holomorphic. Let me just write that down. Similarly, we can argue for the case when dou f by dou X at lambda comma mu is non-zero and dou f by dou X at yeah at t 1; let me use t 1, t 2 and t 1 is not 0 and t 2 is not 0.

So, if similarly you can argue for other case; in which case also the transition function turns out to be identity. So, it is holomorphic. So, the only case that is left is that one of this is with respect to X and the other is with respect to Y which also pretty easy. I'll write it down.

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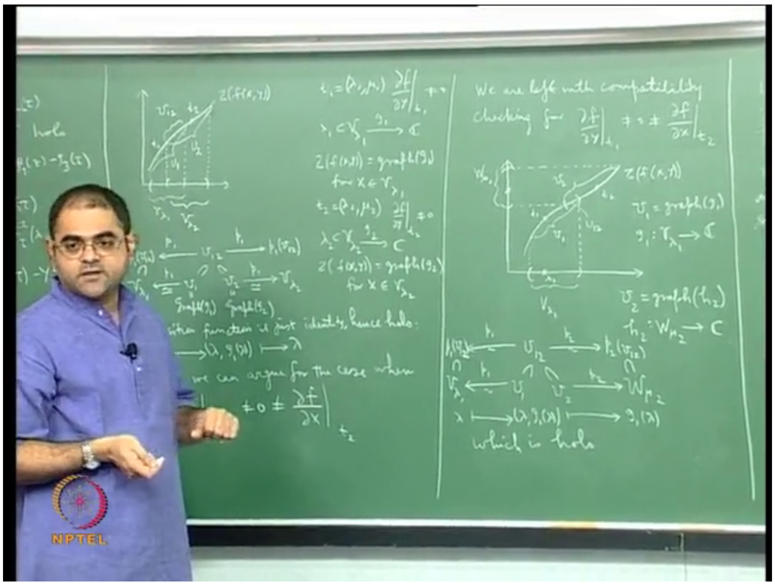

We are left with a compatibility checking for dou f by dou X, let me write dou f by dou Y at t 1 is not 0 and dou f by dou X at t 2 is not 0. So, this is the only case you have to check; that means, the charts are coming from different projection on different variables. So, that case if I draw the same kind of diagram what I will get you can see it pretty easy. So, you see this is my 0 set of f of X comma Y and so I have this. So, here is my t 1 and here is, this is my, I get this my V sub lambda 1 and then and this portion is you U 1 and U 1 is just the graph of g 1, while g 1 has been defined it is a holomorphic map from V lambda 1 to C and then you also have t 2 here, but with respect to t 2 it is only the first variable partial derivative which is non-zero.

So, you have basically you know something like this, I get open neighborhood of this projection of t 2 which is mu 2 and I get it W sub mu 2 which is open neighborhood of mu 2, this is of course, open neighborhood of lambda 2 and this portion is U 2 and U 2 is actually the graph of h 2, where h 2 is from W mu 2 to C. This is lambda 1. So, then you see if I look at this piece which is U12 and try to look at the transition function, it is pretty easy.

So, what you will get is that if I write the same kind of diagram as here, so, U 12 is sitting inside U 1 and U 1 is from and then you have first projection of U 1 into V sub lambda 1. This is holomorphic isomorphism and under this projection you are looking at t 1 of U 12 which is going to be a open sub set of V lambda 1 and this also of course, homeomorphic and then from U 12, you also have U 12 is also sitting inside U 2 and from U 2 it is P 2, it is a second projection because the coordinate chart is in this direction.

So, it is t 2 and this goes to W sub mu 2 and well, this second projection will give me P this was P 2. So, it is P 2 of U 12 which is an open sub set of W mu 2 and now if I look at the transition function, the transition function is well if I start with a lambda it will go to lambda comma g 1 of lambda and then if I take the second projection it will go on to g 1 of lambda, the composite is therefore, lambda going to g 1 lambda which is holomorphic by definition therefore, transition function is holomorphic. So, you are done with this case also. So, which is holomorphic. So, this implies that the moment you take a polynomial f X by which is non singular then the 0 set of that polynomial is automatically a Riemann surface.

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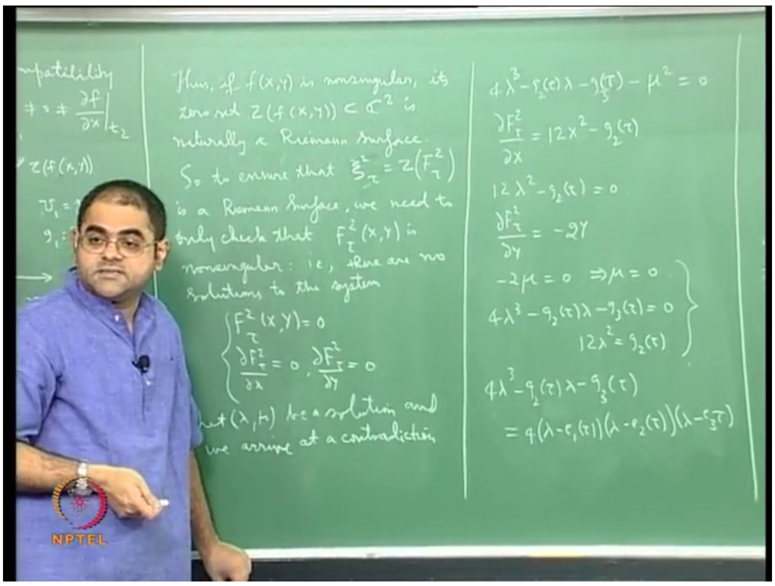

So, let me write that down thus if f of X comma Y is non singular its 0 set is Z f X Y the sub set of C 2 is naturally a Riemann surface. Now, I will verify I just want to say that this E 2 tau I want to say is actually a Riemann surface. The only thing I have to verify that this polynomial capital F 2 sub tau is non singular.

So, to ensure that F 2 tau sorry E 2 tau which is just the 0 set of F 2 tau is a Riemann surface, we need to only check that capital F 2 sub tau is non singular. So, that is there are no common, there are no solutions to the system F 2 tau of X comma Y is equal to 0, dou F 2 tau by dou X is equal to 0, dou F 2 tau by dou Y is equal to 0. So, let us write this out, it is pretty easy and you will see that we will be using again. The fact that E 1, E 2, E 3 are distinct E 1 tau, E 2 tau and E 3 tau which are the zeroes of the derivative to the P function you know they are the 3 distinct zeros of P function and their distinctness is what is going to give you the non singularity in this case, you will see we will see that.

So, let's do the computation; what will get is basically, I will get, let lambda comma mu be a solution and we let us assume there is a solution get a contradiction. Let lambda comma mu be a solution and we arrive at a contradiction. So, what it will mean is lambda comma mu satisfy that equations, so, I will get 4 lambda cube minus g 2 tau, lambda minus g 3 tau minus mu square 0, this is the first equation. The second equation is I differentiate dou F 2 tau by dou X and this is by differentiate it with respect to X, I am going to get 12 X squared minus g 2 tau, that is all.

Then if, so this equal to 0 at lambda comma mu will tell me that 12 lambda squared minus g 2 tau is 0 and then dou F 2 tau by dou Y is 0 for that condition I differentiate partial with respect to Y and I end up with my minus 2 Y. So, the condition I will get is minus 2 mu is 0 alright and this already tells you that mu is 0. The first thing I want to tell you is that you see if; let me go back to the first equation put mu equal to 0, I will get 4 lambda cube minus g 2 tau lambda minus g 3 tau is 0. So, I will get this and then I will add this equation also 12 lambda square is equal to g 2 tau. So, I will get these 3 equations and I am just saying that these 3 equations will give us the contradiction.

See, first thing I want to tell is that you know the. So, as I told you the point come from on the fact that the when you factorize the right side of the equation into 3 linear factors the zeros are E 1 tau E 2 tau E 3 tau. They are the values of P tau at half tau by 2 and 1 plus tau by 2, the 3 fundamental zeros of derivatives of the P function. They are distinctive we have already seen that. So, if you use that factorization you will, actually what happens is that the 4 lambda cube minus g 2 tau lambda minus g 3 tau, this factorizes as 4 times lambda minus E 1 tau into lambda minus E 2 tau into lambda minus E 3 tau and then if you now calculate so called discriminant of this cubic, discriminant of a polynomial equation of one variable is just the square of the difference of the distinct of it is roots. Square of the difference of it is roots taken in some order and it is an important tool in algebra because you can check that the polynomial has distinct roots namely there it is so called separable by checking that discriminant is non-zero.

So, you if you calculate the discriminant in this case, this is the little bit of algebra which you should be able to do, then you will find that it is not a difficult exercise. You can do it.

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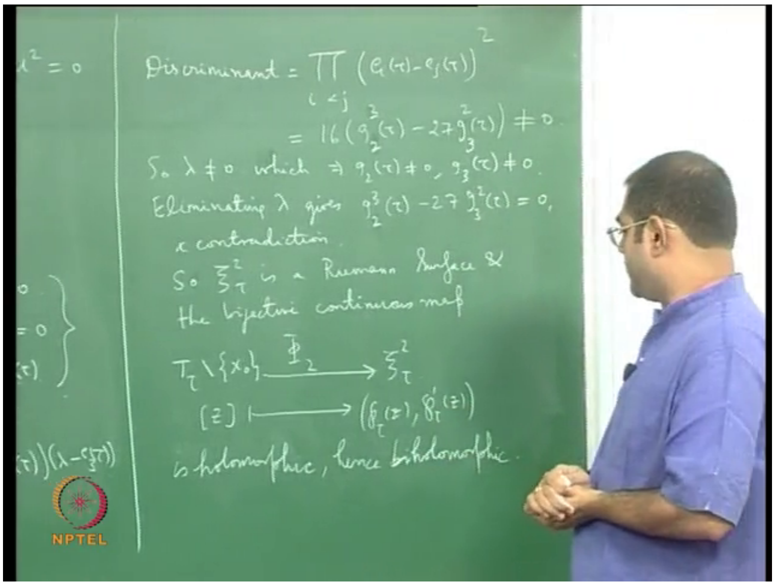

Then you will find that, if I take discriminant; discriminant is just product over i less than j, e i tau minus e j tau, these are the square of the differences. And you can check that this discriminant you compute it, you can see that is discriminant when you compute it, it turns out to be let me write it tell you what it is, it is 16 times g 3 tau. I think squared minus, no cube minus 27 g 2 tau, the whole square. So, this is the discriminant and the discriminant is not equal to 0. So, the discriminant is not 0 because all the e i's are distinct. So, you see you know already mu is 0, if lambda is 0, what will tell you, it will tell you g 2 and g 3 are 0. If g 2 and g 3 are 0 then the discriminant is 0. That is the contradiction. So, the first thing you can tell, you can conclude from this is lambda is not 0. So, you see lambda is not equal to 0.

So, once lambda is not 0 it is also clear that g 2 is not 0 and g 3 is also not 0. So, which implies that g 2 tau is not 0, g 3 tau is not 0; these 2 are not 0 and then what I can do is literally, I can just eliminate lambda from this equations and you know I have 12 lambda squared is g 2 tau and I have you know, I eliminate lambda from these 2 and believe it or not you eliminate lambda from these 2, what you will get is, g 3 cube tau minus 27 g 2 square tau equal to 0. It will give, if you eliminate lambda you will exactly get determinant is 0 which is not which is not true that that is contradiction.

So, eliminating, basically I take a square root of lambda here and equated to a cube root of lambda from here and then I raise both sides to the power 6 and that's it. So, eliminating lambda gives 27, I mean you will get g 3 cube tau, wait a minute I think it is g 2 cube and g 3 square, I should be careful. So, this is g 2 cube and this is g 3 square. Let me check that, yeah g 2 cubed and this is g 3 squared please correct it. So, eliminating will give you g 3 square minus 27 g 2 cube minus 27 g 3 square tau is 0, a contradiction. So, you see the it is the distinctness of the zeroes of the phi, the derivative of the phi function that actually gives you the fact that the corresponding polynomial that defines the affine algebraic elliptic curve is a non singular polynomial therefore, it become a Riemann surface.

So, the moral of the story is that this E 2 tau is elliptic curve is indeed a Riemann surface, naturally, by the construction that we have just seen using the graph construction the implicit function theorem. So, E 2 tau is a Riemann surface and the bijective continuous map that I wrote down namely phi 2 it is from the torus minus special point which is the image of lattice to the E 2 tau is basically this map is z going to, it is just z. So, I put box here to show that this is equivalence class, because the torus is set of equivalence class and this goes to P tau of z, P prime tau z, this map is actually holomorphic because if you project on the first variable it gives a holomorphic map P sub tau and if you project on second variable you get the holomorphic map P prime sub tau, is holomorphic. Hence, bi holomorphic.

So, you because essentially how do I check that this is holomorphic? The method is that I will have to take locally an open set which has a chart. So, locally it will be a graph and then for a graph the coordinate map is just projection onto either the first variable or the second variable. So, I will be essentially be locally getting t tau or P prime tau depending on whether I am projecting on the first variable or on the second variable; therefore, they are holomorphic. Therefore, this map is holomorphic map and that shows you that the punctuate torus is naturally isomorphic as Riemann surface to the affine elliptic algebraic curve, defined by the polynomial equation that comes out of the differential equation satisfied by the Weierstrass P function. So, I'll stop here. The next lecture, I am going to tell how you can extend this map from the torus to one point compactification of this affine elliptic curve which will live in projective 2 space. So, I stop here.