An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 43
Punctured Complex Tori are Elliptic Algebraic Affine Plane Cubic Curves in Complex 2-Space

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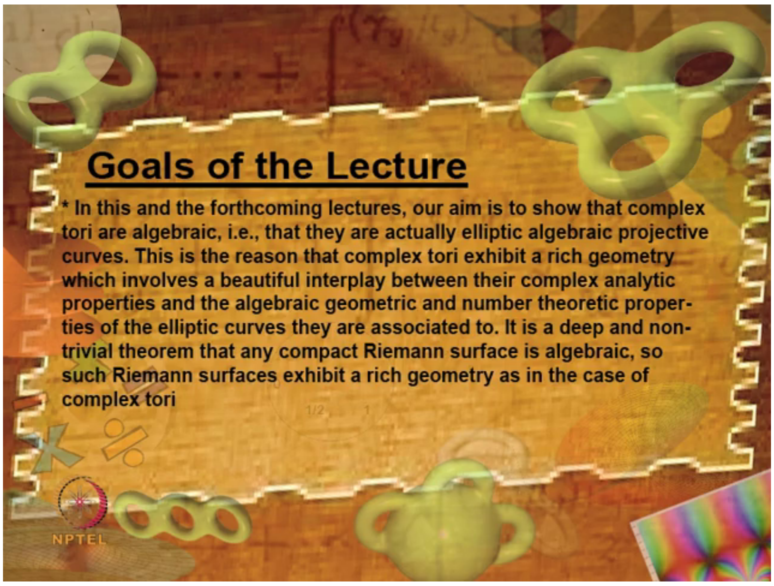

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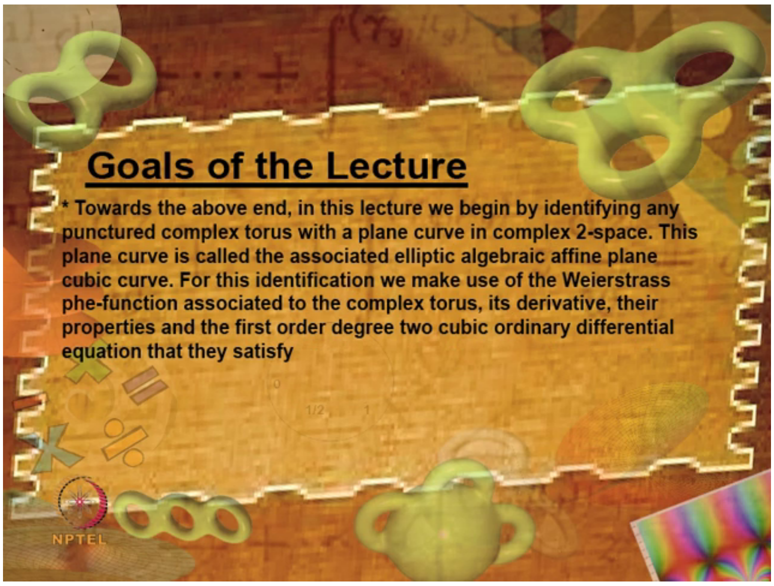

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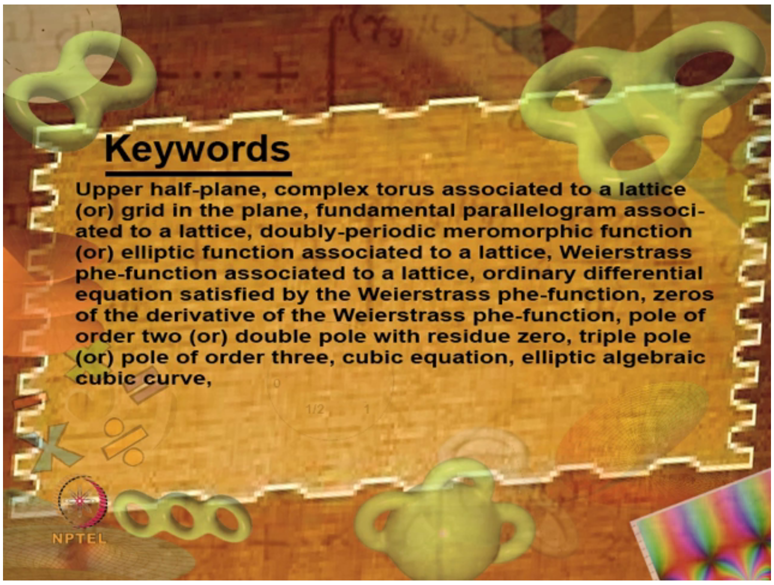

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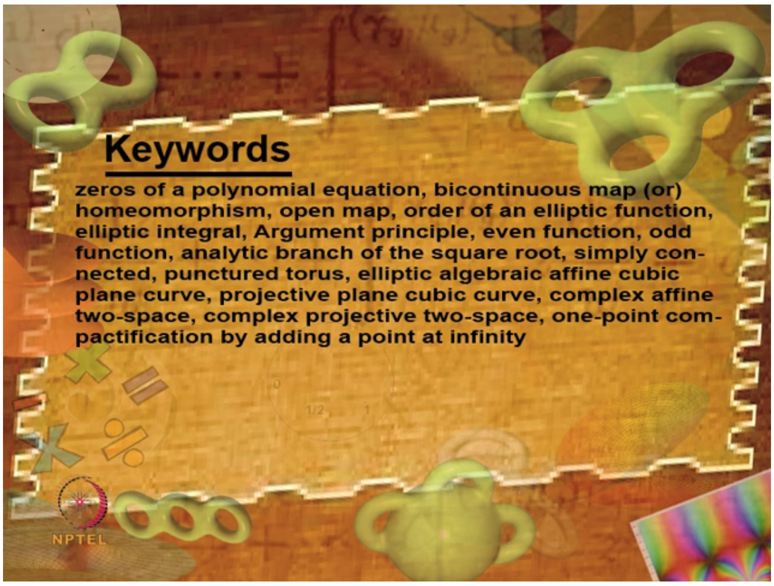

So, the purpose of this lecture and the following lectures is to show that every complex torus is essentially what is called an elliptic curve an elliptic algebraic curve. So, I will start with an element the upper half plane.

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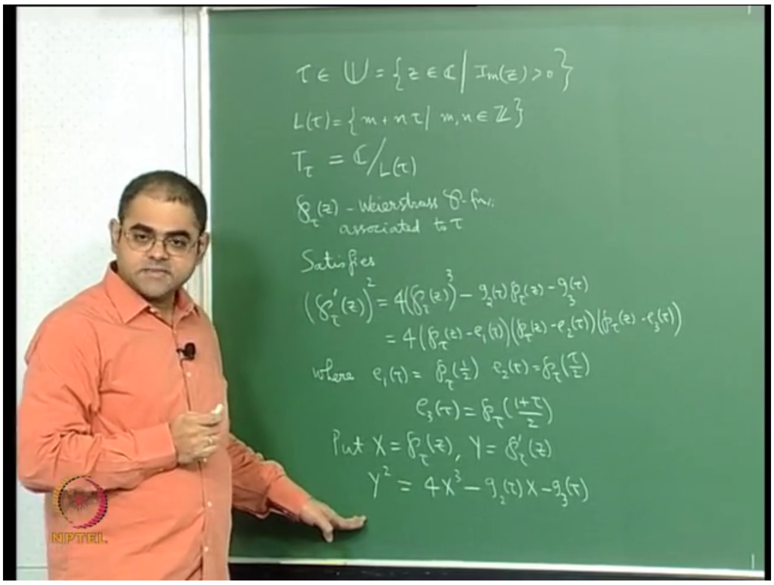

Namely; the set of all complex numbers with imaginary part positive and then you all know that there is there is a lattice associated with tau which is set of all integer linear combination of 1 and tau. This set of all m plus n tau m plus n tau, where m and n are integers, and then you know that there is a torus complex one dimensional torus associated with tau which is just the complex plane model of this lattice.

And here what you mean by this is divided d take the set of e equivalence class of complex numbers, where e equivalence is given by 2 complex numbers are set to b equivalent if they differed by any elemental lattice. We can also think of this lattice, elements of this lattice as translations by the corresponding elements. So, this this can be viewed also as a sub group of mobius transformations of the complex plane and this is just the set of orbits alter the action of this a sub group. So, this also the orbits space set of orbits. And then be you we have seen that that there is a weierstrass phi function associated to this this complex torus. So, it is phi t phi tau of z it is weierstrass phi function associated to tau.

And you know that this satisfies a differential equation. So, it is satisfies following differential equation namely, the derivative of the phi function squared is equal to 4 times the cube of the function minus, yes, it is it was g 2 g 2 times phi minus g 3 minus g 3 tau this is differential equation. And in fact, if you factorize the right side to linear factors you will get 3 linear factors which we write phi tau of phi tau z minus e 1 into so, phi tau z minus e 1 of tau into phi tau of z minus 2 of tau into phi tau of z minus e 3 of tau. Where of obviously, e 1 tau e 2 tau and e 3 tau are the are 0s of the derivative of s v function. And they are the only the distinct 0s in the in any fundamental parallelogram. And e 1 one of tau as you know this was defined by is the way value of the phi a function at half, e 2 of tau is the value of the phi function at tau by 2. And e 3 of tau is the value of the phi a function at 1 plus tau by 2. So, this is the this is the first order second degree differential equation that the s v function satisfies

And in fact, so, I had explained it one of the earlier lectures, and indicated that it is the algebraic nature of this equation, which is if you think of phi tau and phi prime tau is variables, then you are getting a cubic equation. And that should that indicates that hints that there is a there is something algebraic going on behind this. So, what we will do is we put x to be to be called the variable x to be phi tau of z. And then you will put we will put y is equal to a tau phi phi prime tau of z. We will set these 2 variables, and what we do is and if you realize this equation you get you get polynomial equation and algebraic equation. You get y squared is equal to 4 4 x cube minus g 2 tau x minus g 3 tau, we get this equation.

And so, what we can do is this is a polynomial in 2 variable x and y. And therefore, what we can do is you can look at ordered pairs in c 2 which is c cross c Cartesian product of c with itself which satisfy this equation. So, you are looking at this 0s of this polynomial in c 2.

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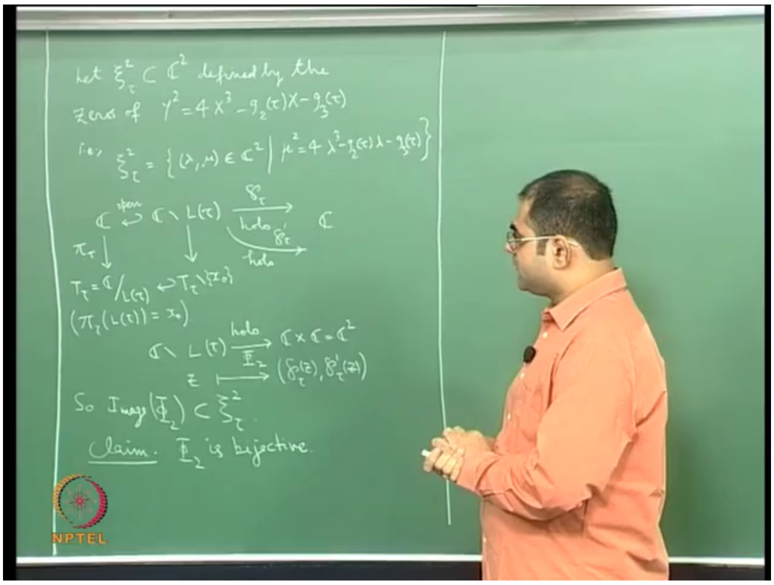

So, let me write that so let so, I will use this notation psi tau 2 I will explain this notation later, let psi tau 2 be the subset of c 2 defined by the 0s this 0s of the of this polynomial y squared is equal to 4 x cube minus g 2 tau into x minus g 3 tau. You look at all the so; that means, that that is psi 2 tau is a set of all lambda mu in c 2 such that lambda comma mu satisfies this equation when you put x equal to lambda y is equal to mu So, I was getting mu cube mu square it should be 4 lambda cube minus g 2 tau lambda minus g 3 tau.

So, in this connection please do not confuse this lambda with the partial elliptic model the partial elliptic modular function that we had defined in sometime ago. These lambda mu are just constant. So, this is the set of 0s of this polynomial phi 2. And now what we can do is that from the. So, you know this. So, so we have the complex plane, and the complex plane and then you have the complex plane minus the lattice, this is the open set that where you taken away all the lattice points. And then you have of course, you have this quotent of this by the lattice to give you the torus complex torus.

And of course, if you if you take this as an open subset here, this is an open subset. If you take the quotient what you will get is an open subset here that will be the torus minus the, let me let me call this as x naught. Where x naught is the image of whole of the whole lattice. Here you know that this map is invariant under the under translations by elements of l tau because after all this quotient map. So, all the points of the lattice they will go to the same point below on the on the torus, and I let me call that point as x naught. So, let me call this map as pi pi sub tau. Pi sub tau of the set l tau is actually x naught ok.

So now the point is that we know that the weierstrass phi function is basically a function which is meromorphic on the complex plane, with it has a double pole at each point of the of the lattice and with residue 0. And the therefore, from c minus from this open set you actually have the phi function as a this is actually holomorphic function. If I if I put phi tau phi tau this is a holomorphic function, this is a holomorphic function. And similarly, you also have you also have the same for the derivative of tau phi prime tau that is also holomorphic. Because you see phi prime tau had poles exactly at the at again at the lattice points in there were there were poles of order 3, divide poles of power 3.

And therefore, and these were the only similarities. Therefore, phi prime tau is also a holomorphic map from this to the complex plane. So, what we will do is that we put these 2 together, and try to and define a map into c 2. So, what we do is; so, let me write this once more. So, you have from complex phi in minus l tau to now I am going to define a map into c cross c which c 2. And the map is you sent z to the order pair phi tau z comma phi prime tau of z you send it to this order pair. And so, you have this map. And it is it is holomorphic in each variable.

Because in the first variable it is just the is map is z going to phi tau of z in, the second variable it is map z going to phi prime of z. And so, long as z varies outside lattice, this is a holomorphic function. So, this holomorphic map these holomorphic maps. And you know the point is that the will if you all this map as; so, let me call this map as phi phi sub phi sub 2. Let me call this map as phi sub 2 then it is very clear that the image of this map goes into this subset. Because if I call the first variable as lambda and second variable as lambda and second variable as I call this as lambda and that is mu, then because of the differential equation lambda mu will satisfy this condition. So, it is very clear that this map lance inside this; so phi 2 of so the image of phi 2.

So, image phi 2 lance inside site of inside this subset. Then and I want to say that I just want to say that this map, I want to say that this image is exactly that, and I want to say that this is this is bijective it is a in fact, it is bi continuous it is a homomorphism. So, I want to say that it is a homomorphism. So, to prove to prove that see the first thing, see first of all the it is a non-constant holomorphic map. And an non-constant holomorphic map is an open map. So, first foremost this is an open map, all right. And secondly, I show that I will I will show that this map is both injective as well as adjective.

So, the claim is phi 2 is bijective, claim is that this is a bijective map. It claims it is bijective map. So, what do I do for that? So, I will be gave improve injectivity and surjectivity. So, the first thing is so, in order to prove injectivity and surjectivity, we will have to recall certain properties of w periodic functions. Which we have already used long before in one of our lectures but I will have to recall those things.

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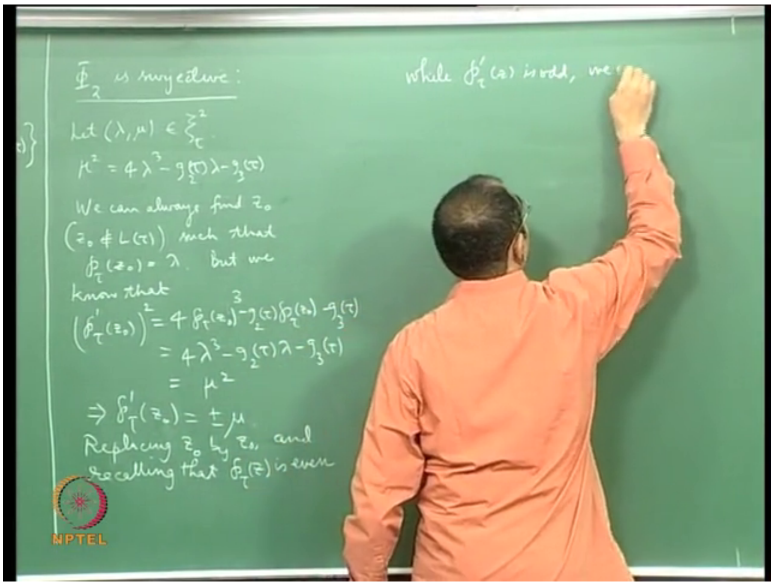

So, the first thing is let me let me tell you the why phi 2 is surjective.

Why is phi 2 is surjective? So, you see so, you take you take a not at lambda comma mu in psi tau psi sub tau of 2. So, please think of this tau is 2 above only as a superscript please do not think of this is square of something, the reasons for this putting this superscript will be will be clear after I introduce projective space which I will do so, very soon. So, you take a point here. And I will have to show that there is a z there is a z here, such that phi 2 of z is lambda and the phi tau prime of z is mu I have to show that. So, the fact that is ordered pair is in this set means that lambda mu satisfy this equation. So, let me write that down first mu squared is 4 times lambda cube minus g 2 tau lambda minus phi 3 tau.

Now, so, let me let me recall the following fact, that if you take the interior of the fundamental parallelogram and translate it suitably so that the boundary does not have any of the lattice points. Then the number of 0s of an elliptic function is equal to the number of poles in that in that region. And what that says is that it the elliptic function takes the value 0 as many times as it takes the value infinity. And replacing the elliptic function by the elliptic function plus an additive constant, will tell you that it will take minus of that constant the value minus of that constant also as many times as it takes the value infinity. So, we know that phi has flow of order 2 at each of the lattice points and if you had if you translate the fundamental parallelogram.

So, you has to include so as to avoid all lattice points on the boundary you will get exactly one lattice point inside. And therefore, the number of times the free function will take the value infinity in the interior will be exactly 2. And it will be at that point it is at that lattice point which is a pole of order 2. And therefore, it will take any other value also 2 times in the interior. So, we can always find z naught such that phi tau at z naught is equal to lambda. So, we can always find z naught such that. So, z naught z naught z naught not in lattice such that phi tau of z naught is equal to lambda.

But, but on the other hand, but you know, but we know that that by virtue of differential equation satisfied by phi we know that phi prime tau at z naught the whole squared will be 4 times phi tau of z naught minus 4 times phi tau z naught the whole cube minus g 2 tau into phi tau of z naught minus z 3 tau so that we 4 times, but phi tau z naught is lambda. So, it will be 4 lambda cube minus g 2 tau lambda minus g 3 tau. And this is by assumption equal to mu squared which will imply that phi prime of tau of the z naught as we plus or minus mu. And recalling the fact that phi prime is a is an odd function we can replace if phi prime of so we are talking also trying to find z naught.

So, that phi prime tau of z naught is actually equal to mu. So, if phi prime tau z naught is minus mu replace z naught by a minus z naught. So, that phi prime tau of z naught will become plus mu. And the value of phi of tau of z naught will not change because phi is an even function. So, replacing z naught by minus z naught and noting and recalling that phi tau of z is even while phi prime tau of z is odd we can get phi tau of z naught is equal to lambda and phi prime tau z naught is equal to mu.

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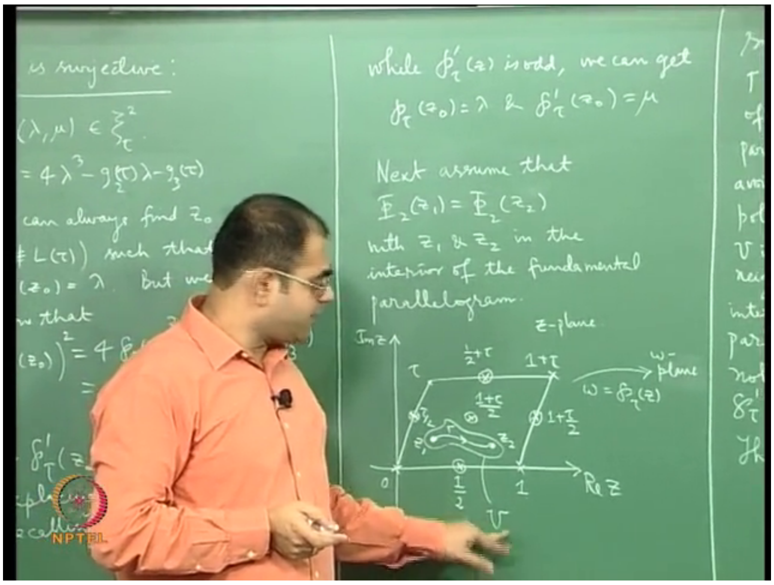

So, this gives you the surjectivity, right. The next thing that one has to do is injectivity.

So, I will have to say something here. So, I think there is a problem. So, there is a problem with so, there is a problem with my statement here again. So, in fact, I have I have not written this right correctly. In fact, what I want it to say is that since the map phi 2 is also invariant under the action of l tau, it goes down to a map from the complex torus minus the point which is the image of the lattice. And I want to say that is the one that is that is the one there is bijective. So, this statement is not accurate. So, I will replace this now. Phi 2 is surjective and for the injectivity when it goes down the statement is that if phi 2 takes the same value at 2 points, then the 2 points differ by a lattice point of the lattice. So, let me make that change. Let me make that change here.

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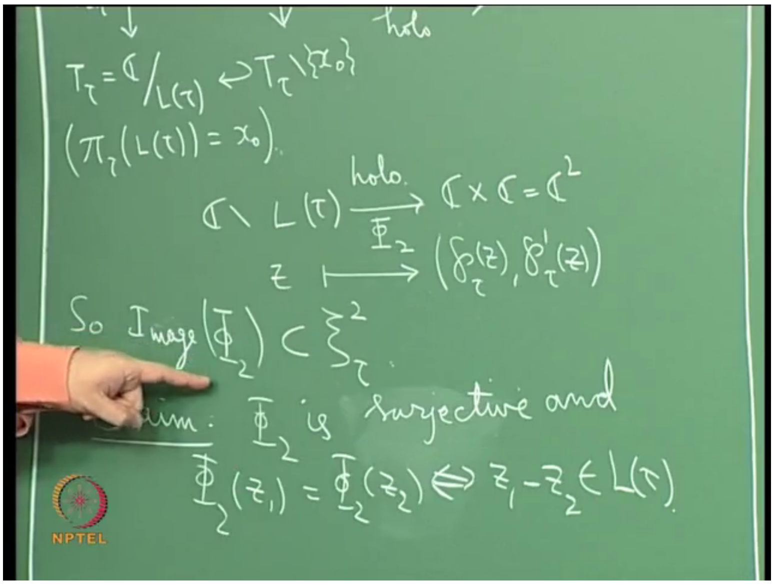

So, phi 2 is inject is surjective, and phi 2 surjective, and phi 2 of z 1 is equal to phi 2 of z 2 if and only if is z 1 minus z 2 is in the lattice. So, basically so, let me again correct myself I am not claiming that phi 2 as I defined it is injective is bijective. The point is that goes down to a map from the torus minus the point the point being the image of the lattice, and it is that map which I want to say is bijective on to this subset. So, surjectivity is there and apart from that you do not have injectivity, but 2 points going to the same point should tell you that the difference rise in the lattice.

So, that is what ill have to prove. Well, here so let me prove this second part. So, what we will do is you can we can argue various cases, we know that the fundamental in the fundamental parallelogram, there is exactly if you take the points in the interior. There are exact there is exactly one point corresponding to the r I mean the equivalence class of a given point under the lattice. The interior there is only one representative. Where is in the boundaries, if it is not the 4 vertices are 4 representatives for the point that goes to x naught for the points that go to x naught.

Then apart from these 4 vertices in along opposite it edges you will have 2 2 points as a representatives for points whose orbits intersect the boundary. So, I will have to look at various cases. So, what I will do is I will first assume that let me first assume that z 1 z 2 are actually inside the fundamental parallelogram. They are in the interior of the fundamental parallelogram. And then I have to literally say that they are equal. Because if a differences is in the lattice, if they are inside the fundamental parallelogram, and the difference is in the lattice, then the only possibility is that they are one and the same. So, next assume. So, let me draw line here. Next assume that phi sub phi is there phi sub 2 of z 1 is equal to phi sub 2 of z 2 with z 1 and z 2 in the interior of the fundamental parallelogram.

So, the fundamental parallelogram is the one that is formed with vertices 0 1 1 plus tau and tau. So, assume that these 2 are in the interior of the fundamental parallelogram. Therefore, then we will have show that z 1 is equal to z 2. In in a other words phi 2 when you restrict it to the interior of the fundamental parallelogram is actually injective. So now for this, we will have to do a little bit of integration. So, what one does is so, one does so, following thing. So, you see so, let me draw picture. So, I have this is the tau plane. This is the z plane. In all the thing tau fixed.

This is the z plane and I am I am just looking at the mapping omega is equal to phi tau of z. And what I am going to do is; so, here is my so, here is 1, this is 0. This is the real axis, this imaginary axis. And so, here is my tau somewhere here upper half plane. And I have this parallelogram 1 plus tau. And you see these so I will put these points at a cross these are precisely these 4 points are precisely the poles of phi as well as phi prime. The phi function has a pole of order 2 with residue 0 at each of these points. And phi prime as a pole of order 3. And then you we also know what these 0s are the 0s of phi prime are.

So, these 0s of phi prime r precisely here there is a 0 at half, there is a 0 at half because you know if you look at that. So, you see when I put z is equal to half, all right. Then e 1 tau is phi tau of half, and then this (Refer Time: 31:32). So, prime manages. So, the 0s of phi prime are at 1 by 2, then there is one 0 at tau by 2, and then there is a 0 here. 1 plus which is 1 plus tau by 2. These are the 3 0s. And let me put a circle around these things to tell you that these a 0.

And of course, if you so, you will also have 0s here and here, because phi prime is after all periodic doubly periodic with periods 1 and tau. So, this point will also be a period. I mean this point will also be a 0 of phi phi prime, and that is going to be 1 plus this is going to be 1 plus tau by 2. And this point will also be a 0 of phi prime. That will be 1 by 2 plus tau. So, these are all crosses which are circled these are the 0s of phi prime. These are all these 0s I have in in this in this fundamental parallelogram including the boundary. And there is only one 0 in the interior. And as far as the poles are concerned the poles are exactly these 4 vertex forms, all right. So, this main this is my diagram.

Now, I have taken z 1 z 2 in the interior fundamental region. And so, one so let me first assume let me first look at the case where neither of z 1 or z 2 is this 0. So, suppose z 1 is not equal to this 1 plus tau by 2 and there is not equal to z 2 also.

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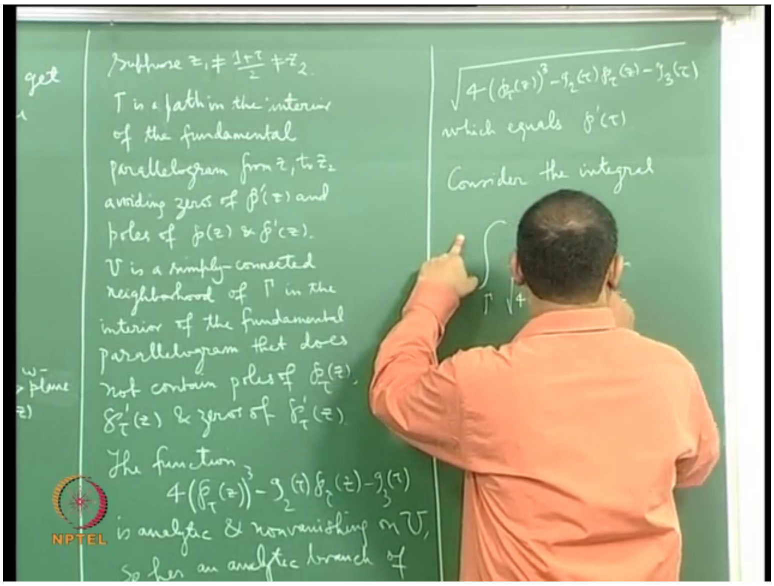

Suppose these are distinct from this point. Then see what you do is so here there is a omega plane is the omega plane. So, I have to omega plane. So, what you do is your situation is that you have you have a point you have z 1.

So, I will I use a thicken dot for z 1, and I will use thicken dot for z 2. And what you can do is you know I can just take I can draw a path from z 1 to z 2, that avoids any of the you know any of this marked points namely the poles of phi and phi prime and the 0s of phi prime. And what I can do is I can I can also find a simply connected neighborhood of that path, which does not which does not contain any of these 0s of phi prime or poles of phi or phi prime. So, I can I can find this path and I can also find. So, let me draw that. So, I can find this let me call this as something let me call this as u.

So, this path is gamma. Let me call this path as gamma. So, gamma is a path in the interior of the fundamental parallelogram from z 1 to z 2 avoiding 0s of phi prime of z and poles of phi of z and phi prime of z. And u of course, you can find u a simply connected neighborhood of gamma, also avoiding the poles of phi and phi prime and 0s of phi prime. U is the simply connected neighborhood of gamma in in the in the interior of the fundamental parallelogram; parallelogram that does not contain poles of phi of z phi tau of z phi prime tau z and 0 of phi prime tau of z now.

So, if you now look at see if you look at the function the function given by the right-hand side of these equations; namely this function right side of this equation. Namely, you take 4 phi tau is that q minus g 2 tau phi tau z minus g 3 tau. That function is an analytical function in this in on u, and the point is that it does not vanish. Because you have avoided because varied vanishes are exactly the 0s of phi prime tau. So, you have an analytic function on a simply connected neighborhood on a simply connected say it does not vanish, then it has a square root.

So, this analytic function will have a square root, and you can find an analytic branch of the square root that is equal actually to phi prime tau of z in on u. So, let me write that down, the function 4 times phi tau phi tau of z cube minus g 2 tau phi tau of z minus g 3 tau is analytic that is holomorphic and non-vanishing on u which is simply connected. So, it has a analytic square root a branch of the analytic square root which is equal to phi prime of tau there. So, hence an analytic branch of so, let me write that here.

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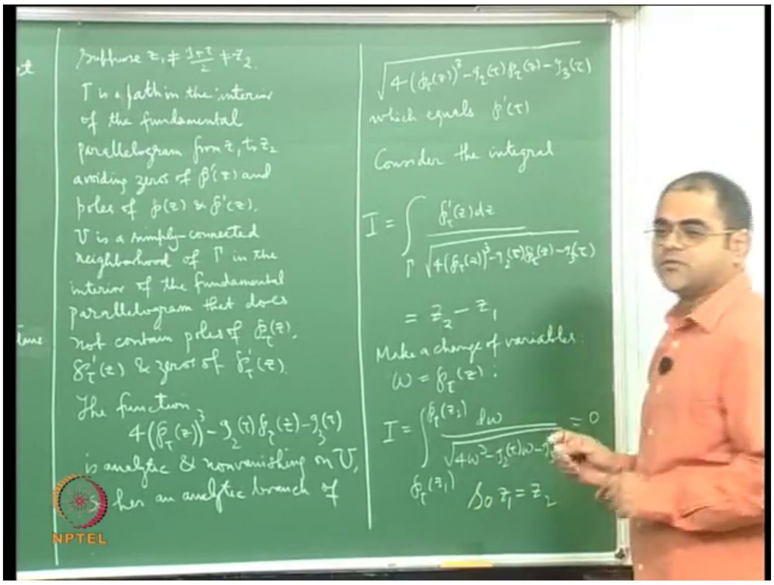

So, it has an analytic branch of square root of this function that is a analytic square root.

In which equals phi prime tau, I can find an analytical branch which equals phi prime tau, right. Now what you do is that you now consider the integral over gamma of phi prime tau of z t z by square root of 4 this quantity. Look at this integral, then this integral is actually because there is an where I where here I have taken in the denominator I take the analytic square root which is equal to phi prime tau of z. So, this this integral is just going to be in so, this going to be the denominators also phi prime tau z. So, this integral d z, and gamma is varying from z 1 to z 2.

So, I will just get z 1 minus z 2 is integral will be z 1 minus z 2 by, because we have taken the analytical branch like this, z 2 minus z 1. Because it is a terminal point is z 2, you will get z 2 minus z 1. But on the other hand, what you will have is that on the other hand actually this integral is 0. These integral is actually 0. Because you see you write omega you make a change of variables, change of variables. You make a change of variables by omega is equal to phi tau of z. You change the variable of integration from z to omega. Then the so, if I call this integral as I, then what you will get is I will also be integral from phi tau of z 1 to phi tau of z 2 of d omega by root of 4 omega cube minus g 2 tau omega minus g 3 tau.

This is the valid change of variables. And but then you have assume that phi tau you have assume that phi 2 of z 1 is phi tau of z 2. So, that means, that you know phi 2 phi tau at z 1 and z 2 are the same phi prime tau at z 1 z 2 are the same. So, what this will tell you z these 2 are the same. Therefore, this integral is 0. So, the point is that in principle you are integrating over the image of gamma in the omega plane, and that would be a closed path starting at phi tau of z 1 which is equal to phi tau of z 2. But the point is that you are able to there is an there is an analytical branch of this square root. So, therefore, this integral is actually 0. So, if you put these 2 together you get z 1 equal to z 2. So, this will tell you. So, this is equal to 0.

So, is a so z 1 equal to z 2; so this proves that when z 1 z 2 are in the interior and different from the 0 of phi prime. Then these 2 be equal will tell you that z 1 and z 2 are one and the same. So, we will have to look also at few other cases. So, suppose I assume z 1 is 1 plus tau by 2 that is the case that I will have to look at.

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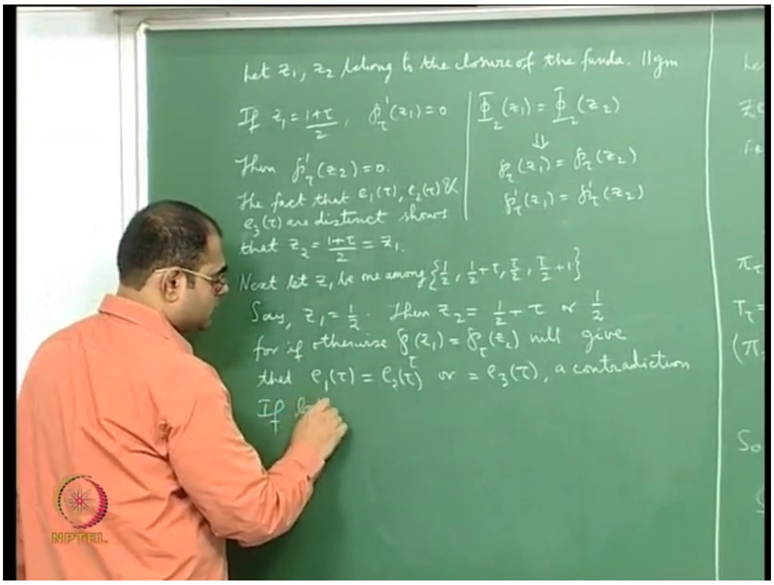

So, if z 1 is 1 plus tau by 2. Since this is a 0 of phi prime of tau, you will get phi prime of phi prime sub tau of z 1 is 0. And but you see phi of phi 2 of z 1 is equal to phi 2 of z 2 is our assumption. And this tells you that that phi tau of z 1 is the same as phi tau of z 2, and phi prime tau of z 1 is equal to phi prime tau of z 2.

And let me assume that z 2 and z 1 are in the closure of the fundamental parallelogram. So, let z 1 z 2 belong to the closure of the fundamental parallelogram. And you take z 1 to be this 0 at the center of phi prime. Then since phi prime is 0 at z 1 phi prime also 0 at z 2. And that for therefore, for z 2 the only choice is are these other points. For z 2 these are the only choices, but at those points, but whichever of those points it is the phi value at that point should be the same as the phi value at 1 plus tau by 2. But remember that the phi value here is e 1, it is e 1 of tau. Phi value here is e 2 of tau. And phi value here is e 3 of tau and we have already seen that these 3 are distinct numbers. They are the distinct complex numbers. Therefore, we will get a contradiction.

So, if for example, z 2 was this. Then you will get the phi value at z 2 the fact that phi value at z 2 is equal to phi value at z 1 will tell you that the phi values that the same here and here, but the phi value here is same as see the phi value here. So, you will that will say the phi value here is the same as the phi value here, but the phi value here is actually e 1 and the phi value here is e 3. And you are saying that you say that phi the e 1 is equal to e 3 which is not possible. So, because of the distinctness of the e I s it will follow that z 2 will also have to be 1 plus tau by 2 in this case. So, let me write that then phi prime tau of z 2 is 0 the fact that e 1 e 1 of tau.

Now, e 2 of tau and e 3 of tau are distinct shows that z 2 also has to be 1 plus tau by 2, and that that has to be equal to z. So, we have settled the case when one of them is this 0 at this center. And then we will also have to look at the case when z 1 z 1 is one of these 0s on the boundary. So, let us let us take that case. Next let z 1 be one among half plus tau by 2 and tau by 2 plus 1. Then again phi vanishes phi prime vanishes at z 1. So, that will tel you that phi prime has to vanish at z 2.

And well then, the, then I then you would like to say that z 1 and z 2 are exactly on the opposite set on the opposite edges either if they are on this off on these auto opposite edges then they differ by they differ by tau. And if they are on these 2 opposite edges they differ by one. So, you will have to look at the situation where they are not you should say that it cannot happen that this is z 1 and that is z 2. That is the case we will have to say, and well essentially in the in that situation the phi value is being the same. Again, will tell you that e 1 is equal to e 2 for example, and that is not possible. So, again essentially the distinctness of the e I s will tell you that if z 1 is one of these then z 2 is the other one which is translate by one or tau.

So, let me write that down. So, let me write say z 1 is half, then z 2 has to be half plus tau. Z 2 is to be half plus 2 are half itself. For otherwise for if otherwise phi of z 1 phi of phi tau z 1 is equal to phi tau of z 2 will contradict that e 1. So, phi tau at half is e 1, e 1 of tau and you will get phi tau at any of the other 2 points which are tau by 2 or tau by 2 plus 1, it is e, e 2 of tau or e 3 of tau. I mean, if other one or if for otherwise, this will give this will not contradict this will give, that this is equal this or this is equal to this which is a contradiction.

So, we have essentially, we have settled all the cases when one of these points is one of these 0s. So, the only other case that is left is you have 2 point both z 1 and z 2 are on the boundary, and not any of these 0s and not any of those poles which are the vertices. And in that case, you can make sure that both of them are on the same edge are an adjacent edge. And in that case again, you can join them by a path, and find the simply connected neighborhood you can translate this whole fundamental parallelogram a little and continue to apply this argument to get z 1 equal to z 2. So, essentially what this will tell you is that either you get z 1 equal to z 2 or you will get z 1 is equal to our translate of z 2 by plus or minus 1 or 2 plus or minus tau.

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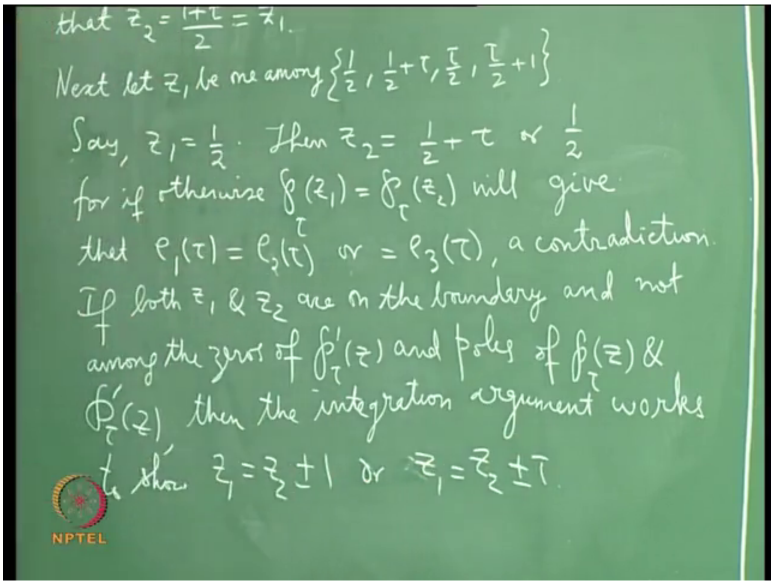

So, let me write that. If both z 1 and z 2 are on the boundary and not among the 0s of phi prime tau of z and poles of phi of phi tau z and phi prime tau of z. Then the integration argument works to show z 1 z equal to z 2 plus or minus 1 or z 1 is equal to z 2 plus or minus tau. The argument will give you this. So, that finishes the proof of the fact that the proof of this claim.

So, I want to just conclude by saying the statement that I wanted to originally say that phi 2 goes down to a map from the torus minus a point it is a bijective map from the torus minus a point to this subset. And you see what you must realize is that so, for the movement this target subset is; it is a close subset of c 2 and a priory you can consider it only as a as a topological case. In fact, it has a manifold structure. In fact, it has a Riemann surface structure, but that has to be explain properly, which I will do in the forth coming lecture, but if you just consider this as a close surface topological surface of c 2.

Then what you get is that you get that phi sub 2 induces the homeomorphism of the torus minus this point on to this on to this subset. You get homeomorphism on this subset because you see basically at least for the movement it is a continuous bijective map. Maybe we will we need to do a little bit more arguing to say that this actually homeomorphism. So, let me record that and stop there. So, let me write here. Let this board be there as it is; so, let me let me write it here so that I keep so, we get.

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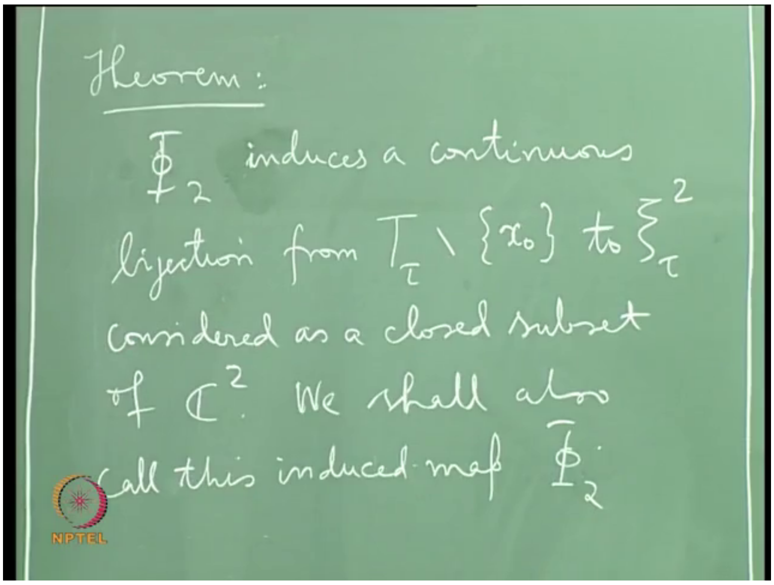

So, we can call this as theorem phi sub 2 from the torus induces a map a continuous bijection from t sub tau minus the special point which is the image of the lattice to this subset psi sub tau upper 2 considered as a closed subset of c 2.

So, essentially so, this is the statement and I will also call by abuse of language I will also call induces map from the punctured torus to psi 2 as phi sub 2. So, we shall also call this induced map phi sub 2. So, I will continue in the in the in the forth coming lecture essentially what I want to say is that you can so, what I what I will be doing is that I will show that this c 2 is a is an open set in the 2-dimensional projective space and this this curve this subset can be compactified by adding just one point. And once you add that one point, then this continuous bijection from the punctured torus to that will extend to homeomorphism of the full torus to the compactification with that one point so called point at infinity added and then we show that this is a homeomorphism.

So, that is what we will do in the next lecture. So, I will stop here.