An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture – 36
The Weight Two Modular Form vanishes at Infinity

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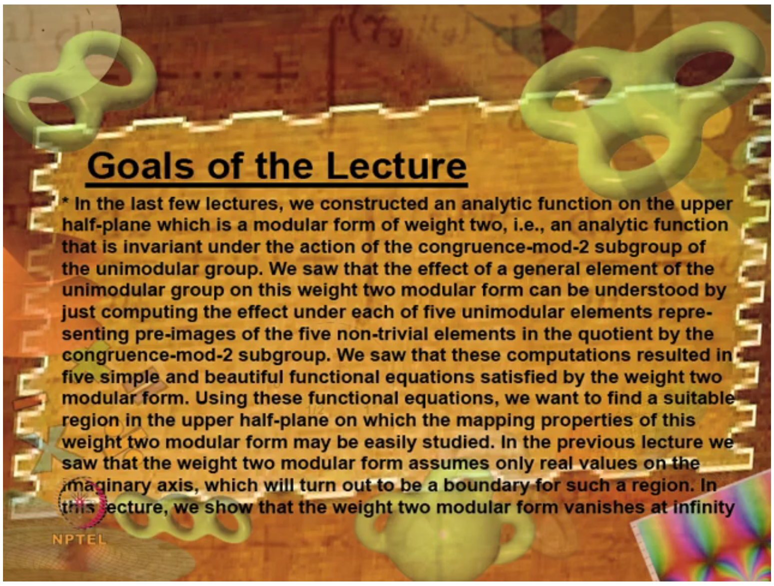

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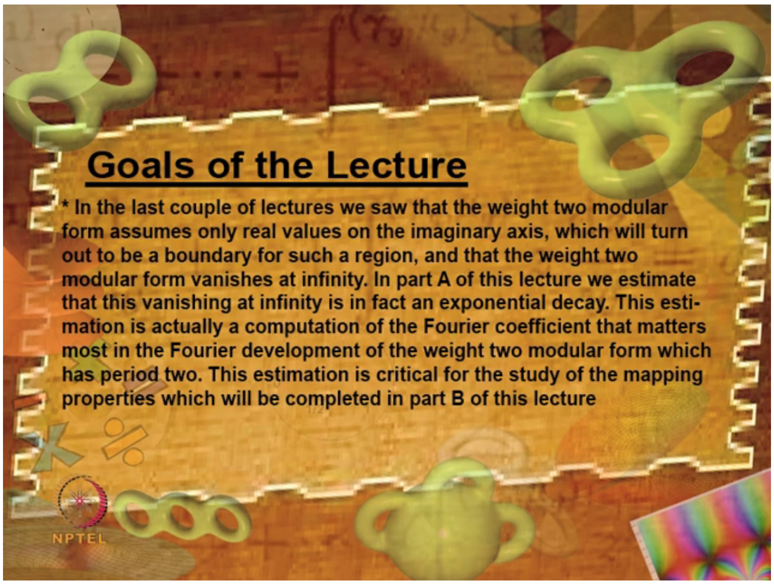

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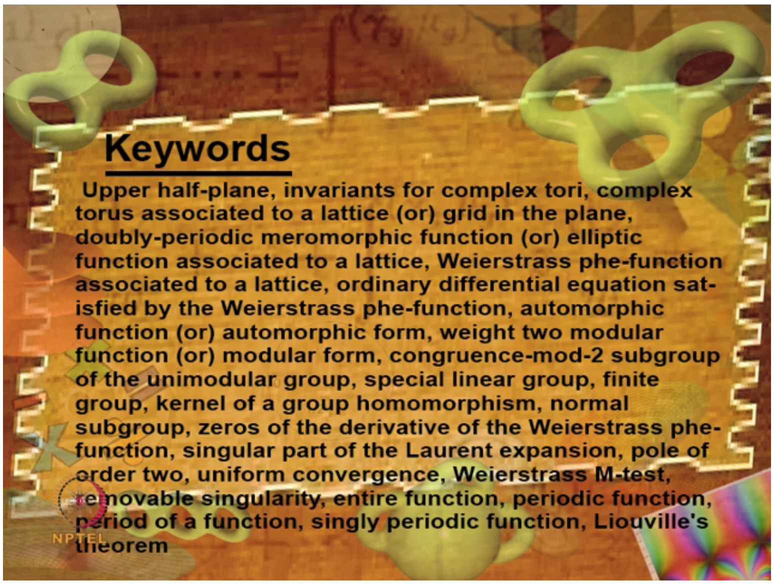

So let me quickly recall where our discussion is at this point of the course.

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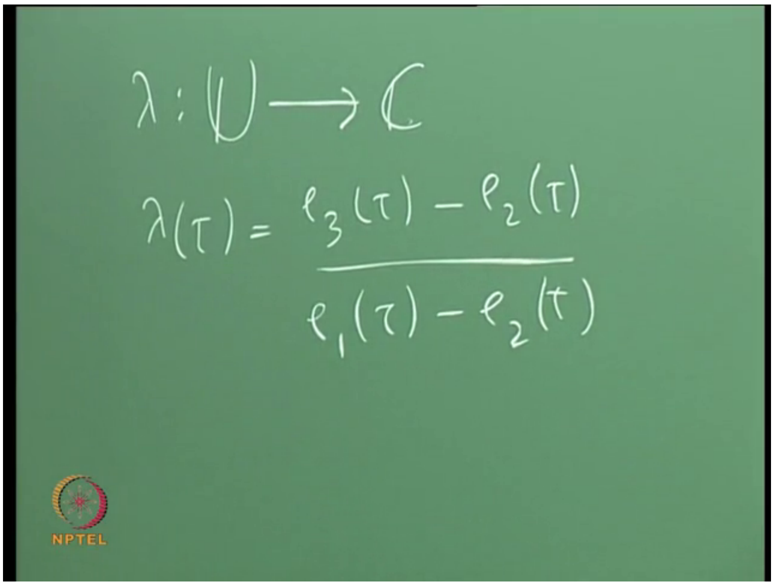

So, we have this function lambda of tau. So, this we have this function lambda defined on the upper half plane with values the complex numbers, lambda of tau was defined as e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau. And we are trying to see so this function is invariant only for the action of the congruence mod to subgroup, and we are trying to study the mapping properties of this function. So, the theorem that we are trying to prove is the following that.

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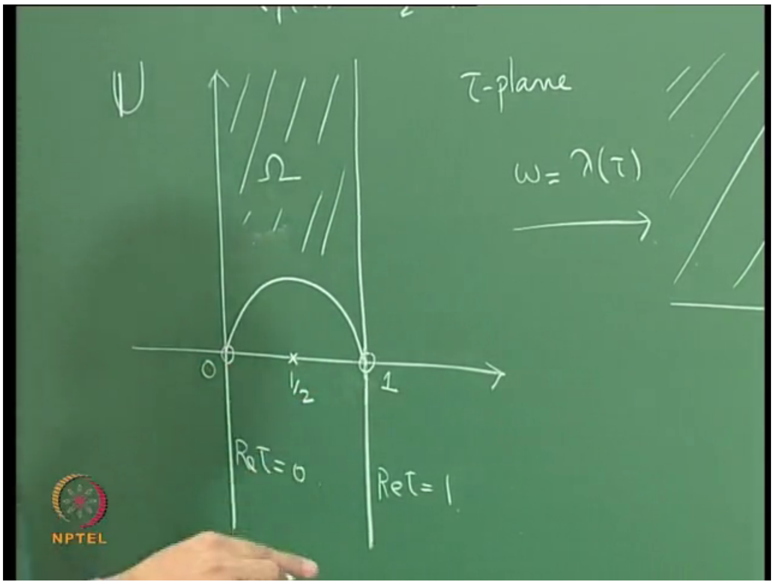

See, if you take the following region the complex plane. So, this the tau plane and we have a so this coordinate is one and well this the line real part of tau equal to 1. This of course, imaginary axis, this real part of tau is equal to 0. And then we take a semi-circular arc which is centered at half and radius half. And we take this region omega, we take this region omega which is bounded on the left by the imaginary axis; and on the right by this line which is real part of tau equal to 1, and below by this semicircular arc. And the claim is that lambda gives us an isomorphism of omega on to the upper half plane and the isomorphism is an; is in such a way that, so it is an isomorphism it is a holomorphic isomorphism. So, it is a holomorphic isomorphism of omega onto the upper half plane. And the under this isomorphism can be extended to the boundary continuously.

So, you can define omega on the board or you can define lambda on the boundary. So, that the boundary is mapped to the boundary of the upper half plane namely the real axis. So, if I draw this diagram, so I am taking the transformation omega equal to lambda of tau and the corresponding the image plane is the omega plane and the claim is that this region omega is mapped into the upper half plane. So, this also the upper half plane here, it is also the upper half plane there. So, it is mapped here onto the upper half plane here. And in such a way that the boundary is mapped onto the real axis, so that is what we have to prove alright this what we have to prove.

So, we started out in the previous lecture by proving that lambda is real on the boundary. The problem with the boundary is only at these two points at 0 and at 1, because you see lambda is anyway analytic on the upper half plane. So, it is defined on the imaginary axis in the upper half plane the problem is at 0. Similarly, there is no problem along all the points of this semicircular arc, the problem is only at 0 here and at the point 1 there. So, what one has to do was what one has to show is that you know lambda approaches a proper limit at 0 and at 1, and the limit and that limit is achieved no matter how you approach it. So, long as you are inside this inside this region or in the boundary of that region that is what you have to verify, and you should verify it for the limit here as well as for the limit there.

And in fact, what I had said yesterday was that I mean in the last lecture was that you see the fact is as tau goes to infinity in particular as tau, so since we are considering only the; this portion of the vertical strip. As tau goes to infinity essentially you are letting imaginary part of tau go to infinity, then the point the fact is that it should give you the value of lambda at infinity and that turns out to be 0. And the value of lambda at 0 turns out to be 1, and the value of lambda at 1 turns out to be infinity. And this how lambda is supposed to map the boundary of omega onto the real axis. So, of course, I have already shown except for the point 0 and one that lambda is real on the boundary of omega that was done in the previous lecture. So, this lecture is essentially to study the behavior of lambda as you approach 0 or as you approach one.

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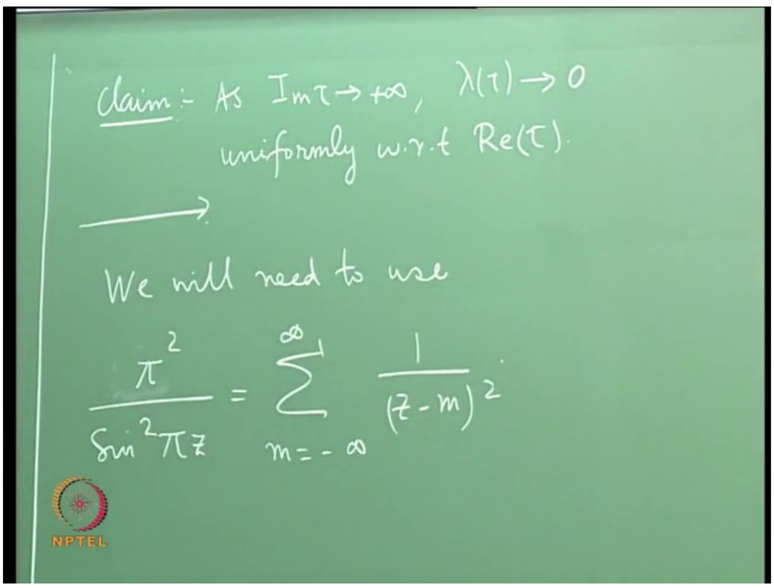

So, let me write down immediately what our claim is, our claim is the following. Our claim is as so yeah I will draw that another diagram if I need one. So, as imaginary part of tau tends to infinity as imaginary part of tau tends to infinity. So, in fact, I should say as imaginary part of tau tends to plus infinity because this is the real variable. Lambda of tau tends to 0, and the way it tends to 0, you see is uniform as a function of real part of tau. So, you see as you let imaginary part of tau tend to infinity, lambda of tau will essentially be thought of as a function of real part of tau. And the way it takes to 0 is uniformly with respect to the real part of tau.

So, let me write that down uniformly with respect to real part of tau. So, this is the claim now. So, this is the first claim. And how are we going to prove this? So, this will tell you that you know the value of you can define the value of lambda at infinity as 0, you can define the value of lambda at infinity as 0 and that is how lambda takes the value 0 in the image. So, and of course, you see if you think of this, this complex plane as the Riemann's sphere, then infinity is actually is the corresponds to the north pole in the Riemann's sphere. So, it is really a point in that sense by the stereographic projection. And all we are trying to say is that lambda can be extended to the point at infinity by defining it to be 0, because this allows us to do that.

So, well, so how does one prove this? So, in order to prove this, one has to do some work with series. So, let me begin with the following statement. We will need to use sin pi square by sin square pi z sigma m equal to minus infinity to infinity 1 by z minus m the whole square. So, we need to use this desired entity. So, we will have to use this identity. You can realize why we will need it because you see this is what partly appears when you write out the formula for the p function right. So, the first thing is let us first justify this identity.

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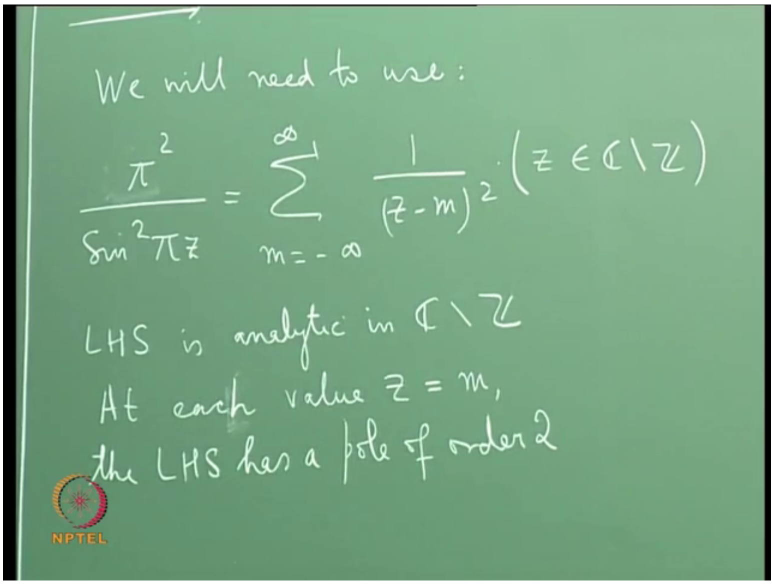

See you see they the LHS the left hand side if you look at the function here this function is analytic everywhere except the points where the denominator vanishes and the points where the denominator vanishes precisely the integers. So, you see LHS is analytic in C minus z in you take away the integers from the complex numbers. So, you see this. So, here I should say z this valid for c minus z this valid for any z, which is not an integer. So, the left hand side is of course, analytic in C minus z. What happens at points of z at points of z, you get by z I mean integers at integer values you get poles of order 2. At each value z equal to m, the LHS the left hand side has a pole of order 2, it has a pole of order 2. And further and you know of course, if you have a isolated singularity then you have a Laurent development. And if you take the Laurent development about z equal to m that means, you try to expand in powers positive and negative of z minus m then the singular part will be precisely 1 by z minus m the whole square. So, at each value z equal to m the left hand side has a pole of order 2 and has singular part one by z minus m the whole square this will be the singular part this will be the singular part.

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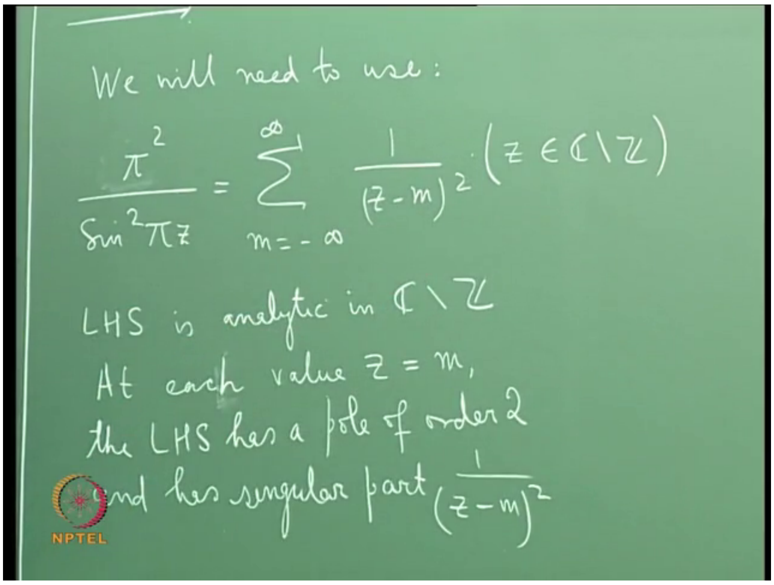

So, this is quite obvious. Now, but you look at the right side, in the right side, you are simply adding all those singular parts together. The right side I have written just by adding all those singular parts, but the nice thing is I cannot just write down something like this and expect it to define something sensible, but in this case it really does. See, this the term on the right side is actually a convergent series in fact if you take mod y greater than or equal to 1, then this will be uniformly convergent I just have to stay away from the integers.

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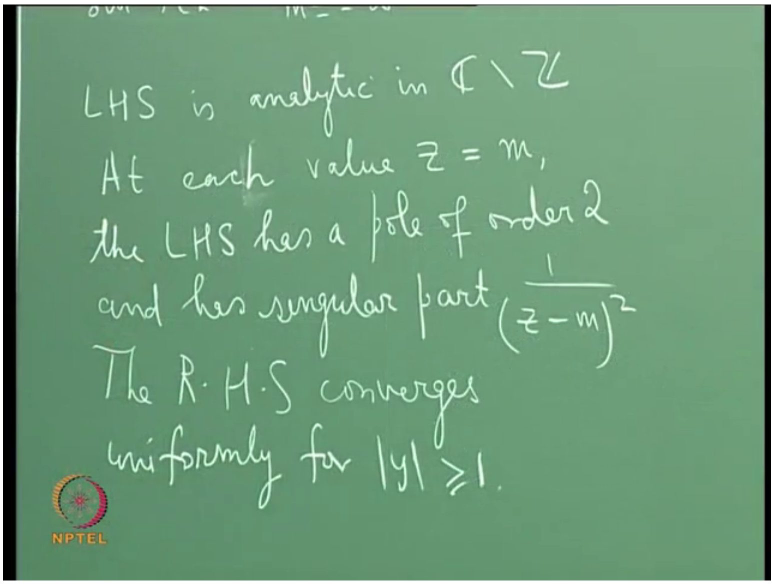

So, the RHS - the right hand side converges uniformly for mod y greater than or equal to 1 that is it. You see the point is you see if you look at if I draw a diagram the situation is like this, I have this is a complex plane.

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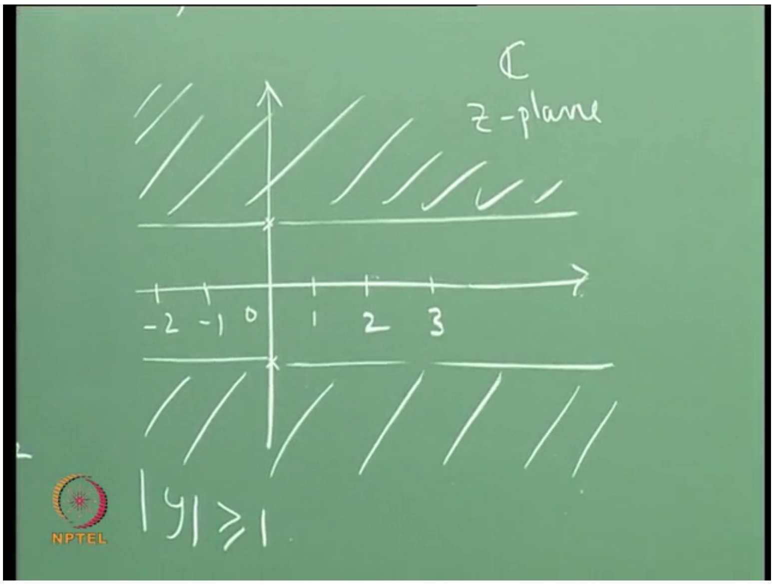

So, this C, this is z plane and you know I have all my integers here. So, I have all my integers here see mod y y equal to one is a; you know it is going to be line parallel to the x axis. So, mod y greater than one is going to be either. So, you know it is going to be this region, it is going to be this and it is going to be this. So, this will be the region this will be the region mod y greater than or equal to 1. And the point is that if you take that region then you can compare the part of the series from 0 to infinity and the other part of the series from say 1 minus 1 to minus infinity separately with sigma 1 by n square which you know is convergent. So, you can by using Weierstrass M-test you can actually confirm that this is uniformly convergent. So, the right side also converges uniformly for mod y greater than 1, alright. Now, well now you see, we want to prove the left side is equal to the right side which is the same as time to show that the left side minus the right side is 0 is identically 0.

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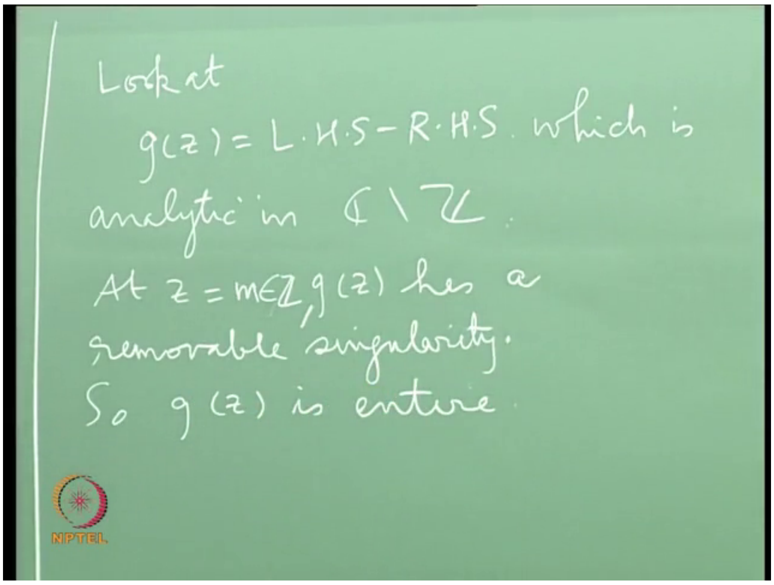

Now let us let us look at the left side minus the right side. Look at g of z is equal to the left hand side minus the right hand side, look at this one. Look at this function. You see this function if you take a point, which is not an integer this going to be an analytic function this going to be an analytic function because both sides are of course, both entries are analytic, so long as z is not an integer therefore, this going to be an analytic function. And the fact that the series is an analytic function is because of uniform convergence and because each term of the series is analytic.

So, this function is going to be analytic in of course, C minus z, but more importantly if you take points of, if you take integer points, you see the singular part of this at z equal to m is already here. So, when you subtract the singular part is removed. So, you see if you concentrate at z equal to m, it will become a removable singularity, z equal to m will be a removable singularity. So, as a result, this function will be an entire function. So, at z equal to m g of z has a removable singularity. In fact, I should it is actually analytic I am saying it is a removable singularity because I do some local analysis and if I take a Laurent expansion then the only the singular part I will get here is 1 by z minus m the whole square the rest of it is all analytic for z in a small neighborhood of m. Because z is going to be is not going to be any other integer other than m.

And this of course, if you take the Laurent development at z equal to m, the singular part is going to be 1 by z minus m the whole square. So, if you take the difference the singular part is going to go away. So, which means you have only the analytic part and that is way of saying that it is actually a removable singularity. In fact, you can actually say it is analytic. So, at z equal to m in z, so g of z is entire, so it is an entire function. This function is an entire function that is a first observation. The second observation is you see if I replace e z by e z plus 1 the left side is not going to change. In other words, you see this function on the left side is periodic with period one and the same is true with the function on the right side because if I replace e z by e z plus 1, I will still get the same series. So, the fact is both the left hand side and the right hand side are single periodic functions with period one and therefore that their difference is also a periodic function with period one.

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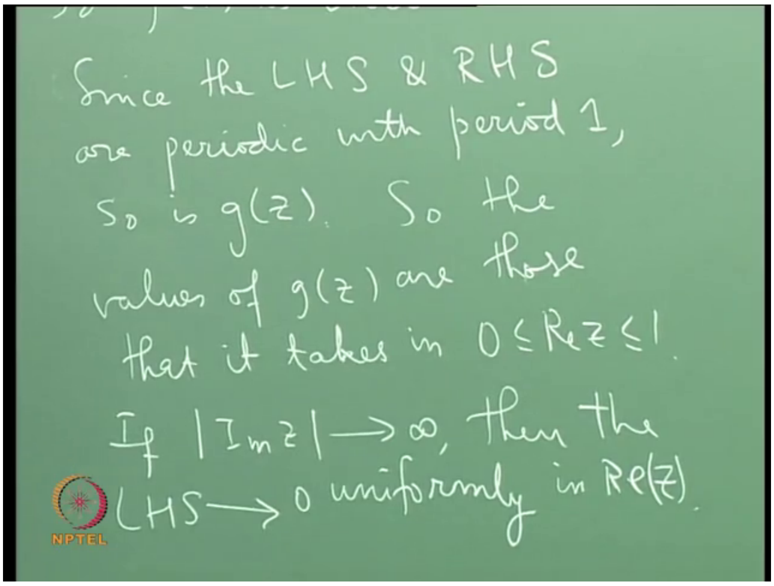

So, let me write that down. Now, since the LHS and the RHS are periodic with period 1, so is g of z. So, you see the difference which is g of z is actually periodic with period 1. Therefore, e 2, so its values are completely controlled by the values in a vertical strip of length one, of horizontal length one. So the values the values of g of z are the same are those that it takes in say 0 less than or equal to real part of z less than or equal to 1 in this vertical strip because it is periodic with period 1. Now, you know now you see what we are trying to do we are trying to show LHS equal to RHS we are trying to therefore, show that this analytic function is 0. So, what we would do you know what we will try to do we will try to show that g of z is bounded in this vertical strip then by periodicity is bounded on the whole plane Liouville's theorem will tell you that g of z is a constant you evaluate that constant. And then show that the constant is 0 that is how you prove g of z is identically 0, so that is the; that is what we are going to do.

So, you see the point is that. Now, you notice the following thing see if modulus of imaginary part of z tends to infinity that is if you write z is equal to x plus i y and you let mod y tend to infinity then you see the left hand side goes to 0 uniformly. Then the left hand side goes to 0 uniformly in real part of z right it goes to 0 uniformly. And that is basically because you see you must all be familiar with this with the fact that we may use.

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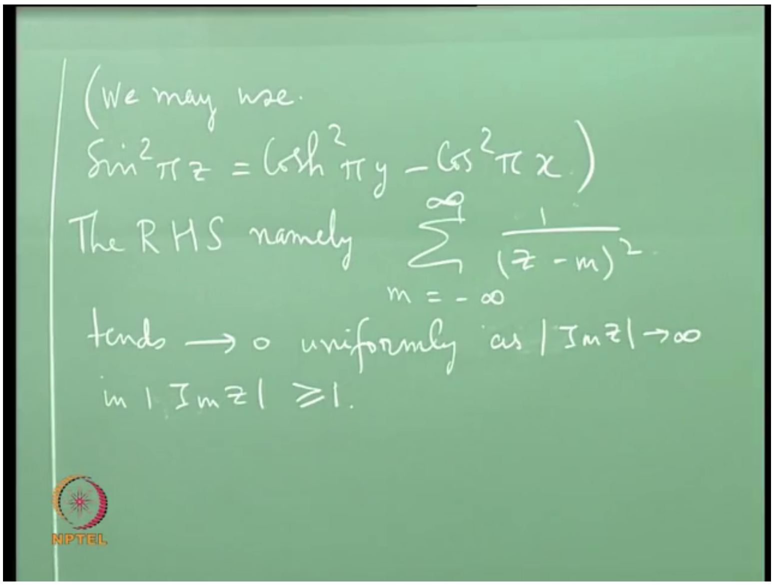

We may use you know sin square pi z well if you expand it, it is cos h square pi y minus cos squared pi x, these are elementary identities. So, you see you can use this to infer that this is very easy to see. On the other hand, look at the right hand side that is look at this series. This series for mod y namely for modulus of imaginary part of z greater than or equal to 1 is uniformly convergent in it is uniformly convergent in x. And you see because of uniform convergence of this series, you can take individual limits instead of taking the limit as mod y tends to infinity of this whole thing. You can take mod y tends to infinity for every term, and then take the sum but you see as mod y tends to infinity of every term it goes to 0. So, obviously, the sum goes to 0. So, the fact is this the right hand side that also tends uniformly to 0 as mod y namely modulus of imaginary part of z tends to infinity.

So, RHS namely so here I should say the of course, the by the LHS I mean pi squared by sin squared pi z. And here by RHS I mean that series sigma m equal to minus infinity to infinity 1 by z minus m the whole square tends to 0 uniformly as a modulus of imaginary part of z tends to infinity in mod y in modulus of part of z greater than or equal to 1. So, the moral of the story is that if you look at this vertical strip 0 less than or equal to real part of z less than or equal to 1.

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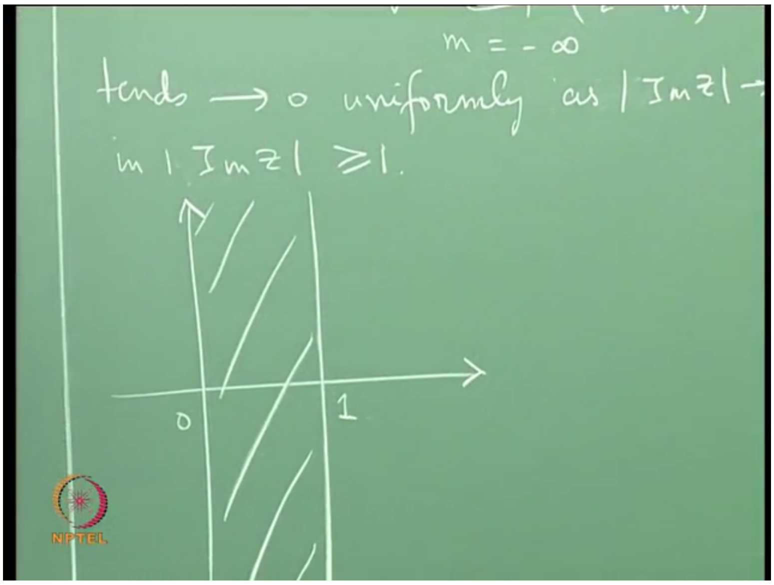

So, you see so it is going to be like this. So, this my vertical strip. If you look at this whole vertical strip, then your function g of z is bounded. See, at infinity, it is going to go to 0; and then so that means, what does it mean it means that you know its value can be the its modulus can be made less than any arbitrarily given small epsilon beyond a portion above and below the strip. And for the remaining portion that is left out that is anyway compact region and you know it has to have a boundary continuous function on a compact set has to have a boundary. Therefore the up short of this the g of z is bounded in this period strip. And since it is periodic with period one consequently it is bounded on the whole complex plane and now Liouville's theorem will tell you that it has to be a constant. And what is that constant that constant has to be the same. And if you now let modulus of imaginary part of z tend to infinity that constant has to remain the same, so it has to be 0. So, g of z becomes identically 0 and we have proved that the left side is equal to the right side.

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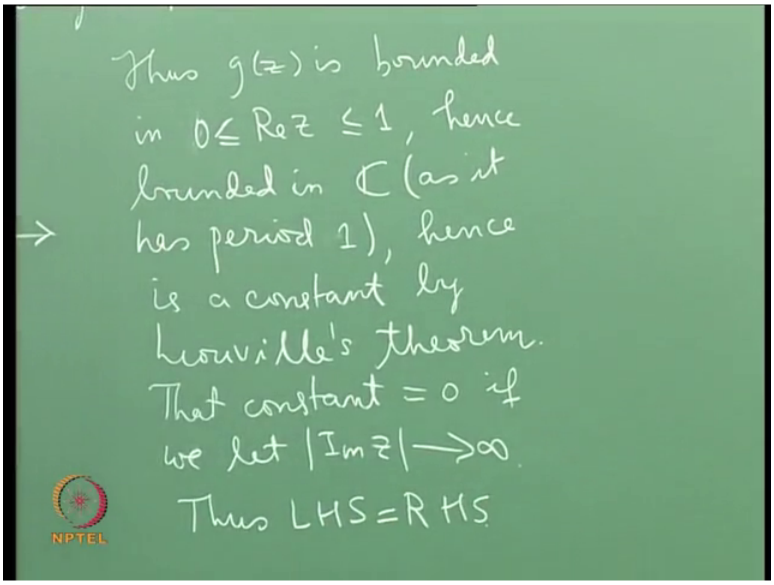

So, let me write that down. Thus g of z is bounded in 0 less than or equal to real part of z less than or equal to 1, hence bounded in C as it has period 1. Hence is a constant by Liouville's theorem and that constant turns out to be 0, if you let mod y tend to infinity that constant is equal to 0 if we let modulus of imaginary part of z tend to infinity. Thus the left hand side is equal to the right hand side, so that proves this formula. Now, the point is that we have to use this formula to get this claim. So, we will do that next.

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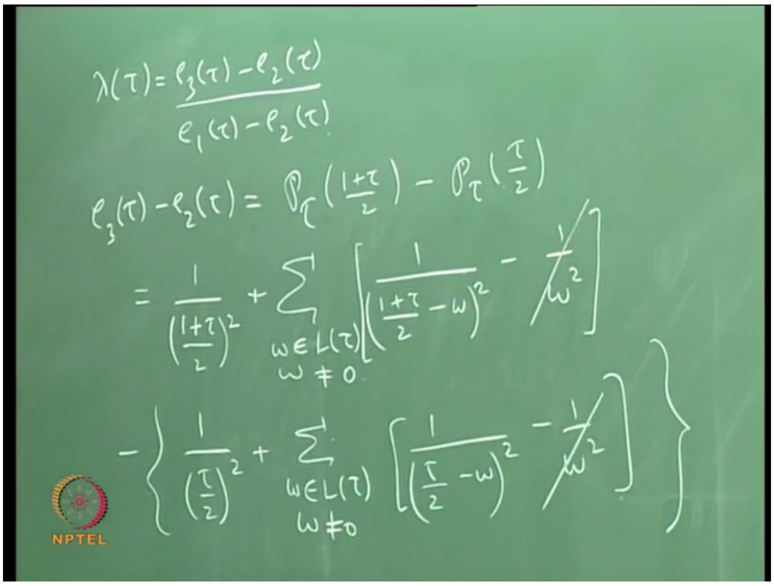

So, what we are looking at is so we are trying to study what happens to a lambda of tau as the imaginary part of tau goes to infinity. So, lambda of tau is e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau. So, let us calculate what the numerator is. So, e 3 of tau minus e 2 of tau is if you remember e 3 of tau is the phi function evaluated at 1 plus tau by 2. So, this p tau evaluated at 1 plus tau by 2 minus p tau the p function associated with tau evaluated at tau by 2 this what it is. And well we can expand it, this turns out to be 1 by 1 plus tau by 2 the whole square plus sigma omega not equal to omega in the lattice, omega not equal to 0, 1 by 1 plus tau by 2 minus omega the whole square minus one by omega square this the expansion of the first term.

And then I will have to write out the second term. So, I will get minus 1 by tau by 2 the whole square plus summation over omega in l of tau omega not equal to 0 of 1 by tau by 2 minus omega the whole square minus 1 by omega square. So, all the series are uniformly convergent; in fact, I can get rid of these terms. And you can notice that when I put omega equal to 0, I can put omega is equal to 0 here, if I put omega equal to 0 here I get this term. So, I can push this term also into the summation and remove the restriction that omega is not 0, the same thing can be done here.

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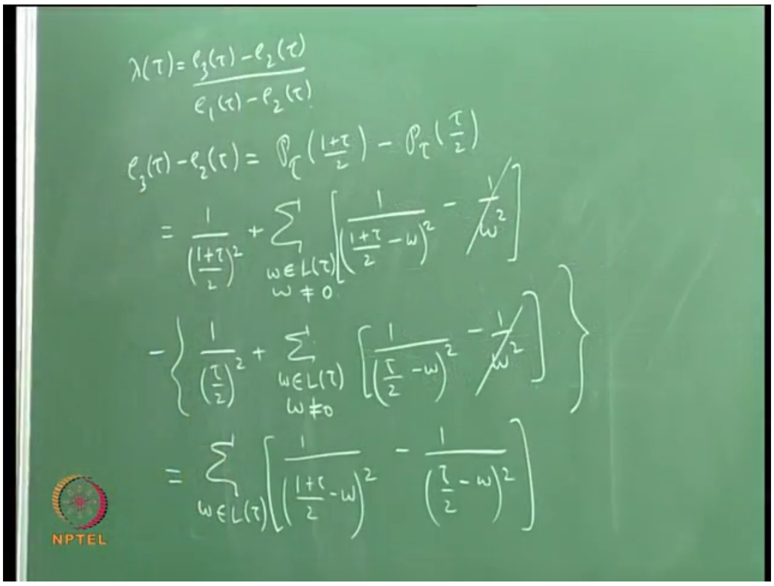

So, this simplifies to summation over omega in l of tau omega, now I remove the restriction omega is not 0. And I simply write 1 by you know 1 plus tau by 2 minus omega the whole square minus 1 by tau by 2 minus omega the whole square, so this is what I get. And now the trick is if you want to use this you change from summing over omega in the lattice to summing over pairs of integers because for omega in the lattice after all omega is of the form n plus m tau. So, we rewrite this as a double summation.

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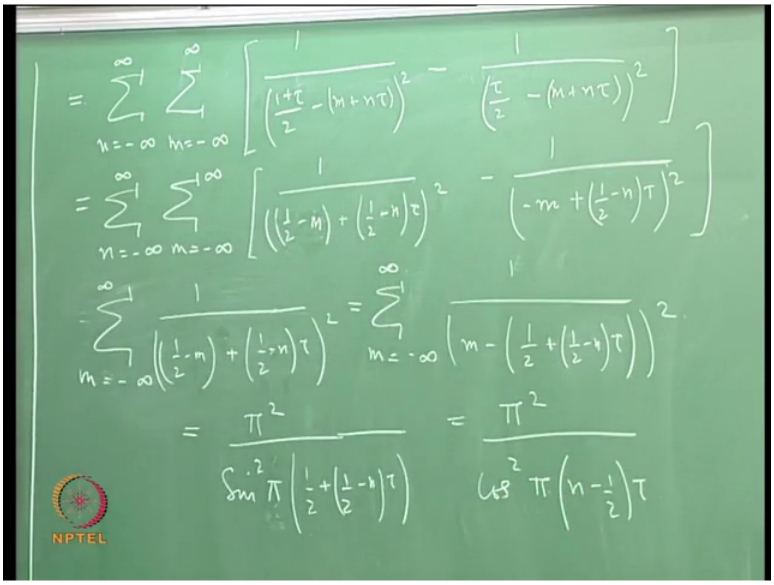

So, what one does is. So, well this equal to so you know now I will write it as c summation over n equal to minus infinity to infinity, another summation over m equal to minus infinity to infinity and I write that as 1 by now you see I am putting omega is equal to m plus n tau. So, what I will get is I will get 1 plus tau by 2 minus m plus n tau the whole square minus the other term which is 1 by tau by 2 minus m plus n tau the whole square. So, this becomes summation n equal to minus infinity to infinity summation m equal to minus infinity to infinity of so I end up with half. So, I can write it as m minus half, so I will get half minus m, I will write it as let me write it first as it is of minus m is the real part. And coefficient of tau is going to be half plus half minus n into tau the whole square minus the other term is going to be well I am going to get minus m plus well half minus n into tau the whole square, this is what I am going to get.

And now you see yeah so now we are in good shape you see just look at look at only the; forget the outer summation in the end look at the summation only in m. And use the fact that summation from m equal to minus infinity to infinity of 1 by m minus z the whole square is pi square by sin square pi z which is what we prove. So, you use that summation m equal to minus infinity to infinity of 1 by half minus m the whole square plus well half minus n into tau the whole square is summation m equal to minus infinity to infinity, what I am going to get is 1 by m minus;

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There is no square here, yeah there is one square here. So, it is going to be m minus half. So, if I switch the sign here its m minus half, if I switch the sign here it is n minus half times tau. So, I guess this I hope this right, and of course, there is a whole square right because. So, I get m minus half which is half minus m, the other one is n minus half it should be n minus half into tau is that right. So, there we have n minus half if I switch it switch the sign I get half minus m, there I get minus of m minus half tau if I switch the sign. So, may be this should be half minus n, here this half minus n.

So, well if I apply that formula then I will get this I square by sin square pi times this whole thing. So, I will get half plus half minus n into tau this what I will get, now if I use the summation that summation formula. And of course, you know if I take sin square if I push the pi inside then I can change this sin square to cos square, and then I can write that as n minus half tau. So, this finally, it becomes pi square by cos square pi into n minus half tau this is what you will get finally, so that is what you get of the first term. And what do you get of the second term, let we write it down.

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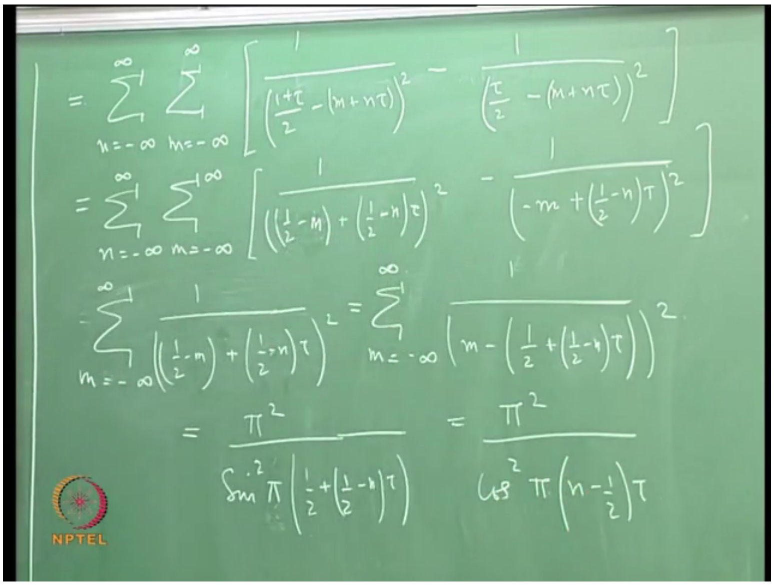

So, I will get sigma m equal to minus infinity to infinity the second term is suppose I calculate 1 by so that is minus m plus half minus n I switch signs throughout. So, I will get m minus half minus n times tau the whole square, if I switch signs that is right. And this going to be pi square by sin square pi by that formula it is going to be half minus n into tau which for uniformity with previous with this one I will write it as pi square by sin square pi n minus half tau. So, this where we have used the formula that we just recently proved.

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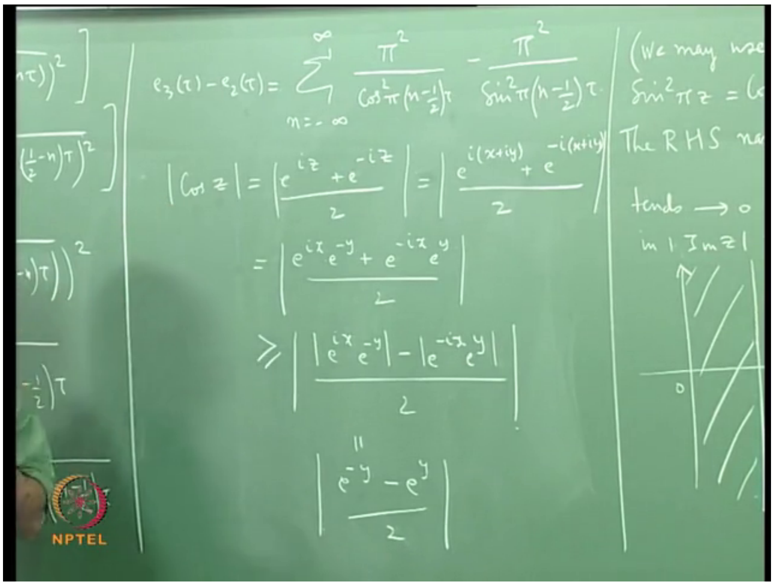

Well, we will end up with e 3 tau minus e 2 tau is equal to there is a summation outside summation n equal to minus infinity to infinity and what you get inside is this minus this. So, it is going to be pi square by cos square pi n minus half tau minus pi square by sin square pi n minus half tau this is what you get. And now you can see that if I let the imaginary part of tau go to infinity then you can see that each term is going to go to 0, and therefore this quantity on the left will also go to 0 given that the series on the right side will converge uniformly. So, you can take the term by its limit.

So, the fact is of course, you can easily verify that the modulus of the denominators that goes to infinity as imaginary part of tau goes to infinity that is very, very simple exercise. Therefore, so what this will tell you is that as imaginary part of tau goes to infinity, this will go to 0; and it will go uniformly to 0 in the variable real part of tau. And the fact is you can take the term wise limit because of uniform convergence. And that term wise limit is 0. So, if you want maybe it will be helpful if I write down that.

So, you see if you for example, you know if I take let us take for example, cos z which is by definition e power i z by c power minus i z by 2 and then you calculate modulus of cos z you will get this. You must have probably seen this in an in a first course in complex analysis and well anyway let me write it down, so that it is perhaps because it is pretty easy. So, I put z is equal to x plus i y. So, I will get and what I will get is e power i x e power minus y plus e power minus i x e power y divided by 2. And then I use a triangle inequality and write that as greater than or equal to modulus of maybe I use yeah I use triangle inequality and write this as modulus of e power i x e power minus y minus e power minus i x e power y by 2. And which turns out to be well e power minus y minus e power y by 2, this is what I get.

And you can see that as the imaginary part of z goes to imaginary part of z is y and as that goes to infinity it is clearly this quantity is going to go to infinity, and therefore, mod cos z is going to go to infinity. Essentially you will get the same kind of inequality if you also put sin z, and therefore, as imaginary part of z goes to infinity both mod cos z and mod sin z go to infinity therefore, the reciprocals will go to 0. And that is the reason why as imaginary part of tau goes to infinity, these term will go to 0 this term will also go to 0 because it will go to 0 in mod in its modulus in its absolute value. And therefore, term wise the limit is 0 and you can take the limit term wise because the convergence is uniform.

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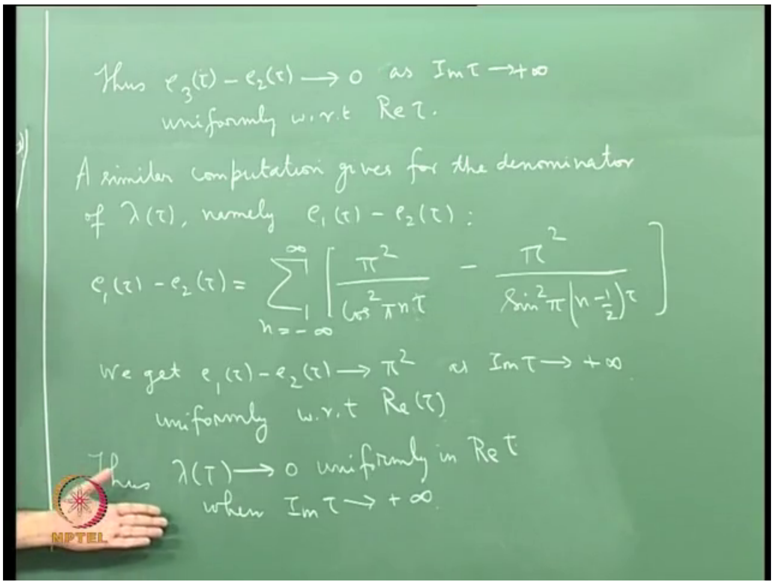

So, sort of all this the following namely that, so let we write that down. So, maybe I can put this in brackets. Thus e 3 of tau minus e 2 of tau tends to 0 uniform as imaginary part of tau tends to infinity uniformly with respect to real part of tau. So, we get this. So, this is what we get for, this is what we get for the numerator of the lambda function as imaginary part of tau goes to infinity. A similar calculation can be done for the denominator as well. A similar calculation computation gives for the denominator of lambda of tau namely it is e 1 tau minus e 2 tau e 1 tau minus e 2 tau.

If you follow the same procedure remembering that e 1 tau is p tau of half and e 2 tau remains as p tau of tau by 2 and do the same thing. What you will end up getting is the following, you will get summation n equal to minus infinity to infinity pi square by cos square pi and tau minus pi square by sin square pi n minus half tau. So, you will get this you just have to follow the same steps and you will get this. And now well the same argument applies as we decided for this case when the imaginary part of tau goes to infinity. The only difference is that there is a term n equal to 0, the term n equal to 0 is going to completely remove this denominator.

So, you take a limit it does not have any effect on this term. So, what you are going to get is you are going to get pi square all other terms are good. So, only the n equal to 0 term and that too only this part is going to be unaffected, all other terms will die to 0. So, what you will get is we get e 1 of tau minus e 2 of tau tends to pi square as imaginary part of tau tends to plus infinity uniformly with respect to real part of tau. So, if I put these two together I get that as tau tends as imaginary part of tau tends to plus infinity lambda of tau goes to 0 uniformly in real part of tau which was the original claim. So, thus lambda of tau tends to 0 uniformly in real part of tau, when the imaginary part of tau goes to plus infinity. So, that tells you that at infinity you can extend lambda at infinity to the point at infinity by defining it to be 0 and that will extend it continuously, so that is the part of the claim.

Now, we need to still prove further claims, and I will do that in the forthcoming lectures.