An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves
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Lecture – 35
The Weight Two Modular Form assumes Real Values on the Imaginary Axis in the Upper Half-plane

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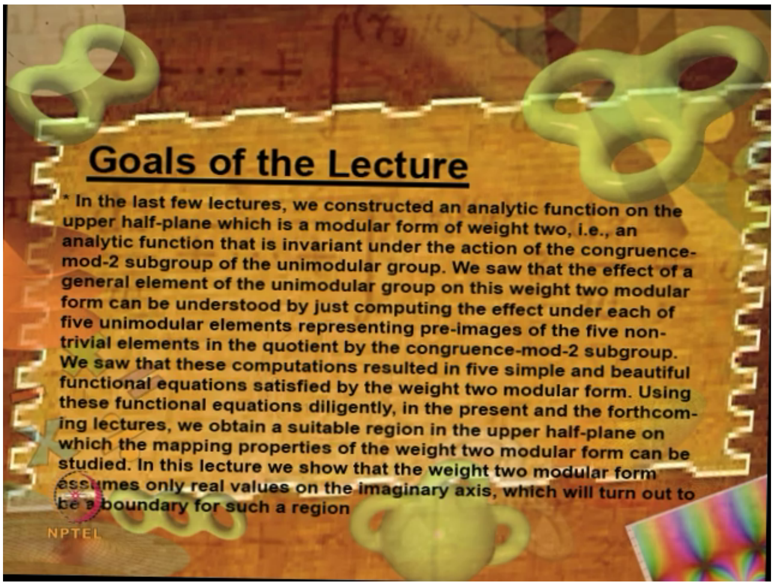

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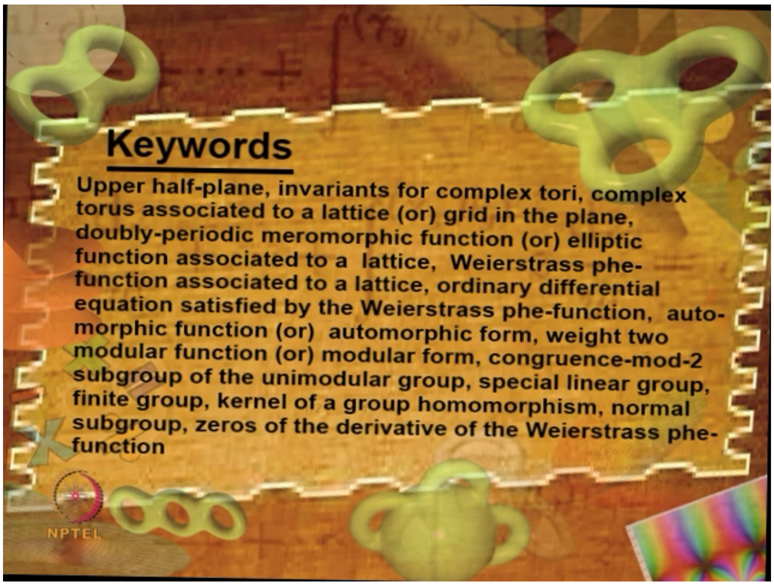

So, last time you see; we were looking at the modular function lambda and you know that this modular function is only invariant under the congruence mod 2 subgroup. It is not invariant under the action of an general element of the uni-modular group PSL 2 Z. So, we ask the question what will happen if you apply you know general element of PSL 2 Z, then we found that this modular function lambda satisfied certain nice functional equations which we derived. So, just to put our discussion in proper perspective, see we are trying to get hold of a modular function which is which is which is modular for the whole group the whole uni modular group PSL 2 Z; what we have at present is the function lambda which is which is invariant only for the congruence mod 2 subgroup alright.

So, the aim is we have to use this lambda to cook up another function which will be invariant under the action of the full uni modular group. So, that leads us to study the mapping properties of lambda. So, let me recall among various recurrence relations satisfied by lambda we have the following. So, in particular you see we have.

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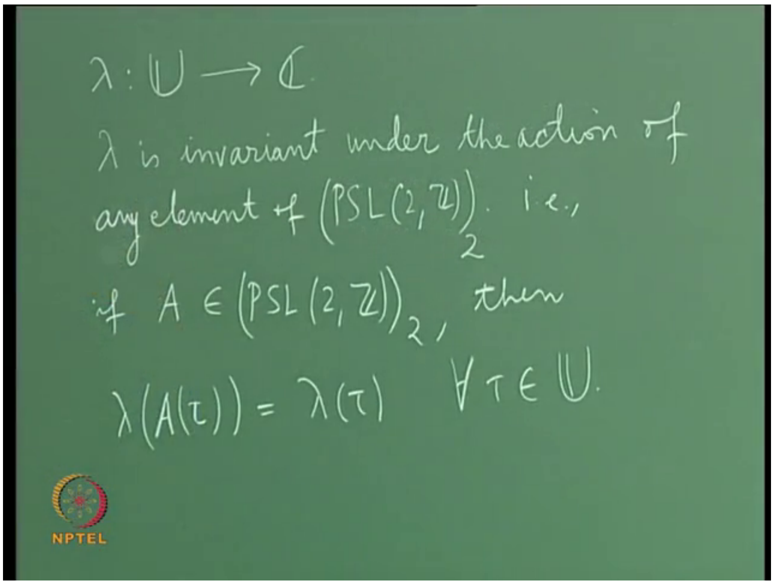

So, you see we have lambda from the upper half plane with values in C this. So, this lambda is invariant under the action the action of a of an element an element of any element of PSL 2 Z subscript 2 this is the congruence mod 2 subgroup; in other words you see if well a is an element of this congruence mod 2 subgroup then you know if I take a tau in the upper half plane and apply a of tau of course, it is going to give me another point in the half plane because after all PSL these are all these are all elements in PSL 2 r which are automorphisms to the half plane and then if I apply lambda to this I will simply get back lambda evaluated at tau. So, this is the invariance property. So, we have seen this and see. In fact, we saw.

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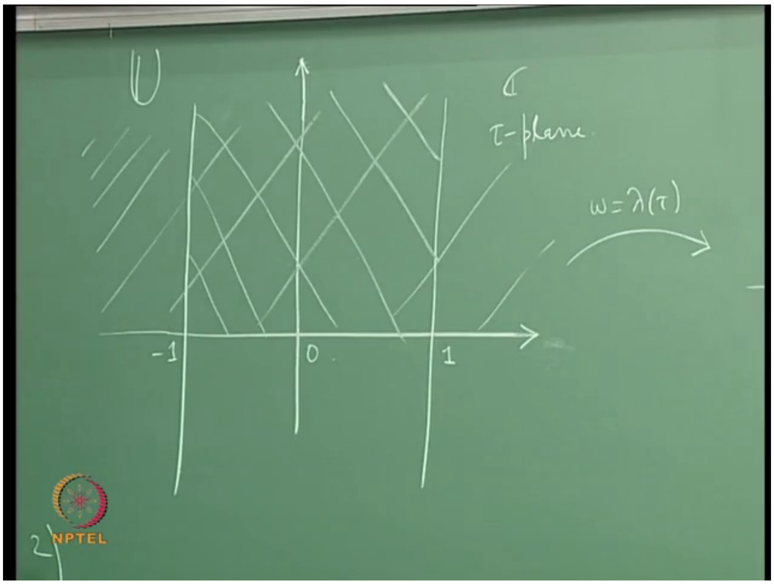

So, you know see we would like to know that if you if you look at the; well the complex plane which I call as the tau plane and then I take the mapping omega equal to lambda of tau.

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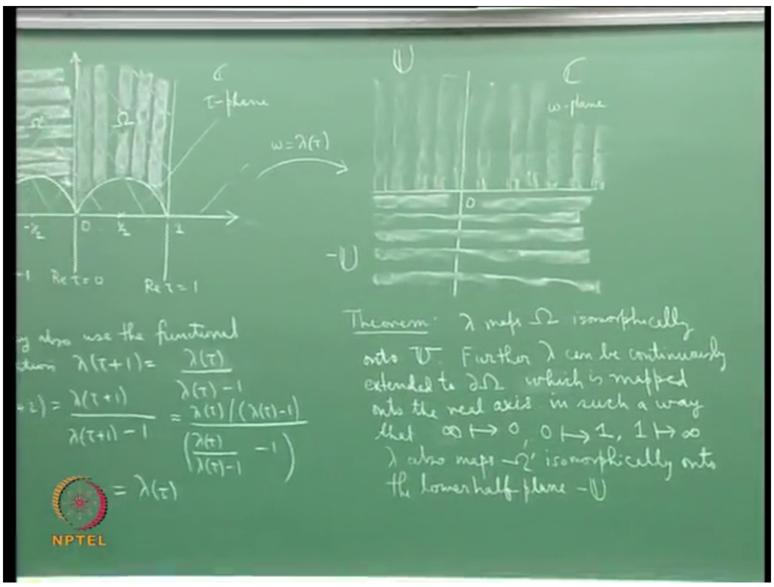

So, this is a mapping that is going to be that is going to go into take values in another complex plane which I will call the omega plane and well we are looking at of course, you see for the movement lambda is defying on the upper half plane, alright, So, lambda is defined here this is this is upper half plane. So, lambda is defined in this shaded region alright.

So, it is defined here and so, lambda is defined here and you know of course, lambda is analytic lambda never takes the values 0 or 1 and we want to know what is the what kind of a mapping lambda is of course, you see lambda is it is an analytic function. So, it will have good mapping properties, but the question is what are those mapping properties we need to we need to understand that.

So, the first thing I want to tell you is that you see lambda.

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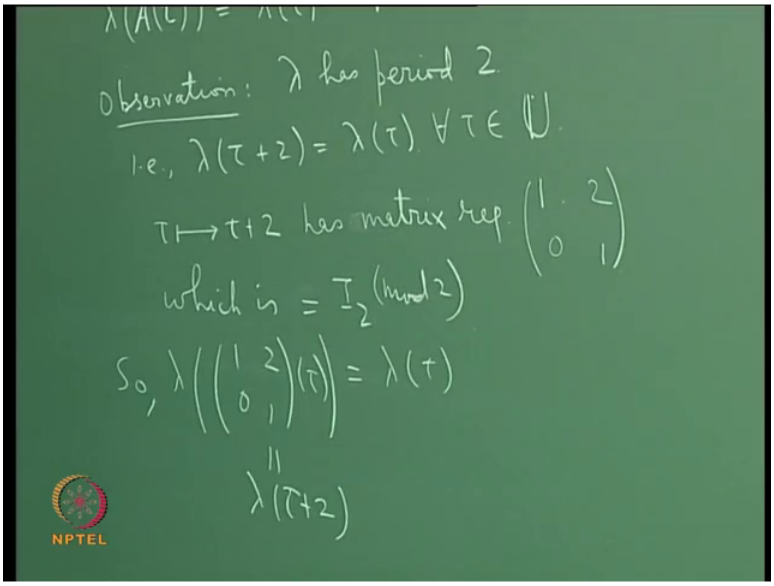

So, the first observation is that lambda has period 2. So, this is the first observation see that is that is lambda of tau plus 2 is lambda of tau for all tau in the upper half plane the first property is that lambda has period 2 now how does one see this well you see there are 2 ways of seeing this one thing is you see the transformation tau going to tau plus 2 this has matrix has representative has matrix representative well the matrix representative will be you know for a general linear transformation a tau plus tau going to a tau plus b by C tau plus b the matrix representation is a b c d the matrix a b c d and of course, you always formulize to make sure that this is the that the determinant of that matrix is one.

So, that it is uni modular and well this is going to be that way it is going to be 1 2 0 1 this is the matrix representative and you see you can see that this matrix if I read it mark 2 it is identity matrix see which is equal to the which is I 2 mod 2.

So, this matrix mod 2 is identity, therefore, you see this is in the this is in the congruence mod 2 subgroup after all the congruence mod 2 subgroup consists of all those elements which when you read mod 2 give you identity that is you have to read every coefficient mod 2 alright this is in the this clearly this matrix is in the congruence mod 2 subgroup and you know that the function lambda is invariant under such an element. So, you see. So, lambda of if I apply 1 2 0 1 to tau these are same as lambda of tau. So, of course, applying 1 2 0 1 to tau means it means that you are applying the Moebius transformation z going to one z plus 2 by 0 z plus 1 to tau. So, which means your; so, the left side is just left side is just lambda of tau plus two. So, lambda of tau plus 2 is lambda of tou.

So, the moral of this story is that if you want since lambda has period 2, alright. So, it is enough to study to study you do not have to study lambda on the whole the effect of lambda on the whole upper half plane it is enough to study on a vertical strip of length 2. So, therefore, what we do is that we actually. So, we do the following we restrict our; it is enough to restrict our attention to the strip. So, I will take this strip. So, I will take this strip namely you see I will take this as one this as minus one. So, it is enough to study lambda only in this region the effect of lambda on this region that will do it is enough to study the effect of lambda on this cross shaded region you can restrict to a vertical strip of horizontal length 2 alright.

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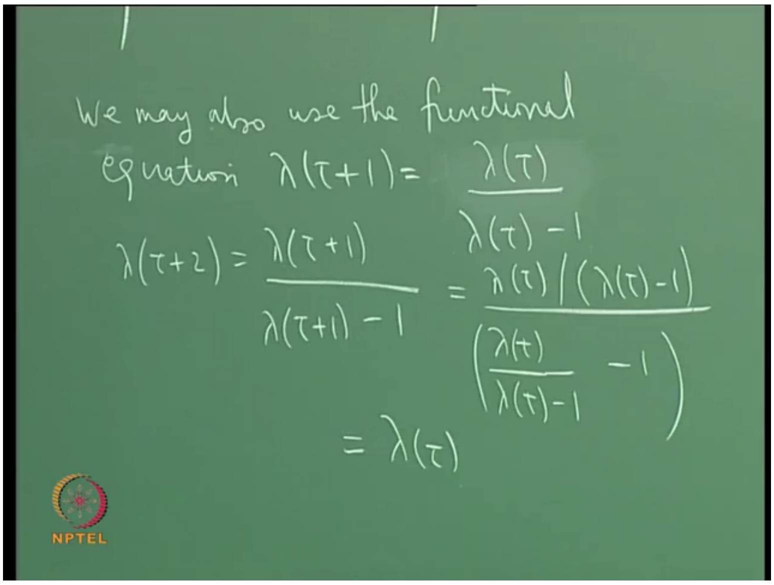

So, it is enough to study lambda only here alright now well of course, there is there is also another way there is another way of looking at it you can also use you can also use the fact we may also use. So, maybe I maybe; I will draw line here we may also use the functional equation the functional equation lambda of tau plus 1 is I guess 1 minus lambda of tau see we proved this last time lambda of tau plus 1 is 1 oh no I think it was it is lambda tau by lambda tau minus 1 sorry.

So, that was lambda of minus 1 by tau. So, this is lambda tau by what was that it was lambda tau minus 1 and now you know you apply this twice alright, then you will get. So, lambda tau plus 2 will be lambda tau plus 1 by lambda tau plus 1 minus 1 and then you apply this to this you will get lambda tau by you know lambda tau minus 1 divided by lambda tau by lambda tau minus 1 minus one and you will see that this is going to give a lambda of tau. So, that there are 2 ways of seeing this right of course, this is this is very straight forward right alright.

So, the situation is now that I it is enough to study the effect of lambda on this on this region then the. So, what is it that what is the kind of result we are going to get? So, let me state what state the result that we will get it is the following. So, let me write the theorem.

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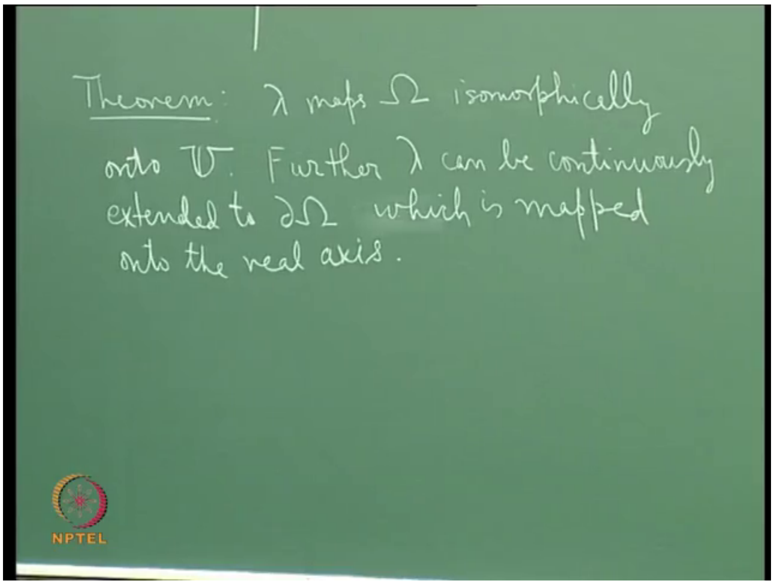

So, this you see let me let me draw the circle here which is centered at half and radius half.

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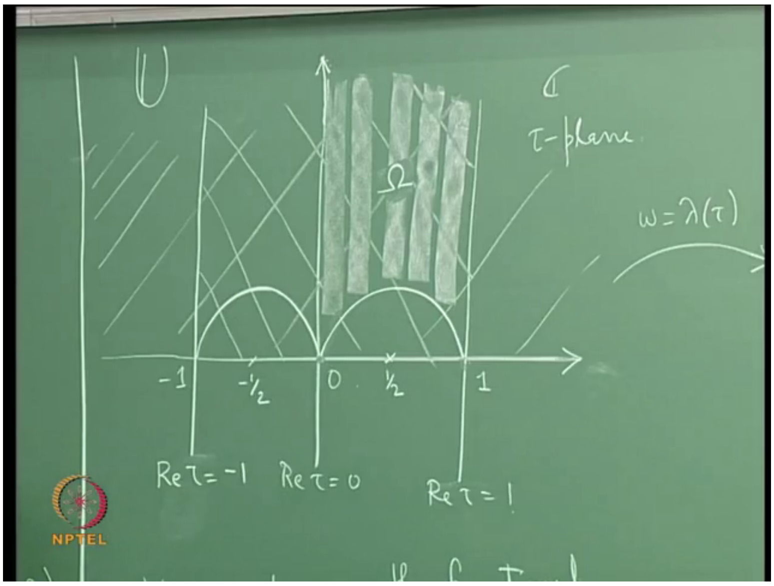

So, I get this; I get this region and similarly I can draw a circle centered at minus half again with radius half. So, I will get something like this and let me look at this region let me look at this region, let me look at this region that is bounded by the imaginary axis by this circle and by this line which is a real part of real part of tau equal to 1.

So, you see this is this is real part of tau equal to 0 this line is real part of tau is equal to 1 and of course, this line is well real part of tau equal to minus 1. So, you look at you look at this you look at this region that I have not shaded it fully, but the boundary is supposed to be consisting of this portion of the imaginary axis followed by this a semicircular arc and then this portion of the line passing through one vertical line passing through one let me call this region let me give a name to this region, let us call this region as omega let me call this region as omega alright then the theorem is following the theorem is lambda max omega injectively. In fact, let me say lambda max omega isomorphically on to the upper half plane.

So, it is very beautiful. So, this region you see this region here this region omega the by omega I mean only the interior of the region do not include the boundary curve; the bounding curves. So, the that interior is of course, an open set and the fact is that this open set is completely mapped isomorphically by lambda on to the upper half plane. So, the image of that is this whole upper half plane and the and the fact is that you see it is the fact is that it is an isomorphic it is isomorphic; that means, it is injective and holomorphic of course, lambda is of course, holomorphic, right. So, to prove this you just have to prove that the map lambda restricted to this open set is one to one and that it takes every value in the upper half plane if you do that; then you will get; you will get this the proof of the statement.

Not only that of course, you know you expect a confirmer map when it maps a certain region to a certain region, then it has to map the boundary to the boundary alright. So, the beautiful thing is that this of course, you know lambda is defined only for the upper half plane, but the fact is that you can extend lambda to the boundary of this region omega in such a way that you can expect where the boundary has to go you see the boundary has to go the boundary here the boundary is the real axis. So, the fact is that lambda can be continuously extended to the boundary.

So, that the boundary of omega goes to the real axis. So, let me write that down further lambda can be continuously extended to the boundary of omega which means you know this portion of the imaginary axis followed by this semicircular arc and then this portion of the of this line. So, that. So, that which is mapped which is mapped by which is mapped onto the real axis which is what you should expect because the boundary of this region has to go to the boundary of that region.

So, you see; so, this whole thing is going to map get mapped to the real axis and the and which values are going to be mapped to which. So, you see lambda the value of lambda at the point at infinity will turn out to be 0; so, in such a way in such a way that 0. So, if the point at infinity goes to 0. So, on this on this complex plane there is a point at infinity you which you must think of as you must actually think of this stereographic projection and think of the point at infinity as that point goes to actually 0, then the point 0 goes to one 0 goes to one this one will go to infinity this one here will go to the point in infinity.

So, you see what is happening as you come from the point at infinity on the on the Riemann sphere to 0 to one and again go back to the point at infinity on the Riemann sphere what happens is the lambda values go from 0 to one to infinity and then back to 0 that is how lambda that is how lambda is extended to the to the boundary. So, this is the theorem. So, I mean the importance of this theorem is that because of the behavior of this region under several of those mappings which were the mappings that you got in the congruence mod 2 subgroup when you read it in z mod 2 you this allows us to extend I mean to cook cup from lambda a function which is modular on the for the whole uni modular.

So, the key lies in this in studying this region the key lies in studying this region and this region is kind of fundamental for the way lambda behaves the way lambda maps. So, we will have to prove this. So, this is roughly the roughly the aim and in fact, I should also say that you know I could have also taken I could have also taken the mirror image of this region I can take the mirror image of this region about there the imaginary axis and you know I will get this region here I will get this region I will get this region alright. Now let us call this region as something; let us call it as let us say omega prime the fact is that this omega prime will be mapped by lambda to the lower half plane ok.

This omega prime will be mapped by lambda onto the lower half plane and again the the mapping will extend to the boundary with the same properties and mind you the values of lambda here are exactly the same as values of the lambda here because this differs by 2 which is a period of lambda alright. So, what will happen is that you see the image of this is going to be is going to be this; the lower half you are going to get this. So, let me write that lambda also maps omega prime isomorphically onto the lower half plane which is which I will call as minus u. So, this is this is well this is u. So, this is the upper half plane this is minus U this is the lower half plane. So, this is how lambda behaves this is how lambda behaves and of course, the. So, the effect of the mapping on these 2 pieces will give you the full image and of course, you will have to extend the mapping to the boundary to this boundary and of course, if you have extended it automatically extends here. So, you have to only extend it here right.

So, this is what we have prove this is what we need we need to prove and as a first step towards that you see. So, I am going to try to prove this. So, the first thing I am going to try to show is that that the value of lambda on this on this boundary I am going to show the value of lambda on this boundary is actually real because after all you see you see what lambda is doing what lambda is doing is that you see it is mapping this whole thing onto the real line now. So, for that let me again draw this draw the diagram here and make a certain observation.

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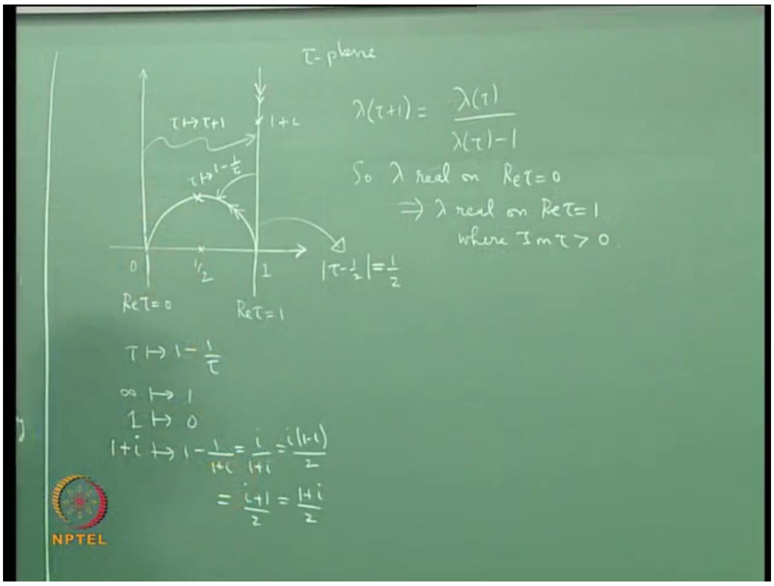

So, here is. So, this is 0 this is one this is the tau plane. So, you have. So, you have. So, you have the semicircle here ok.

So, you see this is real part of tau equal to 1 this is this is real part of tau equal to 0 and of course, you know this, this semicircle is you know mod tau minus half is equal to half this is the this is the equation of a circle centered at half and radius half right now what you must understand is that or what we can observe immediately is that you see this imaginary I mean this imaginary axis is mapped by tau plus 1 onto this.

So, you see if I if I apply tau going to tau plus 1 this every point here goes to a corresponding point here alright and the point in the upper half plane goes to a point in the upper half plane 0 will go to 1 alright and notice that we know that lambda of tau plus 1 is as I wrote it down here lambda of tau plus 1 is lambda tau by lambda 2 minus 1. So, you know what is the advantage of this functional relation it is if I prove lambda is real on this if for tau on this if I prove lambda of tau is real then this will tell me that lambda is also real on this ok.

So, that is the advantage of this functional equation. So,. So, lambda real on a imaginary axis implies lambda real on real part of tau is equal to 1 and of course, of course, I should I will have to worry about the 0.0 and the 0.1 because for the moment lambda is only defined on the upper half plane. So, what I am saying here applies only to only when imaginary part of tau is greater than 0. So, let me write that down where imaginary part of tau is greater than 0.

So, you will have to leave out this point and you have to leave out this point right then notice also that you see this; this line is mapped on to the circle by the transformation. So, there is a transformation that s going like this and this transformation is none other than tau going to 1 minus 1 by tau you see take the transformation tau going to 1 minus 1 by tau; it is a Moebius transformation it is certainly a Moebius transformation and you see and you know that a Moebius transformation will map straight lines to straight lines or circles and circles to straight lines or circles you know that is a fundamental property of Moebius transformations.

So, you see if you take this tau going to 1 minus 1 by tau you see if I put a tau if I put tau equal to infinity. So, infinity will go to one tau minus tau going to 1 minus 1 by tau if you calculate this infinity goes to one the point at infinity goes to one the point at infinity goes to one alright and then well infinity goes to one then if I take the point one with if I take the 0.11 goes to 0 I mean I should maybe I should put now that s all right I mean I am only worried about this transformation I am not applying lambda.

So, if I put tau equal to 1, one goes to 0 alright and then well if I put something in between; say for example, suppose I put tau is equal to suppose I put tau equal to 1 plus i which is which is this point here which is which is this point here 1 plus i somewhere here this length being 1, then 1 plus i will go to what it will go to 1 minus 1 by 1 plus i which if you calculate it is 1 plus i minus 1. So, it is i by 1 plus i and that turns out to be I into 1 minus i by 2 and this is i plus 1 by 2 this is. So, it is 1 plus i by 2 which is this point it is half plus i by 2.

So, you see this Moebius transformation maps infinity 1 plus i by 2 1 2 1 1 plus infinity 1 plus i 1, in that order to 1 1 plus i by 2 0. So, you know. So, this is the confirmality as you go from as you move from infinity to one h the image of this line is traces this semi circle in this order. So, you see this is mapped onto this now you have another functional equation this the second functional equation that we saw was that lambda of minus 1 by tau plus 1.

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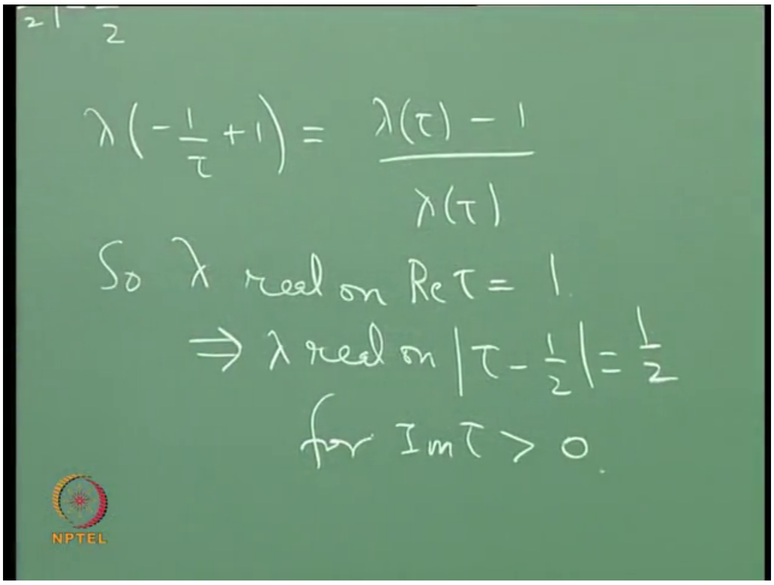

This turned out to be well if you recall that it is it is lambda tau minus 1 by lambda tau we have this, we proved also this functional equation. Now what does this tell you this tells you that if you knew that lambda is real on this you can then conclude that lambda is real on this ok.

So, lambda real on real part of tau equal to 1 implies lambda real on this segment on this on this semi circle for imaginary part of tau. So, you see to show that lambda is actually real on this boundary I have to only show that lambda is real on this imaginary axis and I have to worry about the point 0 I have to worry about the point 1 that s all I will have to do. So, you see. So, the first part of our discussion proceeds to show that lambda is real on the imaginary axis. So, lambda is real on the imaginary axis. So, let me write that down.

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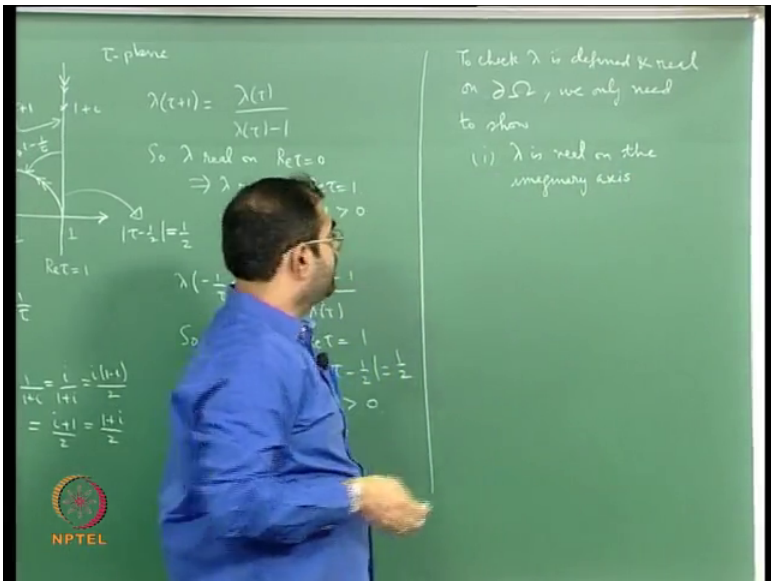

We only need to yeah to check lambda define is defined and real on the boundary of omega del omega we only need to show.

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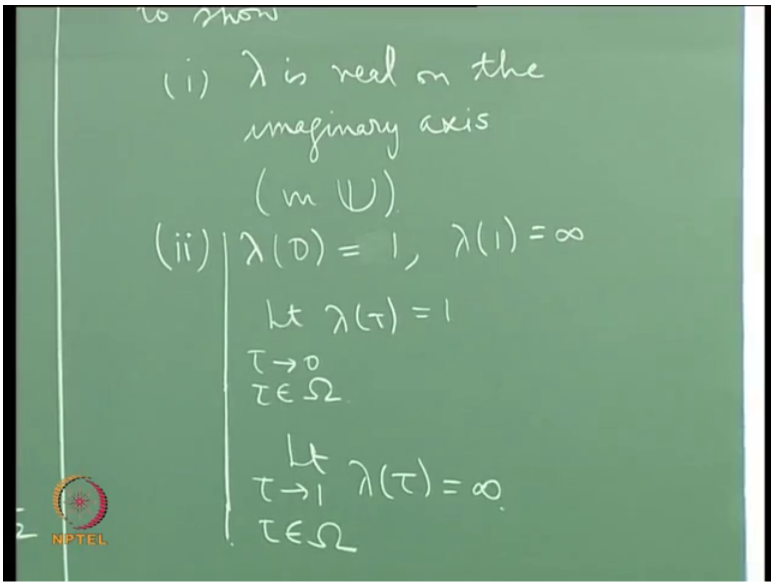

Number 1 lambda is real on the imaginary axis, we need to show this in U of course, and number 2 lambda at 0 and so, you have to show lambda of 0 is 1; lambda of 1 is infinity and you know I need to also show unit as tau tends to 0 lambda of tau a tau inside omega is one limit tau tends to one tau inside omega lambda of tau is infinity. So, this is what I have to show see if I show this. So, if I show this then I will know for sure that you see that this boundary is mapped onto the real axis and that it and that the only 2 troublesome points are these 2 points and there that s those are the points where I will have to check continuously. So, that I make sure that the mapping when you extend it to the boundary is also continuous. So, I will have to verify these 2 limits as well I have to check the limits are these values and then I will have to verify these things. So, this is going to be the first part of our discussion right. So, how does one go about this?

So, to begin with now I need to go down to go back and define a more general function. So, let me do that let us recall that you see we need to define a more general (Refer Time: 36:39) as phi function we need to do this.

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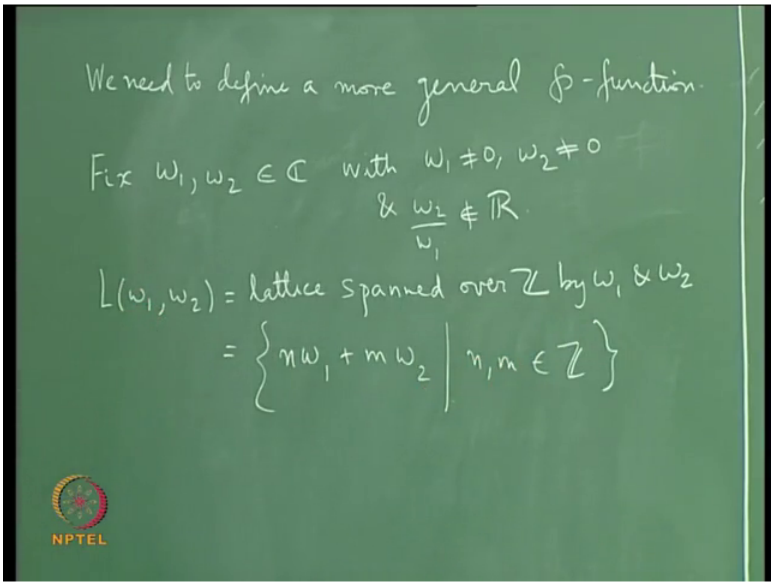

So, what do I mean by that. So, what you do is while you fix a 2 complex numbers omega 1 omega 2; 2 complex numbers with a the ratio with omega 1 of course, both not 0 both not 0 and omega let us say omega 2 by omega 1 is non real that is omega 1 and omega 2 are linearly independent as elements over r the real numbers then associated with this we will have a lattice ok.

So, you get L of omega 1 comma omega 2 this is the lattice spanned over z by omega 1 and omega 2 and what is this? This is just the z span namely it is all z linear combinations of omega 1 and omega 2. So, it is of the form n omega 1 plus n omega 2 where n and m are integers and of course, the lattice that we have been so far considering was L of 1 comma tau where tau was in the upper half plane, but instead of taking one and tau we are simply taking 2 complex numbers such that the ratio omega 2 by omega 1 is non real alright.

Now you if you if you recall you take you take the complex plane and then you go modulo this lattice namely you declare 2 complex numbers to be equivalent if they differ by an element of this lattice in other words you are thinking of the lattice as the group of Moebius transformations that act by translation by elements of the lattice and then you are going modulo that group what is the result the result is again at complex holomorphic torus it is a surface. So, what you get.

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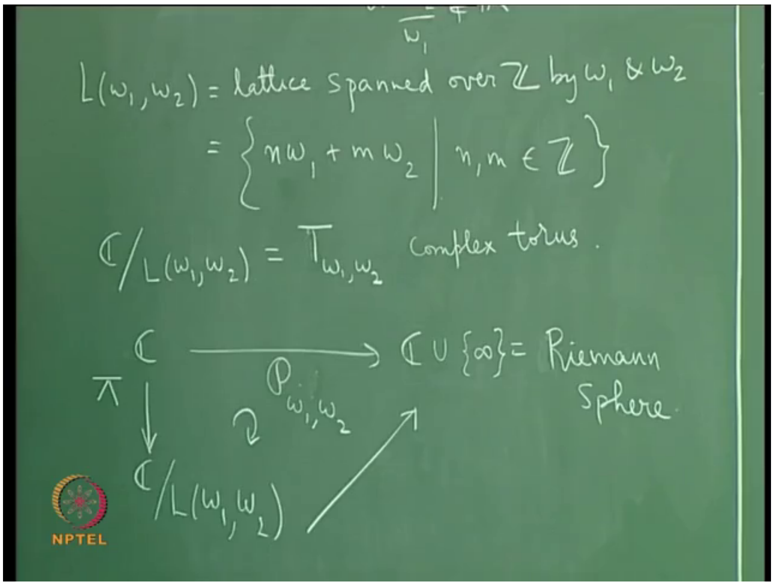

So, this what you get here is a torus t sub well t sub omega 1 comma omega 2 it is a complex torus complex torus and of course, you know you will you will see that the way we got the Riemann surface structure on this torus was such that this mapping pi is a holomorphic map and in fact, this mapping is a holomorphic universal covering for this torus and the fundamental group of this torus can be identified with the lattice above as the deck transformation group of this covering and the deck transformations are precisely elements of the lattice being thought of as translations on which are automorphisms of C ok.

So, this is the picture and again see our aim is to get hold of a simple function on this; simplest possible analytic function on this; of course, you will again not get an analytic function on this because this is compact; any analytic function on this will be constant. So, you will see that you will have the simplest function you can think of this will be a (Refer Time: 40:42) function with which will shows up as a (Refer Time: 40:46) function above which is invariant under the lattice.

Under the translations by the lattice, it will have a double pole just like the phi function, that we saw for the case when this was one and that was tau, it will have again a double pole at each point of the lattice with residue 0 and again what happens is that you will get you will get a (Refer Time: 41:15) function let me call this as P sub omega 1 comma omega 2 and this will take values in C union infinity this is the Riemann's sphere this will be the Riemann's sphere and of course, it will go down to give you will get a function here which is a simplest metamorphic function you can think of on a complex.

Now, and how do you define this P sub omega 1 comma omega 2 it is the formula is literally the same form same as the formula that we that we used to define the phi function when omega 1 was one and omega 2 was tau. So, the formula is pretty the same pretty much the same.

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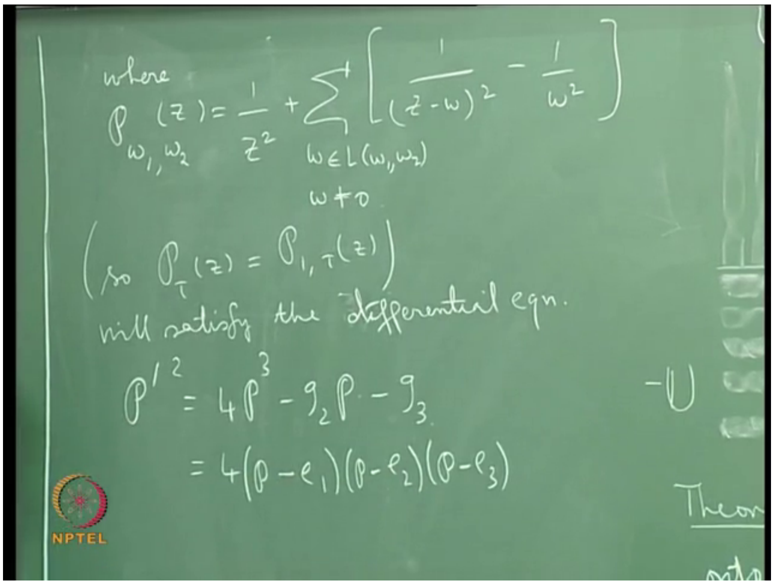

So, you see phi sub free sub omega 1 comma omega 2 of z turns out to be 1 by z square plus summation omega varying over this lattice of omega 1 comma omega 2 omega not equal to 0 1 by z minus omega the whole square minus 1 by omega square, it is a same; it is literally the same of the same form as you as the as the phi function we define then omega 1 was 1 and omega 2 was tau.

So, phi tau of z is just phi in this notation it is phi 1 comma tau the original the function that we defined earlier is just phi 1 comma tau you take omega 1 as 1 and omega 2 as 2 then again you can see that the arguments that we had for P tau of z about its properties we will all hold for this and. In fact, again just like P tau of z satisfied differential equation this will also satisfy a differential equation alright. So, what will happen is that you will get this will satisfy the differential equation?

So, that is the that is the same its literally the same differential equation that we had earlier and that s going to be well P phi prime square is equal to 4 phi cube minus g 2 phi minus g 3 you are going to get the same differential equation where of course, this phi is phi sub omega 1 comma omega 2 you are going to get the same differential equation and you can continue to factorize it as 4 times phi minus e 1 into phi minus e 2 into phi and into phi minus e 3 and to determine what e 1, e 2 and e 3 are we look at the 0s of phi prime and you find that e 1 is actually phi of omega 1 by 2 e 2 is phi of omega 2 by 2 and e 3 is phi of omega 1 plus omega 2 by 2 you get all these things alright and the arguments are exactly the same as they were when omega 1 was one and omega 2 was tau ok.

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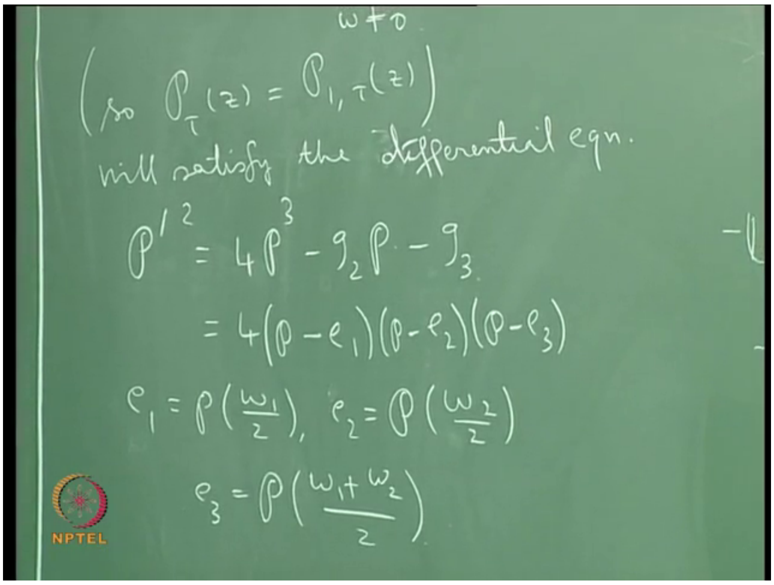

You can do you can try writing that out as an exercise and you will see that literally the same arguments do it alright now well the advantage of this is that I can I mean the advantage of all of this is that you know I can replace omega 1 by one and omega 2 by minus tau where tau is in the upper half plane see in the original phi sub tau that we defined tau was in the upper half plane. So, I could not replace tau by minus tau because if tau is in the upper half plane minus tau is in the lower half plane and I have not defined the phi function when tau is in the lower half plane. So, it is only to overcome that difficulty that I am looking at this more general phi function; now what I want to tell you is that.

So, I want to tell you that if you now look at this these functions yes. So, what I want to tell you is that I just want to say that if you now put omega 1 equal to 1 and omega 2 equal to tau, but now do not assume tau to be just assume tau to be imaginary non real assume it to be complex not real. So, it could lie in the upper half plane or it could lie in the lower half plane then these 3 become functions of tau on the which may lie either in the upper half plane or in the lower half plane alright, then I want to say that these 3 as functions of tau they are real on the imaginary axis. So, you see. So, let me write that down.

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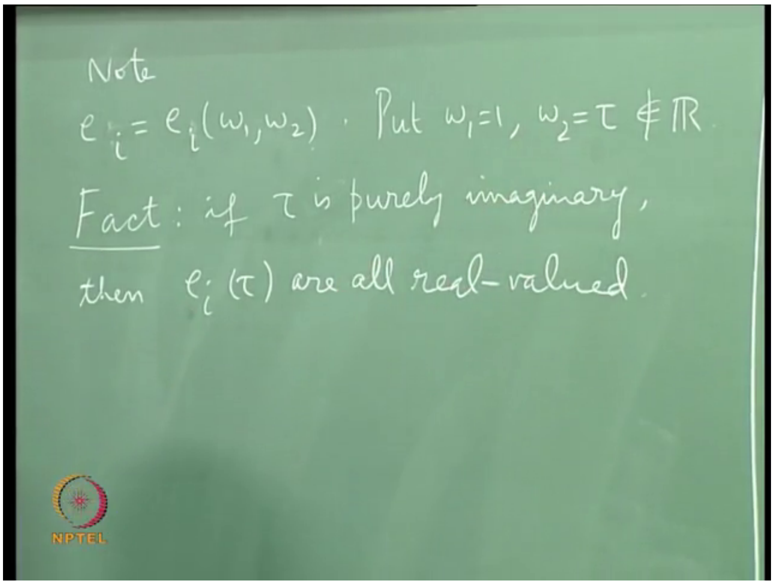

So, you see e 1 note that e 1 is actually a function of omega 1 and omega 2 that is I mean each of the e I s are a function of omega 1 and omega 2.

So, the e 1 is phi. So, from here onwards phi is actually phi sub omega 1 comma omega 2. So, the phi depends on omega 1 and omega 2 alright. So, e 1 is this phi of omega 1 comma omega 2 evaluated at omega 1 by 2. So, what you must understand is though I write e 1 is P of omega 1 by 2 you should not misinterpret it to think that e 1 depends only on omega 1 e 1 also depends on omega 2 similarly e 2, e 3. So, all the a s are functions of both omega 1 and omega 2 alright. Now see the fact is if a tau is if tau is a purely imaginary if tau is purely imaginary; that means, either tau is in the upper half plane or a and in the on the imaginary axis or it is in the lower half plane and on the imaginary axis.

Then the e i of tau or all real they are all real value then these a s are all real value and what is e what is the proof for that. So, let me take let me compute. So, you know so, but before this I need to tell you I need to ask you to put omega 1 equal to 1 omega 2 is equal to tau which is not an element of which is not real. So, put omega 1 equal to 1 put omega 2 is equal to tau where tau is not real and then the fact is if this tau the tau may see tau is not real. So, it could still have a real part tau is not tells you that it has an imaginary part which is not 0 it could have a real part if further that real part is 0; that means, that is the case when tau is purely imaginary then I say that e I of tau are all real value.

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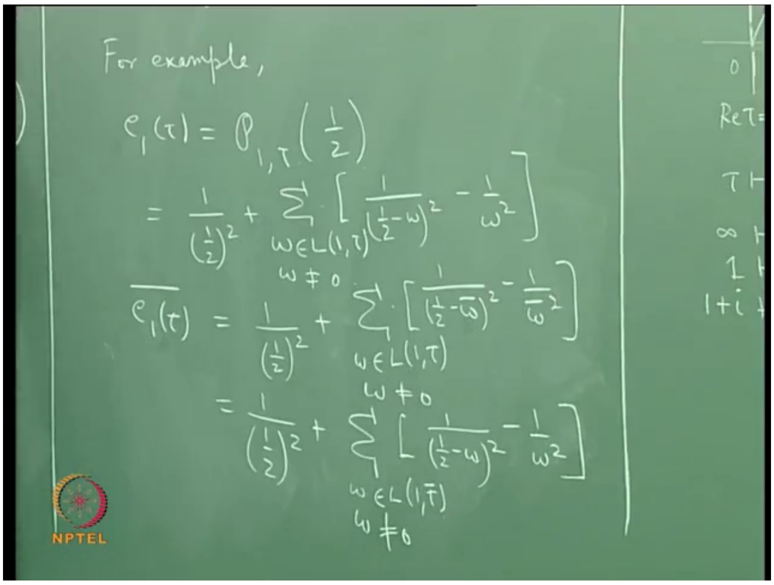

So, you see. So, when I write e i of tau now it is really a function of only tau because the only variable here is tau alright; now let us calculate this for example, what is e 1 of tau e

1 of tau is by definition P of 1 comma tau of 1 by 2 this is what it is by our definition and you see this is what is this; this is by our definition it is one by one by 2 the whole square plus summation over omega in the lattice generated by one and tau omega not equal to 0 one by one by 2 minus omega the whole square minus 1 by omega square this is what it is this is what e 1 of tau is.

Now, you see calculate e 1 of tau conjugate calculate e 1 of tau conjugate e 1 of tau conjugate if i. So, if I take conjugate on this on this side if you watch this is not going to be affected if I take conjugate here you see what s going to happen is that this omega is going to be replaced by its conjugate. So, what I will get is I will simply get one by one by 2 the whole square plus summation over omega in L 1 comma tau omega not equal to 0 I simply get one by half minus omega bar the whole square minus 1 by omega bar square this is what I will get alright now you see that if tau is purely imaginary, then tau bar is minus tau.

So, what you will see is that you see that this will be the same as this which will tell you that therefore, that e 1 of tau is real if tau is purely imaginary. So, because you see what is omega bar see omega bar will be an element of the form n plus m tau I mean omega will be an element or element of the form n plus m tau therefore, omega bar will be an element of the form n plus m tau bar ok.

So, you know this can be written as 1 by 2 the whole square plus summation over well omega belonging to L of 1 comma tau. So, now, you see I replace omega bar by omega if I replace omega bar by omega in the summation I have to replace tau by tau bar alright. So, I have to put tau bar and I will have to put omega not equal to 0 and I can write the same old expression I can now write it as half minus omega the whole square minus 1 by omega square I can do this because I have replaced omega by omega bar and replacing omega by omega bar is to compensate for that I will have to replace tau by tau bar, but you see tau bar is if tau is imaginary tau bar is minus tau. So, L of one comma tau bar will become L of 1 comma minus tau, but the lattice generated by one and minus 2 is same as lattice generated by 1 and tau.

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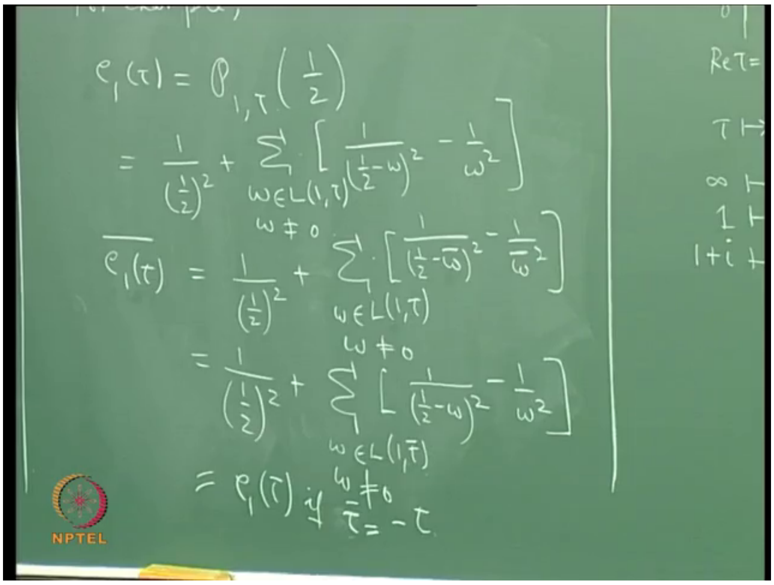

So, this will be simply equal to e 1 of tau if tau bar is equal to minus tau.

So, the moral of the story is that if tau is imaginary then tau is purely imaginary then e 1 of tau is real alright. So, from this it follows that lambda is real because lambda was cooked up from these things. So, let me write that down.

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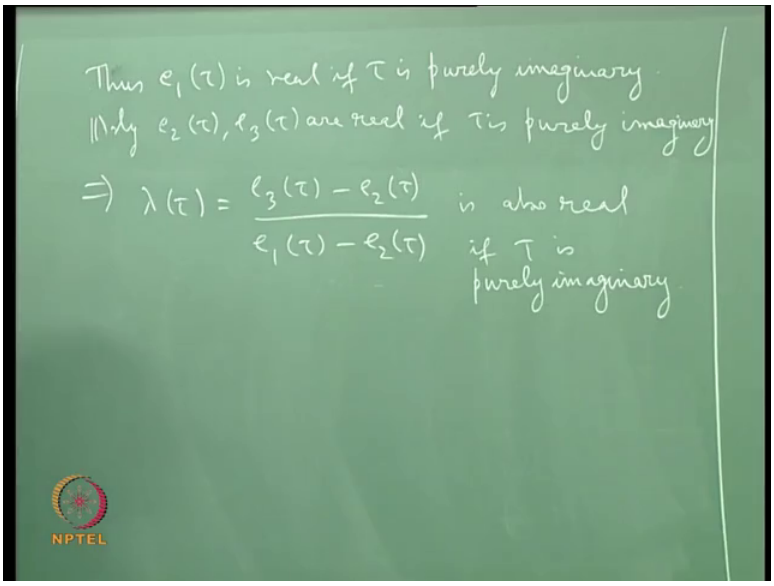

So this is thus e 1 of tau is real if tau is purely imaginary similarly e 2 of tau e 3 of tau are real if tau is purely imaginary. So, this implies that lambda of tau which was defined to be e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau this is how lambda of tau was defined.

This turns out to be real if tau is pure is purely imaginary. So, this completes the proof of this statement that lambda is lambda is real on the imaginary axis. So, this is the next statement that one has to prove and I will do that in the next lecture.