An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Dimensional Tori and Elliptic Curves

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Lecture – 34
The Fundamental Functional Equations satisfied by the Modular Form of Weight Two on the Upper Half-Plane

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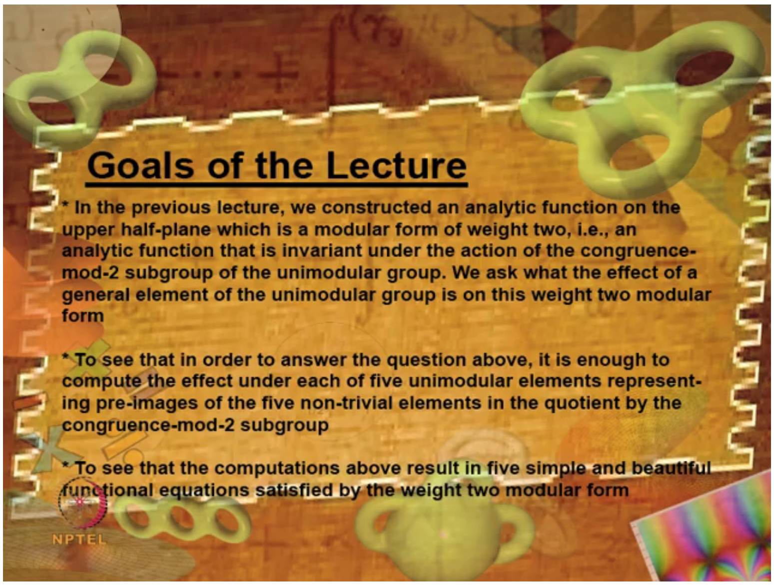

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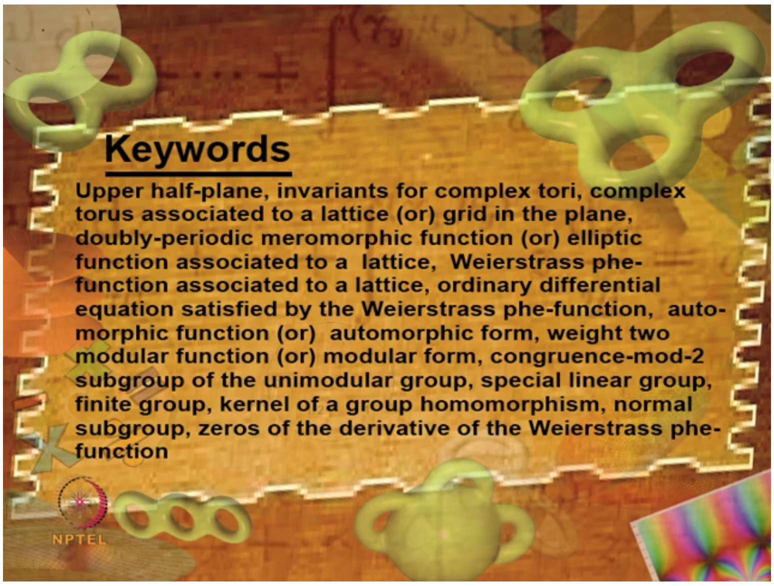

So, you see; let us recall what we did in the last lecture, see just fix up also the notation.

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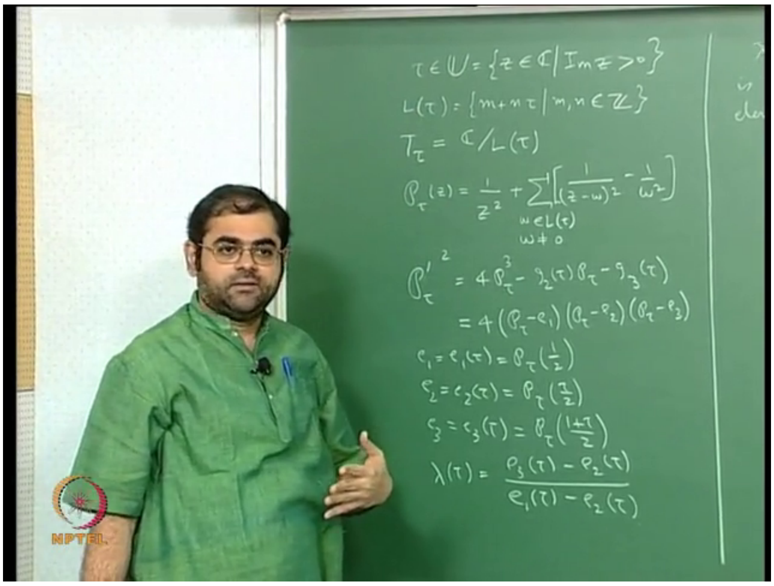

So, we started with tau in the upper half plane the set of all complex numbers (Refer Time: 00:49) positive and then associated to tau the lattice L of tau which is a set of all m plus n tau where m and n are integers and then we have the torus associated to tau the complex one dimension torus associated to tau which is the C module tau and we are able to get hold of the essentially the unique the simplest meromorphic function on the torus which is given by the (Refer Time: 01:33) phi function on the complex plane which is given by phi sub tau of z; it is given by explicitly by a formula of this of this form. So, a summation over, it is 1 by z square which is the singular part at the origin that is a normalisation techniques this function unique, then you sum over one by z minus omega the whole squared minus 1 by omega squared.

We got the phi function and then; we then notice that the phi function satisfies a differential equation which is I guess p; P sub tau the wholes prime the whole square is 4 P sub tau 4 P tau q minus g 2 of tau P tau minus g 3 of tau; this was the differential equation that was satisfied by (Refer Time: 02:46) phi function and its derivative; of course, the variable here is the variable here is the independent variable is z, but that is also dependents on tau and which is what we used to define the modular function lambda. So, if you if you recall we factorise the right side as P tau minus e 1 into P tau minus e 2 into P tau minus e 3 and where you know the. So, e 1 is actually e 1 of tau is actually P tau of half e 2 is e 2 of tau is P tau of tau by 2 and e 3 is also dependent on tau its P tau of 1 plus tau by 2 and then we noticed that with the function lambda of tau which is given by I think its e 3 minus e 2 by e 1 minus e 2, I am not mistaken yeah.

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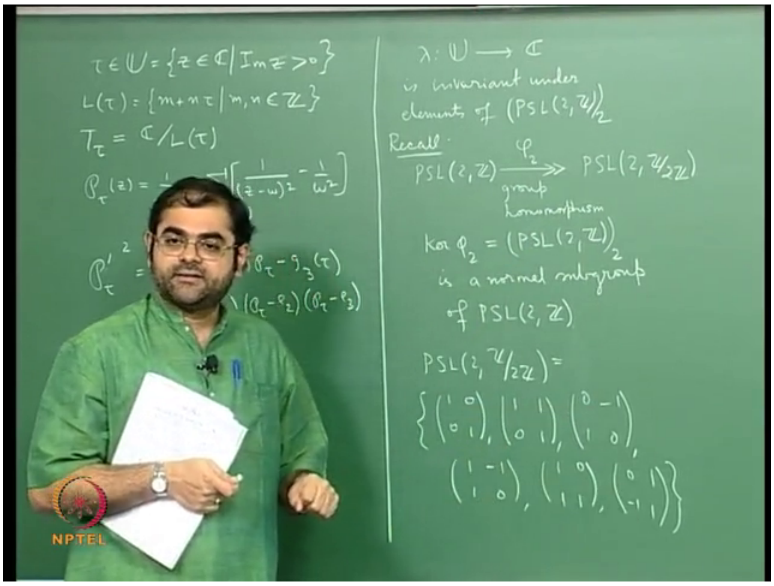

So, it is e 3 minus e 2 by e 1 of tau minus e 2 of tau, we got this; we got this function and what we proved last time was that that this function lambda which is a holomorphic function on the upper half plane is invariant under elements of P s l 2 z the uni modular group which are congruent to the identity matrix when the coefficients are read mod 2. So, this is what we proved last time and so, you see the all this was towards the objective of trying to find a function on the upper half plane which is an invariant for the Tori; that means, a function which is which is constant on orbits of P s l to z for its action on the upper half plane and you know these orbits are precisely the holomorphic isomorphism classes of complex Tori so, but so far we have had partials of success because this is this is only a subgroup it is a normal subgroup of the full uni modular group.

So, you have to somehow from this from this automorphic function from this modular form you have to find out you have to cook up a modular function which is which is modular for the full uni modular group and that is the one that is going to be constant on the orbits for the action of the full uni modular group. So, so how does one do that one cannot do that unless of course, you know the natural thing is that one has to study how lambda is affected by an arbitrary element of the full group the full uni modular group.

So, you see. In fact, the aim about this lecture the aim of this lecture is to show that lambda has satisfies certain functional equations and these functional equations tell you exactly how you can compute the effect of any element of the full uni modular group on lambda. So, how of course, lambda is not invariant under the full uni modular group so, but. So, it changes.

So, one wants to know how it changes in other words what I mean is you see you change tau by a uni modular transformation, then the lambda of tau will change it will no longer be it will no longer be lambda tau unless of course, that element is congruent to identity mod 2 which we have proved. So, for if you take a general uni modular element if you apply to lambda what is the effect; if you apply it to tau what is the effect of that what is lambda evaluate on that that is what we want to calculate and then it turns out that its it can be done in a very very nice way.

So, essentially the first part of the argument is that you should notice that if you recall see you take PSL 2 Z that is that is this there is this group homomorphism there is a group homomorphism which is simply take a representative here which is the 2 by 2 matrix with determinant one with integer entries the representatives is unique up to sign and then you read each entry mod 2.

So, the result is you will get an element of the uni modular group with quotients in z mob 2 z which is just 0 or 1 z mod 2 z is just 0 or 1 is a smallest field what we know and well in fact, this is the same as because plus 1 is equal to minus 1 this is the same as s l 2 z mod 2 as well and. So, and of course this map is of course surjective trivially and the kernel of this map is precisely this congruence mod 2 subgroup. So, kernel of phi 2 is exactly the congruence mod 2 subgroup which under which we know we have proved is in the last lecture that lambda is invariant.

Now, you see now you just take. So, let me write this recall is this is a normal subgroup of well P s l 2 z the uni modular group and if you calculate. So, you know if you calculate PSL 2 Z; PSL 2 Z mod 2 Z if you calculate this; I mean this is essentially if you right down the transformations you may essentially get 6 matrices.

And what are the 6 matrices let them let me write them down. So, I am just trying to write it in a certain order that I have written for myself. So, of course, maybe I will write it here. So, it is of course, you have the identity matrix, then you have the matrix 1 1 0 1, then you have the matrix 0 minus 1 1 0, then you will have 1 minus 1 1 0, you will have 1 0 1 1 and the last one is 0 1 minus 1 1 you will get these matrices. So, I mean of course, you know here where ever there is a minus 1, I can put a plus 1, it is not going to make a difference because the quotients are mod 2, but the point is I am putting minus 1 is because I can use the same matrix as an element here which goes to that and the reason why I put this minus in these 3 places has a significance because after all these are Mobius transformations because you see if you.

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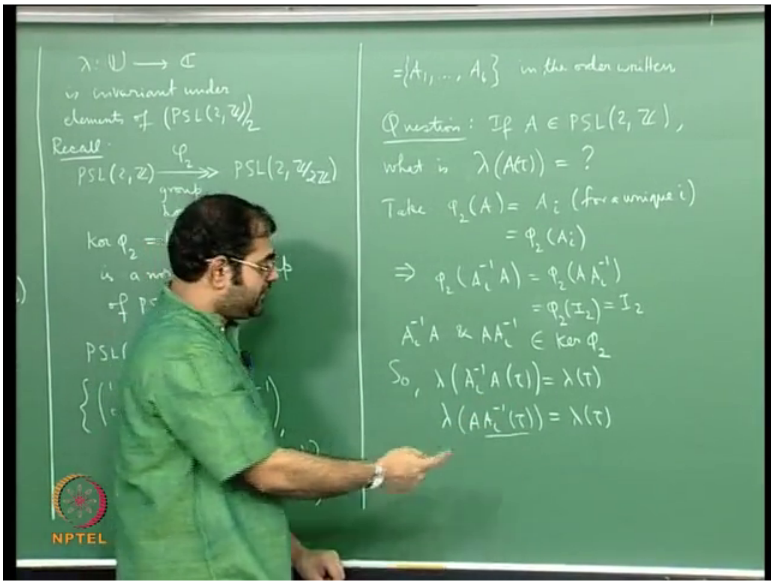

If you notice let me call of course, you know I have. So, let me call these matrices as well A 1, etcetera A 6 in the order written let me call. So, I am calling this as A 1 this is identity matrix this is A 2, this is A 3, A 4, A 5, A 6 and these are all the matrices that you have and. In fact, and of course, I will also think of if the a; I is also is matrices here that is each of these matrices which you consider as matrices here are also matrices here they are pre images chosen pre images which go there that is the advantage of looking at it like this and when I write it here this these signs matter to me because here one is not equal to minus 1.

So, now you see you see take. So, you know the question is the following the question is you want to from lambda; I want to cook up a function which is invariant under the full modular group. So, the question is how is lambda be going to be affected by an element of a general element of the modular group the full uni modular group. So, you see if a is an element of the full uni modular group what is lambda of a tau can you compute this in an easy way.

So, question is I want to know how lambda is affected if I twist it by an element of the general uni modular group. So, see the answer is that well you see take take the image of A under phi 2. So, A is here you take its image here then this image is going to be exactly one of the one of these A is. So, you see. So, so phi 2 of A, suppose it is AI of course, is going to be for a for a unique I for a unique I alright then what is this is this mean. In fact, you know I can also write it as phi 2 of AI, there is a slight abuse of notation I am just saying that I am taking the same AI as representative here which goes to that.

So, what does this mean this means that you see these 2 go to the same? So, it means that you know phi 2 of AI inverse A which is equal to phi 2 of AI inverse will be phi 2 of. So, it will mean that well because phi 2 is a homomorphism you know this will mean that this is equal to phi 2 of identity which is a identity map and what this therefore, tells you is that you know AI inverse a and well you see and a AI inverse see they are all they both belong to the kernel of phi 2 because their map by phi 2 into the on to the identity and you know the elements of the elements here that go to identity there are precisely the kernel of phi 2 and that is the that is the congruence mod 2 subgroup.

So, but we already proved that the effect of the lambda is invariant under the effect of elements of the congruence mod 2 sub group. So, you know. So, so what you will get is you will get lambda of AI inverse A of tau is lambda tau and you will also get lambda of A AI inverse tau is lambda tau. So, so therefore, the you know for example, my aim is to calculate what lambda of a tau is. So, you know in this for example, in this second equation for example, suppose I write replace AI inverse tau by some tau prime, then I will get lambda of A tau prime is equal to lambda of AI of tau prime ok.

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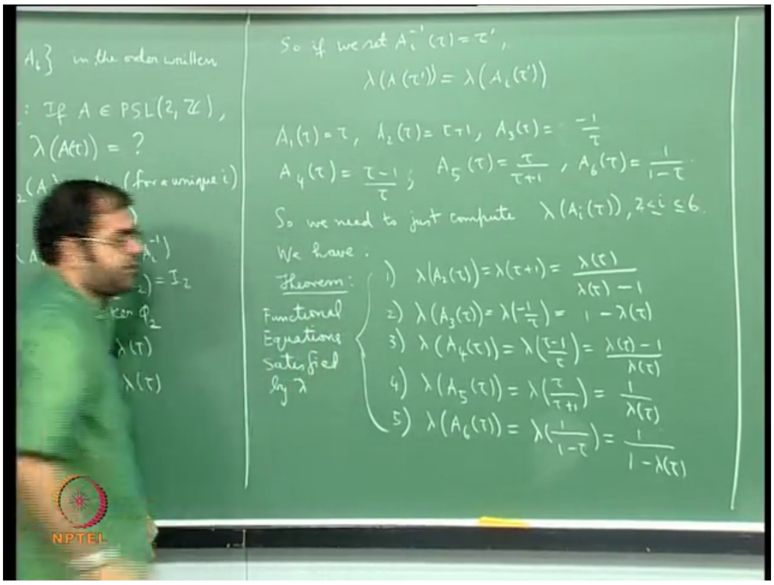

So, you know. So, if we set; let us say AI inverse of tau as tau prime mind you all these A is are elements in PSL 2 Z which are which is which is the subgroup of PSL 2 R and these are certainly automorphisms of the upper half plane. So, therefore, AI inverse of tau is also going to be an element in the upper half plane. So, you see what you will get is that you will get that lambda of A of tau prime is from here you see AI inverse tau is tau prime. So, I will get lambda of A of tau prime is lambda of tau, but tau is AI of tau. So, you see we have the answer to our question tau prime tau prime and well or if you forget if you continue if you want to call tau prime as tau you can in general say that this is the; so, let me not do that just to avoid any confusion.

What this equation tells you is you want to compute the effect of an element a on lambda it is enough to compute the effect of the you read that element mod 2 and compute the effect of that on lambda that is all that is what it says now, but you see these transformations are all very nice transformations. So, you see A 1 tau A 1 tau is just tau you see A 1 tau is just mind you whenever we write an element alpha beta gamma delta alright, then it is thought of as a Mobius transformation that that takes z to alpha z plus beta by gamma z plus delta that is the identification. So, you see A 1 of tau is going to be just tau A 2 of tau is going to be you can see its tau plus 1 A 3 of tau is simply minus 1 by tau and then A 4 of tau is tau minus 1 and A 5 of tau is tau by tau plus 1 and A 6 of tau is tau by minus tau plus one. So, it is minus. So, its tau by; so, it is 1 by minus tau plus 1 which is 1 by 1 minus tau.

So, these are the simple Mobius transformations. So, A 1 tau is this identity A 2 tau is translation by 1 A 2 tau is translation by 1 A 3 tau is minus 1 by tau A 4 tau is translation by minus 1. So, is there. So, A 4 tau is tau minus 1 by tau. So, its tau minus 1 by tau yeah its tau minus 1 by 1 into tau plus 0 and A 5 tau is tau by tau plus 1 and A 6 tau is 1 by 1 minus tau.

So, you see you are just required to find out what is the effect of lambda on one of these and then you will be able to say what is the effect of what is the sorry what is the effect of any general uni modular transformation on lambda, that is what you have to. So, this question reduces our computation to the computation of these things and computing these things actually give us certain equations certain relationships between lambda applied on tau and on these things the images of tau under these maps which are called functional equations. So, In fact, we get 6 functional we get 5 functional equations of course, the first one is trivial there is nothing to do here. So, we get 5 functional equations and all these 5 functional equations can be derived by 2 basic functional equations and these are the functional equations that involve A 2 and A 3.

So, there is something that I want you to notice that you see if tau is in the upper half plane, then 1 by 2 is it not, but minus 1 by 1 by tau is it not, but one by tau is and similarly you can verify that all these guys are certainly in upper half plane so; obviously, if tau is in the upper half plane tau plus 1 is and so, you can calculate this tau plus 1 is minus 1 by tau is. So, this is 1 minus 1 by tau which is translate of minus 1 by 2 by with 1. So, that is also there and so on and so forth; you can justify that these guys are all in continue to be in the upper half plane I mean which is just the fact that these are all automorphism of the upper half. So, they have to be the images of under tau have to be in the upper half plane ok.

So, you see now. So, so we need to just calculate just compute lambda of AI of tau for one less than or for 2 less than or equal to I less than or equal to 5 2 less than or equal to I less than or equal to 6 you have to compute only these quantities now. So, here is the we have. So, let me write this down theorem. So, you have the following. So, let me write this down number one lambda of A A 2 of tau is lambda of tau plus 1 that turns out to be lambda tau by lambda tau minus 1. So, you get all these functional equations mind you lambda the function lambda never takes the function lambda on the upper half plane actually never takes the value one it never takes the value 0.

Therefore, this division by lambda tau minus 1 is never going to create a problem the second one is lambda of A 3 tau is lambda of A 3 tau is minus 1 by tau and that turns out to be 1 minus lambda tau third one lambda of A 4 tau is lambda of a tau minus 1 by tau which is one it is lambda tau minus 1 by lambda tau and lambda of A 5 tau is lambda of A 5 tau I tau by tau plus 1 that comes out to be simply one by lambda tau and the fifth one is lambda of A 6 tau comes out to be 1 by 1 minus lambda tau. So, this is this is lambda of this lambda of A 6 tau is 1 by 1 minus tau that just turns out to be 1 by 1 minus lambda.

So, these 5 together; they are called functional they are called functional equations of lambda because you see the same function with the difference in the arguments there is a relationship. So, these are all functional equations. So, so these are called functional equations satisfied by lambda. So in fact, so, not only does this answer this question, but the fact is these Mobius transformations will help you to single out a certain special region of the upper half plane using which studying which along with the mapping properties of lambda; you can build a modular function which is modular on the whole uni modular group. So, these Mobius transformation are also important. So, we will see that in the forthcoming lectures.

So, you see the fact is that you know we only need to prove one and 2 because you see the others 3, 4 and 5 can be deduced from 1 and 2, 3, 4 and 5 can be deduced by because you see lambda of tau minus 1 by tau is lambda of 1 minus 1 by tau. So, it is lambda of minus tau plus 1 minus 1 by tau plus 1. So, it will be lambda of minus 1 by tau by lambda of minus 1 by tau minus 1 and then you use lambda of minus 1 by tau is 1 minus lambda tau. So, actually 2, 3, 4 and 5 are consequences of 1 and 2; ok.

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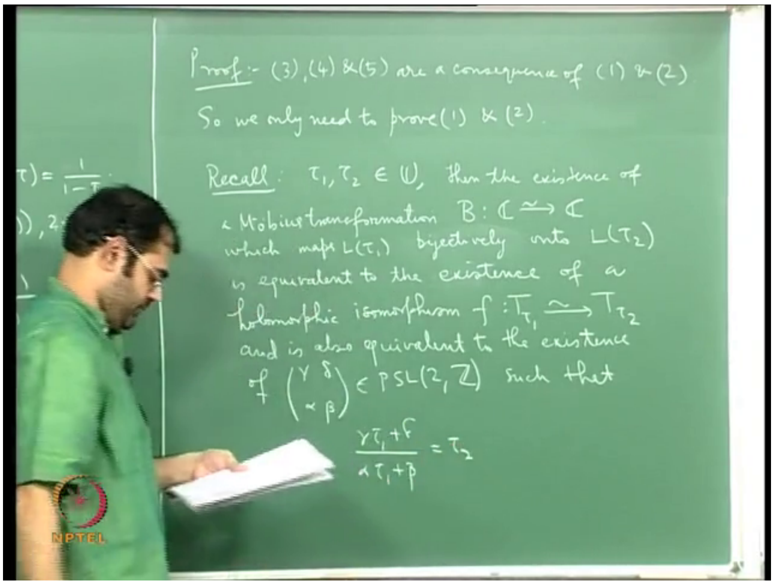

So, proof 3, 4 and 5 are a consequence of A 1 and 2 applied diligently. So, we only have to prove 1 and 2. So, so we only need to prove 1 and 2, well I hope that I hope that these formulas are right because I wrote them down once so. So, now, the point is the point is a following the point is how do you prove; how do you prove them. So, well the to prove them I am again going to go back to this piece of computation that I used in the last lecture which I have borrowed from lecture 24 which is I mean which essentially comes from the statement that you know the complex Tori defined by 2 elements in the upper half plane are holomorphically isomorphic if and only if the 2 elements in the upper half plane differ by uni modular transformation.

So, let me again recall that calculation, it seems to be indispensable. So, you see. So, you see recall you see if we have tau 1 and tau 2 in the upper half plane, then the existence of a Mobius transformation B of the that takes the complex plane on to itself which which maps l of tau 1 isomorphically onto I should let me not say isomorphically I should let me say bijectively on to l of tau 2 is equivalent to the existence existence of a holomorphic isomorphism isomorphism f from the complex Tori is defined by tau 1 to the complex Tori is defined by tau 2 and is also equivalent to the existence of a uni modular element which gamma delta alpha beta in PSL 2 Z. So, I am unfortunately still writing gamma delta alpha beta because I messed it up in lecture twenty four, but really it shouldnt matter such that gamma tau 1 plus delta by alpha tau 1 plus beta is tau 2 ok.

So, this is something that we use indispensably so. In fact, let me write out; let me again recall what this B was so. In fact, the situation was that.

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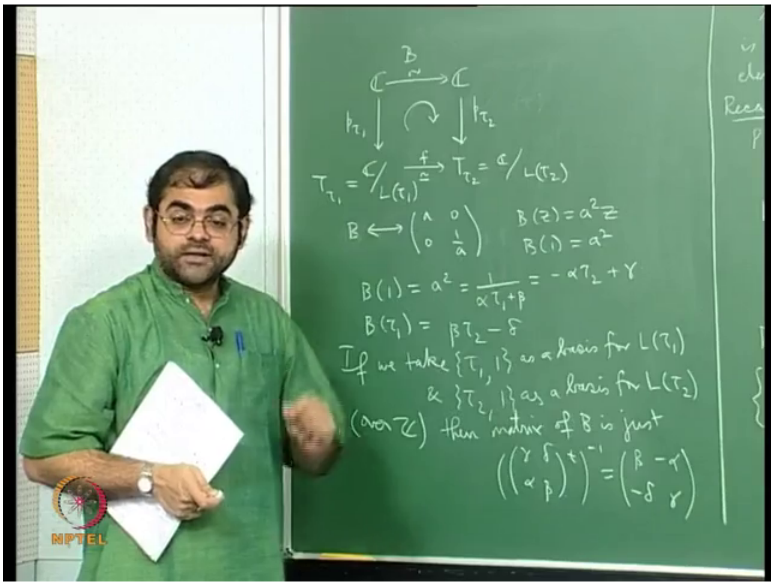

We have this Mobius transformation B which goes down to the complex torus defined by tau 1 namely T sub tau 1 and it goes down to an isomorphism holomorphic isomorphism of T tau 1 to T tau 2 which is c mod l tau 2 and this is a the projection which is a holomorphic universal covering and we notice I mean if you recall B of z B had B had the matrix B the matrix of B was given by a 0 0 1 by a. So, B of z was just a square time z and of course, B of 1 was a square and B is therefore, additive and it takes the lattice defined by tau 1 bijectively on to the lattice defined by tau 2 and what is this a square.

In fact, if you calculate B of one turns out to be a squared which is it is one by alpha tau 1 plus beta and that turns out to be minus alpha tau 2 plus gamma and you can see that B of tau 1 turns out to be beta tau 2 plus beta tau 2 minus delta this is the I mean this is how B is if you calculate and this clearly tells you that B takes and since B is additive because it is just multiplication by a scalar by a complex number complex constant. So, B is additive and it will tell you that therefore, any linear integer linear combination of one and tau 1 is going to land into the lattice defined by one and tau 2 and because it has inverse this has to be a bijective map be is a Mobius transformation you know Mobius transformation are already injective.

So, now, you see and then you know in them. In fact, I mean what is the point the point is you see if you write if we take if we take see you take tau 1 1 as a basis for l tau 1 and tau 2 comma 1 as a basis for l of tau 2 over of course, all over z over z then actually B you restrict B to l tau 1 you get a map you get a bijective map on to l tau 2 that is a its a z module map and if you write the matrix of that map that matrix is just related to this matrix this uni modular matrix in P s l 2 z and what is it. In fact, then you see the matrix the matrix of B is just it turns out to be just that that matrix gamma delta alpha beta well transpose inverse which is which is just you know beta minus alpha minus delta gamma .

So, because you see B of tau 1 B of the first base is vector B is B of tau 1 that gives you beta comma minus delta which is the first column and B of the second basis vector B one gives you minus alpha gamma which is the second column. So, this is exactly the. So, the matrix of B, when I say matrix of B you restrict B to the lattice you restrict B to the lattice l tau 1 you get a bijective map on to the lattice l tau 2 that is a z linear map. So, it is given by matrix with z entries and it is a uni modular group matrix because this an invertible the determinant is an invertible integer and because tau is in upper half plane you know that it cannot be minus 1 we proved that so. In fact, this is the relationship. So, this is actually that matrix its transpose inverse is actually the matrix of B relative to these basis when you restrict B to the lattice.

So, then we proved in the we proved in the last lecture that there is a kind of there is a there is an equation which tells you how the phi function is transformed if you keep the I mean if you apply you know if you apply I mean what is the relationship between the phi function that is defined for tau 1 and the phi function that is defined for tau 2 if tau 1 and tau 2 are related in this way so. In fact, what we proved is that we proved that we proved.

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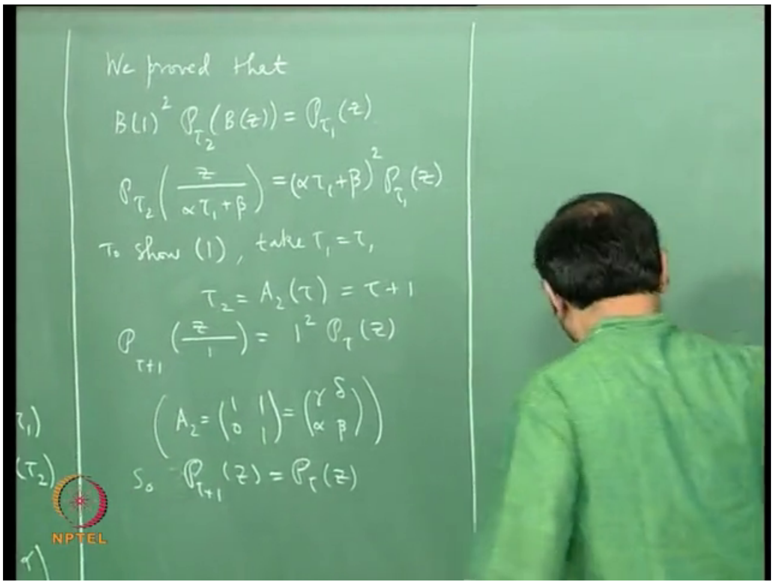

In the last lecture that and I need this B of one the whole square P tau 2 of B of z is equal to P tau 1 of z we proved this. So, we proved this last time and this was very eas I mean this is very easy to prove just using the direct expansion of the phi function and using the fact that B of z is z times B of one and also using the fact that B maps l tau 1 on to l tau 2 and it takes non-zero elements of l tau 1 exactly on to non-zero elements of l tau 2. So, it is very very simple to prove this.

And let us write it out in terms of this alpha beta gamma deltas. So, you see what it will tell you if I write it down is I will get I will get P tau 2 of B z is z times B of one B of one is a square and a squares is. So, it will be B times z by alpha tau 1 plus beta is equal to and I push this B of one whole square to the denominators. So, that I get one by B of one the whole square and that will be one by a square that will be alpha tau 1 plus 1 by a squared the whole square. So, it will be alpha tau 1 plus beta the whole square. So, I will here I will get alpha tau 1 plus beta the whole square into P tau 1 of z. So, yeah. So, this is want I want to use I want to use I need to use this. So, let us now calculate this. So, let us try to prove to show the this is the first formula. So, I just want to calculate lambda tau plus 1 in terms of lambda tau.

So, you see what I do is and you see that A 2 takes tau 2 tau plus. So, basically I take tau 1 as tau and tau 2 as tau plus 1 notice that if I take A 2; A 2 of tau is tau plus 1 tau 2 is A 2 of tau and the point is I should take for A 2 this uni modular element this see I have to take the uni modular element that takes tau 1 to tau 2. So, as I was thinking for a moment once should not get confused with the B there one should not get confuse of the B there. So, well you see tau 2 is A 2 of tau which is A 2 of tau is tau plus 1 and if you do this then what you will get is you will get P tau plus 1 of z divided by.

So, you see our. So, here my A 2 is what is A 2 it is the matrix A 2 is a well it is 1 1 0 1. So, it is it is a matrix for 1 1 0 1 and I have to compare this with gamma delta alpha beta. So, you see. So, if I apply this formula here, I will get z by alpha tau 1 plus beta is going to be 0 tau 1 plus 1. So, it will be 1 is equal to 0 tau 1 plus 1 the whole square that is going to be just one squared into P sub tau 1 is just tau of z. So, I mean finally, you get the P tau plus 1 of z is just P tau of z. So, P sub tau plus 1 of z is simply P tau of z its simply P tau of z.

And now calculate what the corresponding e is are because the lambda is different based on the e is. So, if you want to check how lambda changes you have to check the way e I changes so. In fact, what you will get is so.

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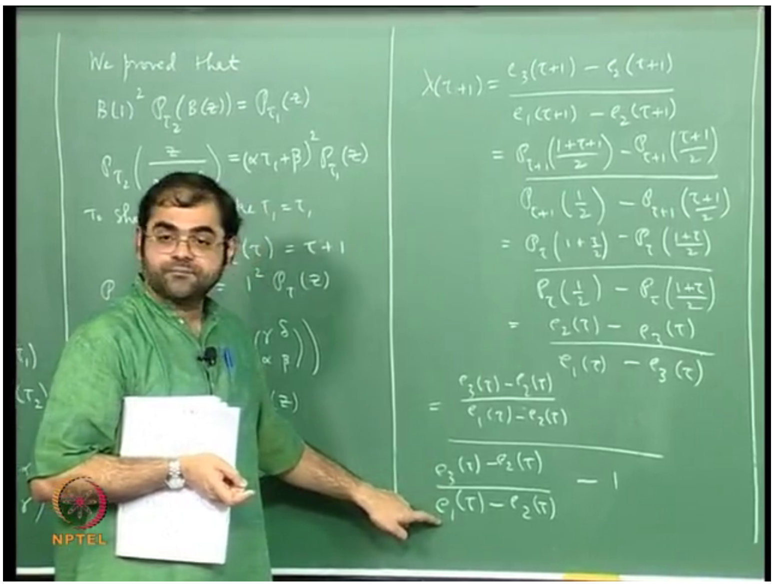

So, let me write directly write this lambda of tau plus 1 is mind you it is e e 3 of tau plus 1 minus e 2 of tau plus 1 by e 1 of tau plus 1 minus e 2 of tau plus 1 I mean this is how lambda tau is defined lambda tau is e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau that is how lambda is defined and you see, but you see what is e 3 of tau plus 1, it is P e 3 of tau is just P tau applied to 1 plus tau by 2. So, this e 3 of tau plus 1 is P tau plus 1 applied to 1 plus tau plus 1 by 2 and e 2 of tau plus 1 will be P tau plus 1 applied to. So, e 2 of tau is P tau applied to tau by 2. So, this will be P tau plus 1 applied to tau plus 1 by 2 and e 1 of tau is P tau applied to half.

So, e 1 of tau plus 1 will be P tau plus 1 applied to half and e 2 of tau is again P tau plus 1 applied to tau plus 1 by 2 this is what you will get, but then you use the fact P tau plus 1 is as same as P tau. So, what I will get is P tau of well if I will get 1 plus tau by 2 minus P tau of 1 plus tau by 2 whole by 2 by P tau of minus P tau of 1 plus tau by 2 and you can you can recognise that you see P of course, P tau the phi function is periodic with respect to 1 and tau. So, you see P tau of 1 plus tau by 2 is P tau of tau by 2 and P tau of tau by 2 is e 2 of tau.

So, this is e 2 of tau and that is e 3 of tau divided by e 1 of tau minus e 3 of tau and you can clearly see that this is you know what I want is lambda tau by lambda tau minus 1 this is. In fact, well it is very easy to easy to see that it is e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau divided by the same quantity e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau minus 1 because if you simplify this you are going to get e 3 minus e 2 on top that is the same as this up to sign and denominator you will get e 3 minus e 1 which is the denominator up to a sign. So, you will get this.

So, you will get therefore, that lambda tau plus 1 is lambda tau by lambda 2 minus 1 which is the first which is the first a functional relation. So, so let me write that down.

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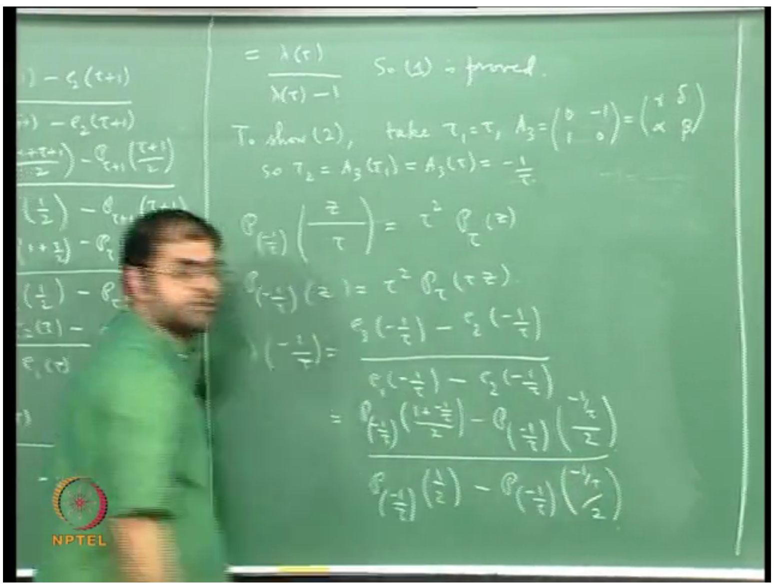

So, you will get. So, so we get and that turns out to be lambda of tau by lambda of tau minus 1. So, one is literally the same kind of exercise if you carry it out you can prove the second one. So, let me write that down. So, for the second one to show 2 take again. So, if you look at 2. So, you see the transformation that I have to take is e 3 it takes tau to minus 1 by tau. So, I will take tau as tau 1 and I have take A 3 for this transformation. So, that when I apply tau to it I get minus 1 by tau which will be tau 2. So, you take tau 1 is equal to tau A 3 is equal to which is A 3 is just. So, it is. So, it is 0 0 minus 1 1 0 which is which has to be compared with gamma delta alpha beta and. So, tau 2 will be just A 3 of tau 1 which will be which is A 3 of tau which is A 3 of tau is minus 1 by tau minus 1 by tau. So, tau 2 is minus 1 by tau.

And then the you apply. So, let me rub this off well and then you literally do the same thing you again apply this formula, I mean this formula right. So, you will get P sub tau 2 which is minus 1 by tau of z divided by alpha tau 1 plus beta. So, alpha tau 1 plus beta is going to be one into tau plus 0. So, I am going to get z by tau is equal to again alpha tau alpha tau 1 plus beta is going to be 1 into tau plus 0. So, it is just tau squared into P sub tau 1 which is tau and of z and for convenience I can write this is as P of minus 1 by tau of z is well you know if I replace z by tau by z, then I have to replace z by z tau.

So, this will become tau squared P tau of tau z and now go through the now you go through the same computation you calculate what is lambda of minus 1 by tau you calculate what is lambda of minus 1 by tau it is by definition e 3 of minus 1 by tau minus e 2 of minus 1 by tau divided by e 1 of minus 1 by tau minus e 2 of minus 1 by tau and well now I make use of the fact that e 3 of minus 1 by tau is P of minus 1 by tau applied to e 3 is P tau applied to 1 plus tau by 2.

So, it is 1 plus minus 1 by tau divided by 2 minus e 2 well e 2 of tau is just P tau applied to tau by 2. So, this is going to be P of minus 1 by tau applied to minus 1 by tau by 2 divided by I will get again here I will get e 1 of tau is just P tau applied to half. So, this is going to be P of minus 1 by tau applied to half and here again I am going to get P of minus 1 by tau applied to minus 1 by tau by 2 this; what I am going to get.

And now I make use of this formula that I have derived and we will get what we want. In fact, we will get. So, let me write that down. So, we will get that is equal to.

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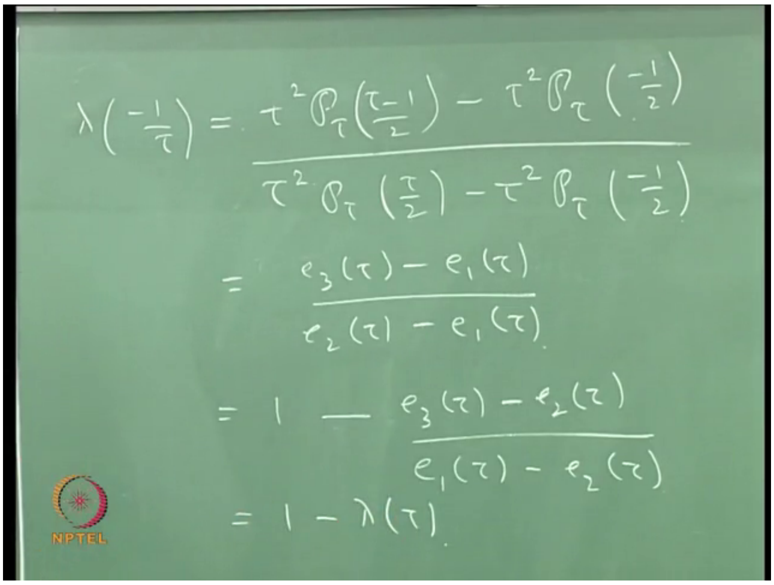

So, let me write down; what is on the left side lambda of minus 1 by tau is equal to well P see P minus 1 by 2 of z is just tau square beta of tau z. So, I will get tau square P sub tau of tau time z. So, I have to multiply this whole thing by tau if I multiply this by tau I will get tau minus 1; minus 1 by 2 minus again I will get tau square P tau of multiplying this quantity inside by tau I will get minus half divided by tau square P tau of multiply the quantity inside by tau I will get tau by 2 minus tau square P tau of minus half ok.

And then you have to now remember that I can add one to this and this becomes P tau of 1 plus tau by 2 which is e 3 of tau of course, the tau squares will cancel throughout you can take tau square common numerator and denominator that will just go away and P tau of minus half is same as P tau of half because I can add 1 and its periodic with period one and P tau of half is e 1. So, and P tau of tau by 2 is anyway e 2.

So, what you get is you get you get actually e 3 of tau minus e 1 of tau by e 2 of tau minus e 1 of tau and this will turn out to be what you want which was well I rubbed it out it is this 1 minus 1 minus lambda of tau. So, it is 1 minus e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau because if I simplify this the numerator I will end up with e 1 minus e 3 and denominator I will get e 1 minus e 2. So, which is just 1 minus lambda of tau.

So, similarly you can now you can manipulate these things to get the other to get the other functional equations. So, so that finishes so, that answers the question as to how lambda behaves if you try to apply an arbitrary element of the uni modular group to the argument of lambda. So, it gives all this functional equations.

So, in tomorrow's lecture, I mean in the next lecture we will see how all this data can be used to study the mapping properties of lambda and studying these mapping properties will help us to single out certain special domain in the upper half plane on studying which you can cook up a function which is a modular for the full modular group and which is of which is going to be the elliptic modular function and that is the invariant we are looking for. So, we will see that in the next lecture.