An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves
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Lecture - 32
The Values of the Weierstrass Phe-function at the Zeros of its Derivative are nonvanishing Analytic Functions on the Upper Half-Plane

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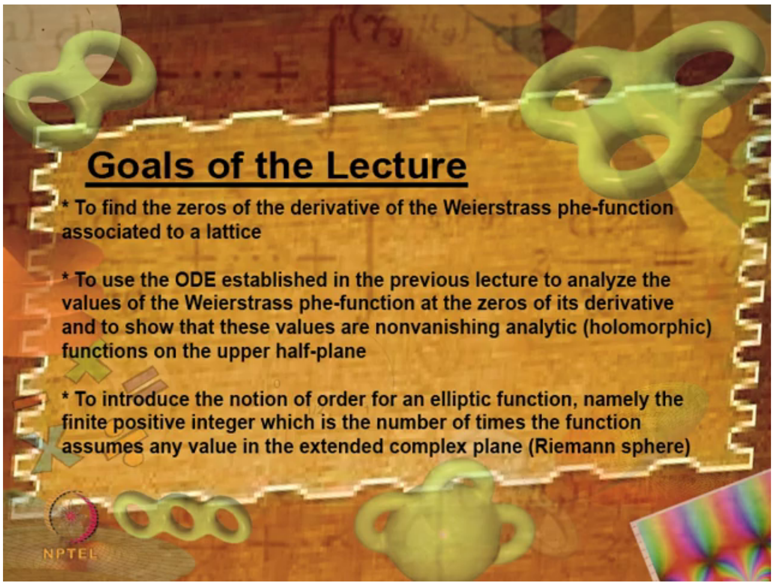

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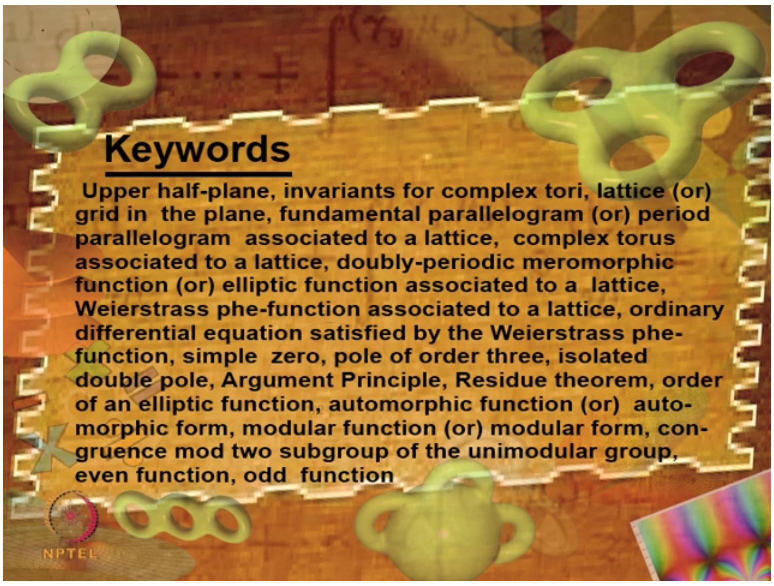

So, in the last lecture you see we got the differential equation that is satisfied by the Weierstrass phe-function for a torus. So, let me write that differential equation down. So, you see.

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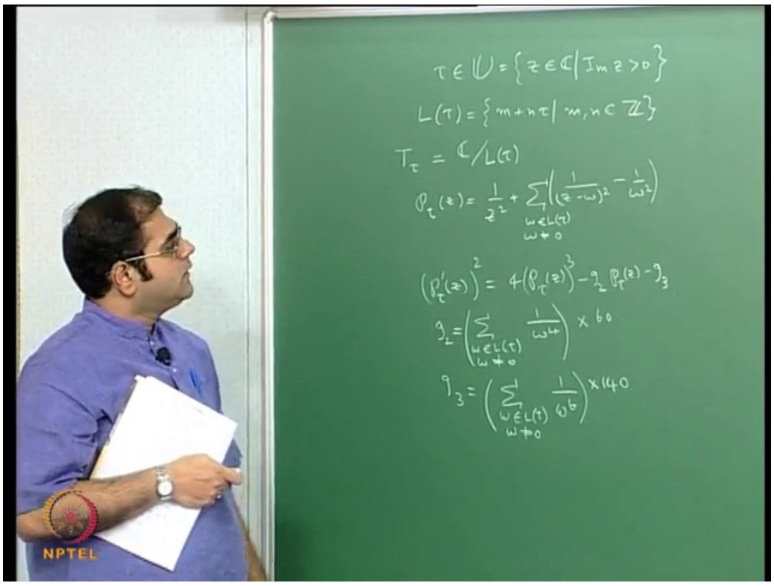

So, we have a tau in the upper half plane set of all complex numbers is it imaginary part of this z is 0, and corresponding to tau we have the lattice defined by tau which is the set of all m plus n tau, where m and n are integers. Then we have the torus defined by tau which is simply the complex plane modulo L tau and this is going model the equivalence relation if given by translation by an element of L tau.

So, in other words 2 complex numbers are said to be related, if one is a translate of the other by an element of L tau and this is you know this is the this is the complex torus associated complex one dimensional torus. It is a compact Riemanns surface of genus one right which is associated to this torus I mean which is associated to this this element of the upper half plane. And of course, we associated to this tau, the Weierstrass phe-function p tau of z which is 1 by z square plus sigma omega in the lattice omega not equal to 0, 1 by z minus omega the whole square and so on by omega squared. So, this is a Weierstrass phe-function.

And then we showed in the last lecture that this phe-function satisfies a differential equation now, and that differential equation is just to write it correctly. P tau z the derivative the whole squared is 4 times p tau z whole cube minus g 2 times beta of z, minus g 3 where g 2 is summation of summation over omega in the lattice omega not equal to 0, 1 by omega power 4 and I think there is a constant which is going to be I guess 6 there is this whole thing in to 60, and g 3 is summation over omega in the lattice omega not equal to 0 of 1 by omega power 6 in to 1 4 0 140 these were the constants that we got and this was the.

So, I was telling you last time that somehow this differential equation I mean if you call p of tau as a variable x and p dash of tau as another variable y, then this is the equation y squared equal to 4 x cubed minus g 2 x minus g 3 which is the equation of a cubic. It is a cubic polynomial equation and that is the first hint of the fact that of the fact that the torus is actually algebraic we will see that later. But for the moment you see the aim of our discussion has been to show that you wanted to show that the set of holomorphic isomorphic some classes of tori namely the set u mod PSL to zee, which we have managed to make in to a Riemanns surface; the upper half plane modulo the action of the unimodulor group PSL to zee ok.

We have managed to make that in to a Riemann's surface and we want to show that that is holomorphic isomorphic to the complex plane as with the natural Riemann's surface structure. So, you see this as I was explaining last time, this leads us to find invariants for tori; quantities which depend only on the holomorphic isomorphism class of a torus. So, you try to attach to every torus a quantity, which depends only on the holomorphic isomorphism class of the torus. But you see this means that you are trying to find a function on the upper half plane which is invariant under PSL to zee, because the orbits of PSL to zee in the upper half plane are precisely, the holomorphic isomorphism some classes of tori.

So, the aim is somehow that you want to cook up your function on the upper half plane, which is invariant at the action of PSL to zee that is what you want. So, and I told you that because of the philosophy you are trying to find invariants for tori and therefore, somehow the value of the invariant at a given point tau, has to do has got to do with the geometry of the torus that it defines, and then you know we took in by the general we took in the general philosophy that you see the geometry of the torus should be dictated by the functions that it allows. And searching for such functions is what led us to get the Weierstrass phe-function, and then we got this differential equation alright. Now the reason we got it is because we can use this to cook up a function on the upper half plane a complex function on the upper half plane in fact, an analytic function ok.

Which will be a kind of invariant to begin with what I am going to construct today is what is called as the it is called a modular function alright, and this modular function will finally, I want a function which is invariant under the whole group PSL to zee, but for the moment I will begin by constructing a function, which is invariant under what is called a congruence mod to subgroup of PSL to zee; namely a subgroup of matrices which are congruent to identity, when you read the coefficients mod 2. So, I will explain that. So, to begin with what you do is.

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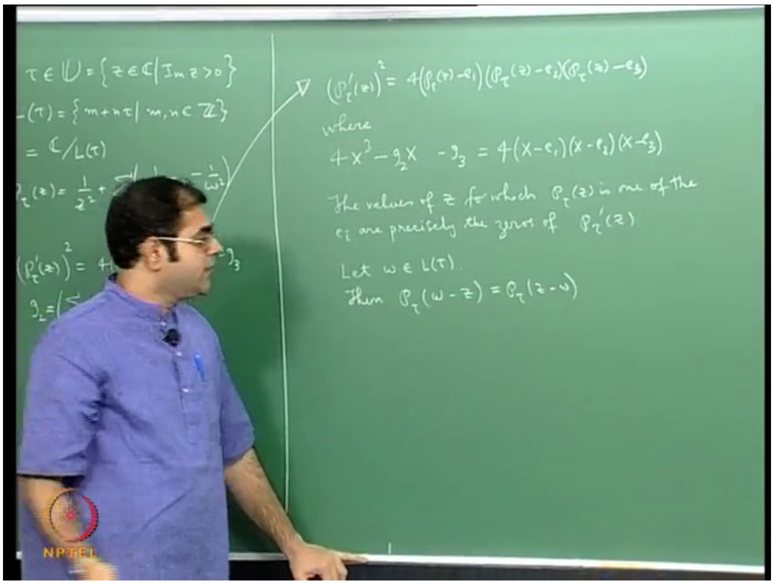

So, this is to recall but now, let us take this differential equation and you know write it in the following form; p prime tau of z the whole squared is equal to let me write this as well 4 times p tau z minus e 1 in to p tau z minus e 2 in to p tau z minus e 3 let me write it in this form alright. So, in other words you see the I am thinking of the right side as the polynomial 4 x cube minus g 2 x minus g 3. So, you take 4 x cube minus g 2 x minus g 3 this is the polynomial alright of degree 3, and it will have 3 roots alright. So, if you write those roots if you write the factorization in terms of the roots, the factorization will look like this it will look leading coefficient into x minus the first root, into x minus the second root in to x minus the third group. Of course, it is true that you see x is p tau therefore; it is obvious that since x depends on tau your e 1 e 2 e 3 will also depend on tau. You must understand that e 1 e 2 e 3 are also functions of tau ok.

So, they will change if you change tau alright. So, enter and the point is the key is to find what this e 1, e 2, e 3 are. So, to find you see we would like to look at when for what for what values of z this side vanishes. Because on this side I have p tau prime whole squared if this vanishes, this will vanish only for values of z for which either one of the 3 has to one of the 3 vanishes. So, I am just trying to find out for what values of z will be tau of z be a equal to one of the e i and this will be precisely the values of z, for which the derivative vanishes because of this differential equations.

So, this is where and this is how I am using this differential equation. So, let me write that down, the values of z for which p tau of z is one of the e i are precisely the zeroes of p tau prime of z. This are precisely zeros of this function which is on the left alright now let us try to. So, I first try to tell you that there are 3 easy zeros that you can guess. So, let omega be an element of that lattice. So, then you see if I take p tau of omega minus z; let me calculate p tau omega minus z, see this is you see therefore, you know omega is in the lattice. Therefore, omega is of the form n plus m tau alright. So, I take p tau of omega minus z and this is you see this is p tau of z minus omega that is because p tau is an even function alright the Weierstrass phe-function is an even function which you can anyway see by looking at this this series expansion it involves only even powers right. And you see since omega since omega is a period this also equal to P tau of z.

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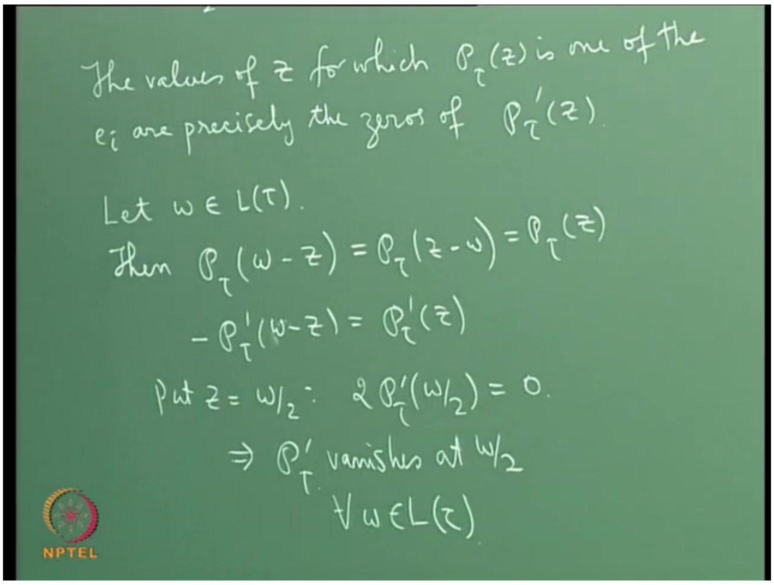

So, now, you take the first one and the last one and differentiate with respect to z. What you will get is, you will get if you difference is the first one I will get minus p tau prime of omega minus z is equal to p tau prime of z. And now you see suppose I plug in z zee is equal to omega by 2 in this. Put z zee equal to omega by 2 what you will get is you will get that omega by 2 is a 0. So, you will get 2 times p tau prime of omega by 2 is 0. So, this implies p tau prime vanishes at omega by 2, for every omega in the lattice.

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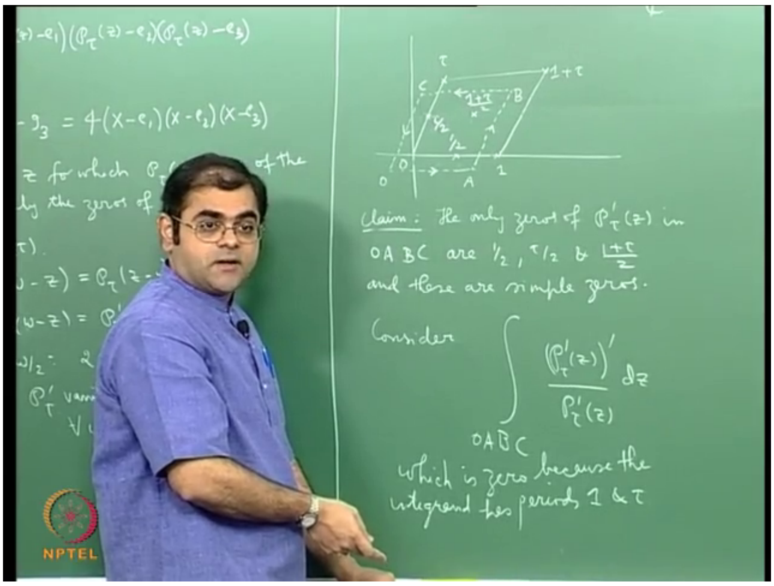

So, in particular if you know if you look at the.

So, let me draw this this period parallelogram, this fundamental parallelogram namely the parallelogram formed by the generators of this lattice. So, you see you have 1 here and well you have tau somewhere here. So, this is a complex plane and this is upper half plane and tau is there and I have this parallelogram here, this point is 1 plus tau and well the 3 zeroes I can immediately 3 distinct zeros I can immediately spot the closure of that parallelogram are of course, half tau by 2 and 1 plus tau by 2.

So, I will get a 0 here this will be tau by 2 and of course, I will get a 0 here which is half and then I will get a 0 here this is going to be 1 plus tau by 2. Of course, you have to be careful that here omega is not 0 because I cannot put omega equal to 0 here, because the derivative at 0 is not defined. So, the point I want to make is that, you see if you shift if you translate this parallelogram a little bit so that these 2 zeros are not on the boundary of the parallelogram, then my claim is that these are the only 3 zeros and they are simple zeros for p prime tau ok.

So, the claim, let me draw the parallelogram which is slight shift of this parallelogram so that these two zeros come inside the parallelogram translate of this parallelogram. So, let me draw it. So, it should look something like this, let me let me put tau by 2 here and so that I can. So, let me draw a parallelogram like this, and let me write the half here and here is translate of this parallelogram let me call this parallelogram as O, A, B, C let me call this parallelogram. So, A, B, C and well and let me if I want to integrate along the edges of this parallelogram as a closed path, then the orientation I take is the counter clockwise orientation.

So, the orientation is like this so, but the claim. So, the claim is the following. The claim is the only zeroes of p prime tau of z in this translate in O, A, B, C are these 3 1 by 2 tau by 2 and 1 plus tau by 2 and these are simply zeros. So, this is the claim; the claim is that these are simply zeros. So, how does one see that, the first thing that one needs to use is the so called. So, you will have to use the argument principle and you also have to use the residue theorem at the same time. So, for that what you do is, consider you consider this integral I integrate over O, A, B, C which is a closed path with the orientation the way I have drawn it, and I am going to integrate the function p tau prime the whole prime by p tau prime or z d z ok

I am going to look at this interval alright. Now on the one hand you see because p sub tau the phe-function is doubly periodic, this function continues to be doubly periodic and with periods 1 and tau therefore, if you calculate the, if you break this integral in to 4 pieces one along each edge of the parallelogram, then you see the integrals the car the contributions for the opposite edges will cancel that is because of periodicity. So, this is equal to 0 which is. So, let me write that which is 0, because the integrand is periodic has periods 1 and tau.

So, I am again and again using this fact that you see the I mean another way of stating this is trying to say that the sum of residues of an elliptic function is always 0. And this is an elliptic function it is meromorphic and it is periodic with respect to both 1 and tau and that result also comes in the same way, because you integrate over a suitable parallelogram like this and then you see that the integral automatically becomes 0 because of periodicity.

On the other hand the residue theorem tells that this is 2 pi i in to sum of residues of the function, at the poles inside this parallelogram right. So, this is on the one hand 0, on the other hand what is the argument principle. See the argument principle tells you that integral of f prime by f d z will be 2 pi i times number of zeros counted with multiplicities minus number of poles counted with multiplicities.

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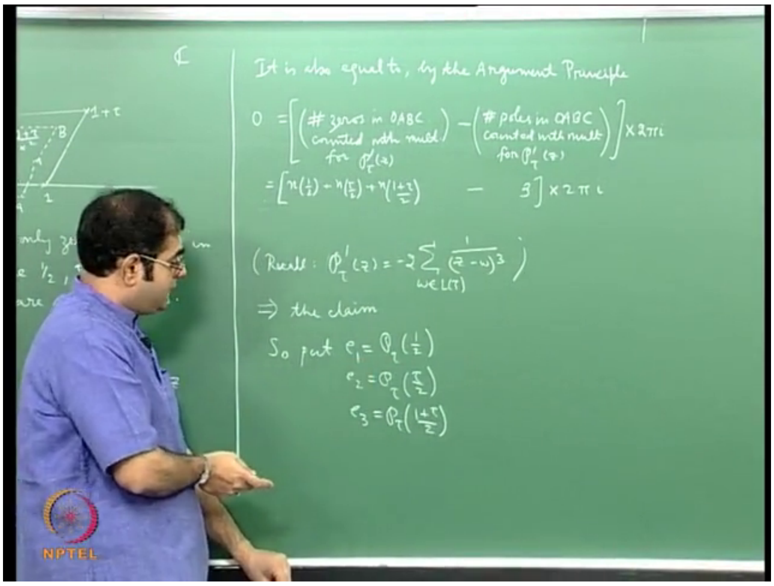

So, let us apply that it is also equal to by Riemann's principle this will be. So, this is our this integral is already 0 I have. So, I will get on the left side I have 0, that is equal to. So, number of zeros in O, A, B, C and when I of course, you know when you take number of zeros you have to count them with multiplicity you have to count zeroes with multiplicities, when you come poles you have to count poles with multiplicities. So, it is number of zeroes in O, A, B, C counted with multiplicity minus number of poles in O, A, B, C counted with multiplicity, but for which function for the function. So, this is for phe prime sub T of z.

So, here let me write that for phe prime sub T of z and here also it is for T prime sub T of z and it will be this in to well 2 pi i this is argument principle. Now you see you know p prime tau of z has you know that it is it has precisely a pole of order 3 at each lattice point; because you see if you recall we have this p prime tau of z is equal to minus 2 sigma omega in the lattice 1 by is at minus omega the whole cube. We have this we get this by differentiating this term by term, which you can do because of uniform convergence right and this clearly tells you that at each lattice point you have a pole of order 3 alright. So, in this parallelogram the only pole is at 0, and it has to be counted 3 times.

Therefore, the number of poles this is 3 alright minor and here I will get number of zeros, and the zeros the only zeros are these 3 and they will occur with some multiplicities it is no. So, let me write that suppose I write n sub n of half as a multiplicity of 0 at half plus n of tau by 2 as multiplicity of the 0 at tau by 2 and n of 1 plus tau by 2 as a multiplicity of the 0 at 1 plus tau by 2, then this is the sum this is the number of zeros counted with multiplicity, and this in to 2 pi i this is equal to 0 what does it imply? It means that each one has to be at least 1 so, but the sum is 3 therefore, each one has to be 1.

So, the moral of the story is it tells you that these are 3 zeros, and each of these zeros are simple zeros plus it also tells you that you cannot have any other zeros because otherwise this you want this will because the sum of all the zeros counted with multiplicities has to be equal to 3. So, this implies. So, that implies this claim. So, this implies claim. So, what we have proved is that the zeroes of p prime are precisely these. And now the next thing that I want to say is well. So, where are we? So, we are we know that for z equal to half and tau by 2 and 1 plus tau by 2, p tau of z will be either e 1 or e 2 or e 3. So, let us put. So, we can put. So, put e 1 to be you know p tau of half, e 2 to be p tau of tau by 2, and e 3 to be p sub tau of 1 plus tau by 2. Because e 1 e 2 e 3 are precisely the values that p tau has to take in order for the right side to be 0 and that means, these have to be zeroes of p tau prime p prime.

So, we do this. So, see we have managed to find what these for what values you will get zeroes on the left side on the right side by looking at the zeroes on the left side alright. Now the next thing that I want to tell you is I want to make another statement I want to make the statement that none of. So, you see now already what you see here is that there is a dependence on tau. See this is what we are originally looking for because we are trying to cook up a function on the upper half plane and tau is varying in the upper half plane. So, you see tau is showing up already alright, but we have to make sure. So, first thing is I claim that e 1, e 2, e 3 are all distinct. I claim that even can never no matter what the value of tau is e 1 can never equal e 2; e 2 can never equal e 3; e 1 can never equal e 3. So, that is the next claim alright. So, let me write that down here claim.

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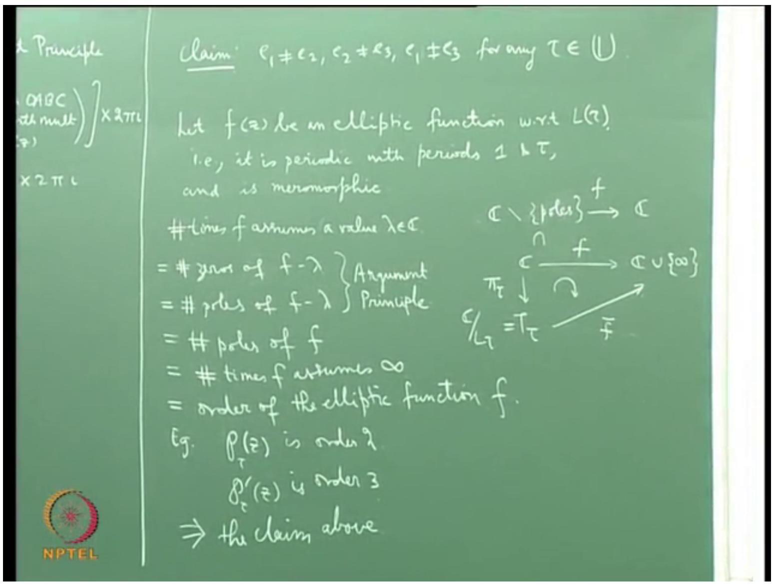

E 1 is not equal to e 2; e 2 is not equal to e 3 and e 1 is not equal to e 3 for any tau in the upper half plane. I came that all these values are different, e 1 if you fix a tau then you get 3 values of e 1, e 2, e 3 and these 3 are three distinct values and how does one prove that. So, that involves trying to understand what is called the order of an elliptic function. So, let me define this. So, you see let. So, let me do the following thing let maybe. So, let f of z be an elliptic function with respect to L of tau.

What does it mean? It means that it is w periodic function, which is meromorphic which is that is it is analytic except for pores ok. That it is periodic with periods 1 and tau and is meromorphic. Meromorphic means the only singularities that are allowed are poles. So, you take a general elliptic function, this elliptic function connected to this lattice alright. Now of course, you know what this means is that you are considering on the complex torus, a general meromorphic function because they a function a function like this. So, what is this? This is a function which is in which goes from C minus the lattice to so in fact; I should say C minus set of poles, it goes from C minus set of poles to C. So, this is my function f and if you want you can also consider it as you can extend it to a function f from C to C union infinity, you allow you set the value of f at a pole as infinity.

So, you get you can also think of it as a holomorphic map in to the Riemann's sphere and the point is that since it is invariant under the lattice, it will go down to this torus that is this projection map which is going modulo L tau that goes to T tau which is just C model tau and you get that. Therefore, you get a map like this. So, you get a map like this. So, maybe I can call this as f bar and this diagram permits. And of course, the fact that because this is a locally by holomorphic map because this is a universal covering it is a holomorphic universal covering it is locally by holomorphic therefore, this guy is holomorphic

So, in per in other words what is f bar? F bar is a holomorphic map from this complex torus in to C you in infinity, that in other words it is a meromorphic function on the torus I which means it is going to be holomorphic, at the torus minus image of all these poles and you see all these poles if it has a pole at one point, because of periodicity, it will have a pole at every translate of that point by the lattice and all these will go down to a single point in the torus below.

So, it will have and b and you see therefore, we get a meromorphic function alright and. So, I am and such a function is what is called an elliptic function, and we our standard example is the basis p function and it is derivative alright. So, take a general elliptic function and see I am trying to define what is meant by the order of such an elliptic function.

So, you see, I make the following claim. So, let me write this number of times f assumes a value lambda in C; look at the number of times the function assumes a value lambda in

C, now that will be equal to the number of zeros f minus the number of zeros of f minus lambda. So, the centerfolds z where f of z equal to lambda is precisely the set of all zeros of f of z minus lambda, and the fact is that the number of zeros of f minus lambda. So, you know suppose I am looking at only number of times f assumes a certain value, and suppose I look at that only in a translate of this fundamental parallelogram in a translate such as O, A, B, C of course, I am sorry to call this also as O because this is one that is O. So, I did not confuse. So, this is this is the origin and this is it is bad notation.

So, you see you take a translated parallelogram like this and I am considering my I am confining my attention only to number of values number of places inside this parallelogram, where the function is taking the value lambda. So, let me write that. So, I am not trying to write that because it will be too much write down, but let me again repeat when I say number of times f assumes a value, I am looking at only values inside a translate of the period parallelogram that this basic parallelogram.

So that I translate it so that the edges of parallelogram do not have do not hit any poles I mean that is all I want. I do not want the edges of the parallelogram to hit any poles of this function and I look at the number of times f assumes that value. Then this will be equal to the number of zeros of f minus lambda, but you see we have just, but you see f minus lambda is I claim that this is number of poles also of f minus lambda. That is because you see the number of 0 is minus number of poles will be integrating f minus lambda derivative by f minus lambda, which continues to be doubly periodic.

So, again I am using the argument principle alright. So, the number of zeros of f minus lambda will be equal to the number of poles of this will be equal to the number of poles of f minus lambda, and you see translation is not going to affect your pole then the number of poles of f minus lambda I mean these are the points where f is going to infinity; and if f is going to infinity then f minus any constant complex number will also go to infinity. So, this is also going to be equal to number of poles of f.

So the moral of the story and mind you and this is exactly number of times f assumes the value infinity. The number of times at a pole f assumes value infinity. So, what this calculation tells you. So, of course, at this point I am using argument principle, literally replacing the numerator by f minus lambda derivative and put here f minus lambda. So, whatever I proved; I proved that an elliptic function will assume each complex value including the value infinity, exactly the same number of times. It cannot assume one value many more times than it assumes another value and this number is called the order of the elliptic function. This is called this is equal to order of the elliptic function; this is the order of the elliptic function f alright. So, we have already we have examples of this. So, you see. So, the examples that we have seen or you know phe. So, you see zeta. So, phe of z this order 2 well and it is derivative is order 3 and so on.

So, now, you see now this is enough to show this is enough to prove my claim. See because if e 1 is equal to e 2, then you are saying that this value common value which is e 1 or e 2 is assumed at half and at tau by 2, but you see at half already it is assumed twice because you see at half it is already assumed twice, and at tau by 2 it is assumed twice because order is 2 because order is 2 it will assume each value twice. So, at half it will assume the value e 1 twice at tau by 2 it will assume the value e to twice that is another way of saying that p tau of z minus e 1 has a 0 of order 2 at half P tau of z minus e 2 as a 0 of order 2 at tau by 2.

But if the e 1 is equal to e 2 what it will tell you is that, it will assume that common value which is equal to e 1 and e 2 4 times that is not possible because of because it is order 2. So, this is the argument that will imply the claim, that none of the e i is the e is are all distinct and this is extremely important alright. So, this implies the claim above see the point is that since f see the elliptic function f suppose. So, at least phe it assumes the value infinity twice because it has a pole of order 2.

So, what will happen is that you will have the same. So, it will assume each value twice, it cannot assume a value 4 times and the point is each value it assumes at a point with multiplicity 2, it add it assumes each value at a given point with multiplicity 2 right. So, that is the reason why these 3 are all why the e i's are all distinct. Now what you do is now we really take advantage of the dependence on top. So, we do the following thing. So, let me use this side of the board. So, you see what I am going to do is a I mean it is very clever trick. So, what you do is the following. So, you take. So, you again look at this carefully.

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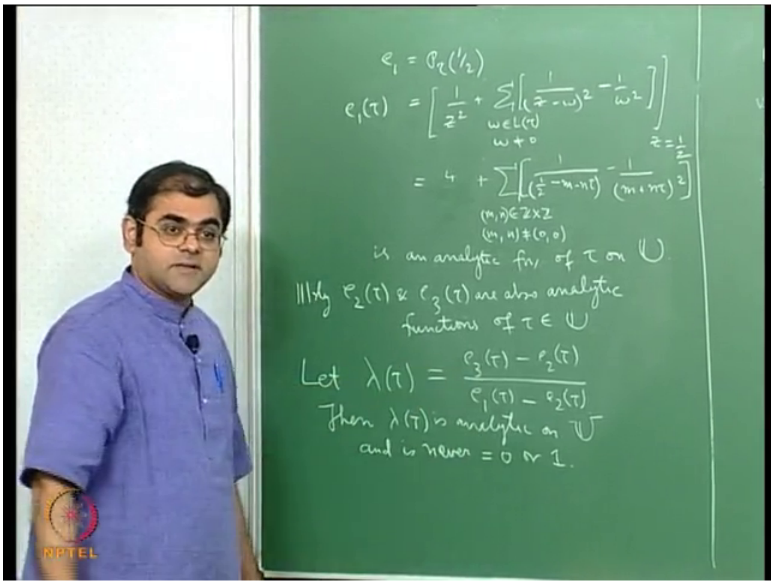

E 1 is p sub tau of half and well what is it let us expand it this is well I have to substitute z zee is equal to half. So, p tau of z is 1 by z squared plus sigma omega in the lattice of defined by tau omega not equal to 0, 1 by z minus omega the whole squared minus 1 by omega squared.

And I have to substitute z zee is equal to half right well. So, if I do that well I will get 4 by z squared here plus well here what I will get is, let me rewrite it as sigma over m comma n an ordered pair of integers and m comma n not equal to 0 comma 0, and here I will get 1 by said this half. So, it is half minus m minus n tau, because omega is supposed to be m plus n tau as m meant and vary that is how you get the lattice and of course, you do not include 0 here because the contribution for 0 is already out there. So, it sorry I think this there is no z here.

So, I put z equal to half. So, I will simply get 4 and that will just be 4 and then I get minus 1 by well omega squared is going to be m plus n tau the whole squared this is what I am going to get if I put z equal to half. Now look at it carefully, you will see that if you forget the lattice, that is the reason why I wrote the summation over zee cross zee forget the lattice. Notice that this never vanishes see because purposely half is not a point of the lattice and this of course, converges. So, you can see very clearly that in the upper half plane as tau varies, this is an analytic function. This is an analytic function of tau on a upper half plane see the clever thing is. So, you see e 1 depends on tau; e 1 is e 1 of tau. So, if you want I should write I should be careful and write e 1 is equal to e 1 of tau.

Now, look at it as a function of tau it is very clear that this is an analytic function of tau. In the same way you plug in for the other 2 roots this other 2 zeros of the derivative of the phe-function. So, you will get e 2 of tau and e 3 of tau are also analytic functions of tau in the upper half plane alright they are also analytic functions of tau. So, by looking at the zeros of the derivative of the weierestrass function, you have produced 3 analytic functions on the upper half plane, and now the trick is your defined lambda of tau to be I think it is e 1 e 3 minus e 2 by e 1 minus e 2. So, you define it as e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau you take this ratio. Notice that I told you for no value of tau these are all the same, the I am short of that is e 1 of tau will never be e 2 of tau. So, this will always be analytic, e 3 of tau will never be e 2 of tau for any given tau.

So, it is never going to be 0 alright. So, what I am going to get and you see e 1 is not equal to e 3. So, this is never going to be 1. So, what I get is, I get a analytic function in the upper half space on the upper half plane which is never 0 which is never 1 and this is going to be the modular form that is going to first give you something that is not invariant under the whole unimodular group, but it you I will prove in the forthcoming lectures that this will be invariant under the under subgroup of PSL to zee, the congruent subgroup mod 2 alright. So, let me write that down.

So, then lambda of tau is analytic on U on the upper half plane, and is never equal to 0 or 1 and why this function is so important is because it will you will see that if you compose this function with an automorphism upper half space given by a unimodular matrix which is whose entries if you read mod 2 you get the identity matrix, then this function is invariant. So, such see usually functions which are analytic are meromorphic functions holomorphic or meromorphic functions are invariant under certain group of mobius transformations, they are called automorphic functions or automorphic forms and those which are invariant under the unimodular group PSL to zee, they are called modular forms. So, and this is a modular form which is not a modular form for the whole uni modular group, but it is a modular form of for only the congruence of group mod 2; namely those matrices in PSL to zee, whose entries if you read them mod 2 you get the identity matrix 2 by 2 identity matrix.

So, that is the importance of this function and. So, this is how you see we cook up one part this is a partial invariant you can think of it as a partial invariant and then use this to. So, what we will do is, we will use this to build a function which is invariant under the full modular group, and that is a function we are looking for. Because we are trying to get hold of a analytic function on the upper half space which is invariant under the whole or unimodular group PSL to zee this is. The first step is to get this modular form of way 2.

So, we will do that in the forthcoming lectures.